INTUITIONISTIC FUZZY d-ALGEBRAS

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Abstract. The intuitionistic fuzzification of a d-algbera is considered, and related results are investigated. The notion of equivalence relations on the family of all intuitionistic fuzzy d-algebras of a d-algebra is introduced, and then some properties are discussed. The concept of intuitionistic fuzzy topological d-algebras is introduced, and some related results are obtained.

1. Introduction

Y. Imai and K. Iséki ([7]) and K. Iséki ([8]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5, 6], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([14]) introduced a new notion, called a d-algebra, which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. In [11], Y. B. Jun, J. Neggers and H. S. Kim introduced the notions of fuzzy d-subalgebra, fuzzy d-ideal, fuzzy $d^\sharp$-ideal and fuzzy d^*-ideal, and investigate relations among them. They also discussed d-ideals in d-algebras (see [13]). The concept of a fuzzy set, which was introduced in [16], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D. H. Foster (cf. [4]) combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld (cf. [15]), to formulate the elements of a theory of fuzzy topological groups. In 1993, Y. B. Jun ([9]) combined the structure of a fuzzy topological spaces with that of a fuzzy BCK-algebras to formulate the elements of a theory of fuzzy topological BCK-algebras. Y. B. Jun and H. S. Kim [10] introduced the concept of fuzzy topological d-algebras of d-algebras and applied some of Foster's results on homomorphic images and inverse images to fuzzy topological d-algebras. After the introduction of fuzzy sets by L. A. Zadeh [16], several researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of intuitionistic fuzzy set was first published by K. T. Atanassov [1], as a generalization of the notion of fuzzy sets. In this paper, using the Atanassov's idea, we establish the notion of intuitionistic fuzzy d-algebras, equivalence relations on the family of all intuitionistic fuzzy d-algebras, and intuitionistic fuzzy topological d-algebras which are a generalization of the notion of fuzzy topological d-algebras, initiated by Jun and Kim [10]. We investigate several properties, and show that the d-homomorphic image and preimage of an intuitionistic fuzzy topological d-algebra is an intuitionistic fuzzy topological d-algebra.

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2. Preliminaries

Definition 2.1 (see [14])**.** A *d-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:
(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$
for all x, y, z in X. A non-empty subset N of a d-algebra X is called a *d-subalgebra* of X if $x*y \in N$ for any $x, y \in N$. A mapping $\alpha : X \to Y$ of d-algebras is called a *d-homomorphism* if $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all $x, y \in X$.

Definition 2.2 (see [1])**.** An *intuitionistic fuzzy set* (IFS for short) D in X is an object having the form

$$D = \{\langle x, \mu_D(x), \gamma_D(x)\rangle \mid x \in X\}$$

where the functions $\mu_D : X \to [0, 1]$ and $\gamma_D : X \to [0, 1]$ denote the degree of membership (namely $\mu_D(x)$) and the degree of nonmembership (namely $\gamma_D(x)$) of each element $x \in X$ to the set D, respectively, and $0 \le \mu_D(x) + \gamma_D(x) \le 1$ for each $x \in X$.

For the sake of simplicity, we shall use the notation $D = \langle x, \mu_D, \gamma_D\rangle$ instead of $D = \{\langle x, \mu_D(x), \gamma_D(x)\rangle \mid x \in X\}$. Let f be a mapping from a set X to a set Y. If

$$B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle \mid y \in Y\}$$

is an IFS in Y, then the *preimage* of B under f, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle \mid x \in X\},$$

and if $D = \{\langle x, \mu_D(x), \gamma_D(x)\rangle \mid x \in X\}$ is an IFS in X, then the *image* of D under f, denoted by $f(D)$, is the IFS in Y defined by

$$f(D) = \{\langle y, f_{\sup}(\mu_D)(y), f_{\inf}(\gamma_D)(y)\rangle \mid y \in Y\},$$

where

$$f_{\sup}(\mu_D)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu_D(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_{\inf}(\gamma_D)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \gamma_D(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

for each $y \in Y$ (see Çoker [3]).

3. Intuitionistic Fuzzy d-algebras

Definition 3.1. Let X be a d-algebra. An IFS $D = \langle x, \mu_D, \gamma_D\rangle$ in X is called an *intuitionistic fuzzy d-algebra* if it satisfies:

$$\mu_D(x * y) \ge \min\{\mu_D(x), \mu_D(y)\} \text{ and } \gamma_D(x * y) \le \max\{\gamma_D(x), \gamma_D(y)\}$$

for all $x, y \in X$.

Example 3.2. (1) Consider a d-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Let $D = \langle x, \mu_D(x), \gamma_D(x)\rangle$ be an IFS in X defined by $\mu_D(0) = \mu_D(a) = 0.8$, $\mu_D(b) = \mu_D(c) = 0.3$, and $\gamma_D(0) = \gamma_D(a) = 0.03$, $\gamma_D(b) = \gamma_D(c) = 0.08$. Then $D = \langle x, \mu_D, \gamma_D\rangle$ is an intuitionistic fuzzy d-algebra.

(2) Consider a d-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Let $D = \langle x, \mu_D, \gamma_D\rangle$ be an IFS in X defined by $\mu_D(0) = \mu_D(a) = \mu_D(c) = t_1$, $\mu_D(b) = t_2$, and $\gamma_D(0) = \gamma_D(a) = \gamma_D(c) = s_1$, $\gamma_D(b) = s_2$, where $t_1 > t_2$, $s_1 < s_2$, and $s_i + t_i \in [0, 1]$ for $i = 1, 2$. Then $D = \langle x, \mu_D, \gamma_D\rangle$ is an intuitionistic fuzzy d-algebra.

In what follows, let X denote a d-algebra unless otherwise specified.

Proposition 3.3. *If an IFS $D = \langle x, \mu_D, \gamma_D\rangle$ in X is an intuitionistic fuzzy d-algebra of X, then $\mu_D(0) \geq \mu_D(x)$ and $\gamma_D(0) \leq \gamma_D(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then $\mu_D(0) = \mu_D(x * x) \geq \min\{\mu_D(x), \mu_D(x)\} = \mu_D(x)$ and $\gamma_D(0) = \gamma_D(x * x) \leq \max\{\gamma_D(x), \gamma_D(x)\} = \gamma_D(x)$. □

Theorem 3.4. *If $\{D_i \mid i \in \Lambda\}$ is an arbitrary family of intuitionistic fuzzy d-algebras of X, then $\cap D_i$ is an intuitionistic fuzzy d-algebra of X, where $\cap D_i = \{\langle x, \wedge\mu_{D_i}(x), \vee\gamma_{D_i}(x)\rangle \mid x \in X\}$.*

Proof. Let $x, y \in X$. Then

$$\wedge\mu_{D_i}(x * y) \geq \wedge(\min\{\mu_{D_i}(x), \mu_{D_i}(y)\}) = \min\{\wedge\mu_{D_i}(x), \wedge\mu_{D_i}(y)\}$$

and

$$\vee\gamma_{D_i}(x * y) \leq \vee(\max\{\gamma_{D_i}(x), \gamma_{D_i}(y)\}) = \max\{\vee\gamma_{D_i}(x), \vee\gamma_{D_i}(y)\}.$$

Hence $\cap D_i = \langle x, \wedge\mu_{D_i}, \vee\gamma_{D_i}\rangle$ is an intuitionistic fuzzy d-algebra of X. □

Theorem 3.5. *If an IFS $D = \langle x, \mu_D, \gamma_D\rangle$ in X is an intuitionistic fuzzy d-algebra of X, then so is $\Box D$, where $\Box D = \{\langle x, \mu_D(x), 1 - \mu_D(x)\rangle \mid x \in X\}$.*

Proof. It is sufficient to show that $\bar{\mu}_D$ satisfies the second condition in Definition 3.1. Let $x, y \in X$. Then

$$\begin{aligned} \bar{\mu}_D(x * y) &= 1 - \mu_D(x * y) \leq 1 - \min\{\mu_D(x), \mu_D(y)\} \\ &= \max\{1 - \mu_D(x), 1 - \mu_D(y)\} \\ &= \max\{\bar{\mu}_D(x), \bar{\mu}_D(y)\}. \end{aligned}$$

Hence $\Box D$ is an intuitionistic fuzzy d-algebra of X. □

Theorem 3.6. *If an IFS $D = \langle x, \mu_D, \gamma_D\rangle$ in X is an intuitionistic fuzzy d-algebra of X, then the sets*

$$X_\mu := \{x \in X \mid \mu_D(x) = \mu_D(0)\} \quad \text{and} \quad X_\gamma := \{x \in X \mid \gamma_D(x) = \gamma_D(0)\}$$

are d-subalgebras of X.

Proof. Let $x, y \in X_\mu$. Then $\mu_D(x) = \mu_D(0) = \mu_D(y)$, and so

$$\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\} = \mu_D(0).$$

By using Proposition 3.3, we know that $\mu_D(x * y) = \mu_D(0)$ or equivalently $x * y \in X_\mu$. Now let $x, y \in X_\gamma$. Then

$$\gamma_D(x * y) \leq \max\{\gamma_D(x), \gamma_D(y)\} = \gamma_D(0),$$

and by applying Proposition 3.3 we conclude that $\gamma_D(x * y) = \gamma_D(0)$ and hence $x * y \in X_\gamma$. □

Definition 3.7. Let $D = \langle x, \mu_D, \gamma_D\rangle$ be an IFS in X and let $t \in [0, 1]$. Then the set

$$U(\mu_D, t) := \{x \in X \mid \mu_D(x) \geq t\} \text{ (resp. } L(\gamma_D, t) := \{x \in X \mid \gamma_D(x) \leq t\})$$

is called a *μ-level t-cut* (resp. *γ-level t-cut*) of D.

Theorem 3.8. *If an IFS $D = \langle x, \mu_D, \gamma_D\rangle$ in X is an intuitionistic fuzzy d-algebra of X, then the μ-level t-cut and γ-level t-cut of D are d-subalgebras of X for every $t \in [0, 1]$ such that $t \in \mathrm{Im}(\mu_D) \cap \mathrm{Im}(\gamma_D)$,* which are called a *$\mu$-level d-subalgebra* and a *γ-level d-subalgebra* respectively.

Proof. Let $x, y \in U(\mu_D, t)$. Then $\mu_D(x) \geq t$ and $\mu_D(y) \geq t$. It follows that $\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\} \geq t$ so that $x * y \in U(\mu_D, t)$. Hence $U(\mu_D, t)$ is a d-subalgebra of X. Now let $x, y \in L(\gamma_D, t)$. Then $\gamma_D(x * y) \leq \max\{\gamma_D(x), \gamma_D(y)\} \leq t$ and so $x * y \in L(\gamma_D, t)$. Therefore $L(\gamma_D, t)$ is a d-subalgebra of X. □

Theorem 3.9. *Let $D = \langle x, \mu_D, \gamma_D\rangle$ be an* IFS *in X such that the sets $U(\mu_D, t)$ and $L(\gamma_D, t)$ are d-subalgebras of X. Then $D = \langle x, \mu_D, \gamma_D\rangle$ is an intuitionistic fuzzy d-algebra of X.*

Proof. Assume that there exist $x_0, y_0 \in X$ such that $\mu_D(x_0 * y_0) < \min\{\mu_D(x_0), \mu_D(y_0)\}$. Let

$$t_0 := \frac{1}{2}\Big(\mu_D(x_0 * y_0) + \min\{\mu_D(x_0), \mu_D(y_0)\}\Big).$$

Then $\mu_D(x_0 * y_0) < t_0 < \min\{\mu_D(x_0), \mu_D(y_0)\}$ and so $x_0 * y_0 \notin U(\mu_D, t_0)$, but $x_0, y_0 \in U(\mu_D, t_0)$. This is a contradiction, and therefore

$$\mu_D(x * y) \geq \min\{\mu_D(x), \mu_D(y)\}$$

for all $x, y \in X$. Now suppose that

$$\gamma_D(x_0 * y_0) > \max\{\gamma_D(x_0), \gamma_D(y_0)\}$$

for some $x_0, y_0 \in X$. Taking

$$s_0 := \frac{1}{2}\Big(\gamma_D(x_0 * y_0) + \max\{\gamma_D(x_0), \gamma_D(y_0)\}\Big),$$

then $\max\{\gamma_D(x_0), \gamma_D(y_0)\} < s_0 < \gamma_D(x_0 * y_0)$. It follows that $x_0, y_0 \in L(\gamma_D, s_0)$ and $x_0 * y_0 \notin L(\gamma_D, s_0)$, a contradiction. Hence

$$\gamma_D(x * y) \leq \max\{\gamma_D(x), \gamma_D(y)\}$$

for all $x, y \in X$. This completes the proof. □

Theorem 3.10. *Any d-subalgebra of X can be realized as both a μ-level d-subalgebra and a γ-level d-subalgebra of some intuitionistic fuzzy d-algebra of X.*

Proof. Let S be a d-subalgebra of X and let μ_D and γ_D be fuzzy sets in X defined by

$$\mu_D(x) := \begin{cases} t, & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\gamma_D(x) := \begin{cases} s, & \text{if } x \in S, \\ 1, & \text{otherwise,} \end{cases}$$

for all $x \in X$ where t and s are fixed numbers in $(0, 1)$ such that $t + s < 1$. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$. Hence $\mu_D(x * y) = \min\{\mu_D(x), \mu_D(y)\}$ and $\gamma_D(x * y) = \max\{\gamma_D(x), \gamma_D(y)\}$. If at least one of x and y does not belong to S, then at least one of

$\mu_D(x)$ and $\mu_D(y)$ is equal to 0, and at least one of $\gamma_D(x)$ and $\gamma_D(y)$ is equal to 1. It follows that

$$\mu_D(x*y) \geq 0 = \min\{\mu_D(x),\, \mu_D(y)\},$$
$$\gamma_D(x*y) \leq 1 = \max\{\gamma_D(x),\, \gamma_D(y)\}.$$

Hence $D = \langle x, \mu_D, \gamma_D\rangle$ is an intuitionistic fuzzy d-algebra of X. Obviously, $U(\mu_D,\, t) = S = L(\gamma_D,\, s)$. This completes the proof. □

Theorem 3.11. *Let α be a d-homomorphism of a d-algebra X into a d-algebra Y and B an intuitionistic fuzzy d-algebra of Y. Then $\alpha^{-1}(B)$ is an intuitionistic fuzzy d-algebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\mu_{\alpha^{-1}(B)}(x*y) &= \mu_B(\alpha(x*y)) = \mu_B(\alpha(x)*\alpha(y))\\ &\geq \min\{\mu_B(\alpha(x)), \mu_B(\alpha(y))\}\\ &= \min\{\mu_{\alpha^{-1}(B)}(x), \mu_{\alpha^{-1}(B)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\gamma_{\alpha^{-1}(B)}(x*y) &= \gamma_B(\alpha(x*y)) = \gamma_B(\alpha(x)*\alpha(y))\\ &\leq \max\{\gamma_B(\alpha(x)), \gamma_B(\alpha(y))\}\\ &= \max\{\gamma_{\alpha^{-1}(B)}(x), \gamma_{\alpha^{-1}(B)}(y)\}.\end{aligned}$$

Hence $\alpha^{-1}(B)$ is an intuitionistic fuzzy d-algebra in X. □

Theorem 3.12. *Let α be a d-homomorphism of a d-algebra X onto a d-algebra Y. If $D = \langle x, \mu_D, \gamma_D\rangle$ is an intuitionistic fuzzy d-algebra of X, then $\alpha(D) = \langle y, \alpha_{\sup}(\mu_D), \alpha_{\inf}(\gamma_D)\rangle$ is an intuitionistic fuzzy d-algebra of Y.*

Proof. Let $D = (x, \mu_D, \gamma_D)$ be an intuitionistic fuzzy topological d-algebra in X and let $y_1, y_2 \in Y$. Noticing that

$$\{x_1 * x_2 \mid x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\} \subseteq \{x \in X \mid x \in \alpha^{-1}(y_1 * y_2)\},$$

we have

$$\begin{aligned}&\alpha_{\sup}(\mu_D)(y_1*y_2)\\ =\ &\sup\{\mu_D(x) \mid x \in \alpha^{-1}(y_1*y_2)\}\\ \geq\ &\sup\{\mu_D(x_1*x_2) \mid x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\}\\ \geq\ &\sup\Big\{\min\{\mu_D(x_1), \mu_D(x_2)\} \mid x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\Big\}\\ =\ &\min\Big\{\sup\{\mu_D(x_1) \mid x_1 \in \alpha^{-1}(y_1)\}, \sup\{\mu_D(x_2) \mid x_2 \in \alpha^{-1}(y_2)\}\Big\}\\ =\ &\min\{\alpha_{\sup}(\mu_D)(y_1), \alpha_{\sup}(\mu_D)(y_2)\}\end{aligned}$$

and

$$\begin{aligned}&\alpha_{\inf}(\gamma_D)(y_1*y_2)\\ =\ &\inf\{\gamma_D(x) \mid x \in \alpha^{-1}(y_1*y_2)\}\\ \leq\ &\inf\{\gamma_D(x_1*x_2) \mid x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\}\\ \leq\ &\inf\Big\{\max\{\gamma_D(x_1), \gamma_D(x_2)\} \mid x_1 \in \alpha^{-1}(y_1) \text{ and } x_2 \in \alpha^{-1}(y_2)\}\Big\}\\ =\ &\max\Big\{\inf\{\gamma_D(x_1) \mid x_1 \in \alpha^{-1}(y_1)\}, \inf\{\gamma_D(x_2) \mid x_2 \in \alpha^{-1}(y_2)\}\Big\}\\ =\ &\max\{\alpha_{\inf}(\gamma_D)(y_1), \alpha_{\inf}(\gamma_D)(y_2)\}.\end{aligned}$$

Hence $\alpha(D) = \langle y, \alpha_{\sup}(\mu_D), \alpha_{\inf}(\gamma_D)\rangle$ is an intuitionistic fuzzy d-algebra in Y. □

Let $\Omega(X)$ denote the family of all intuitionistic fuzzy d-algebras of X and let $t \in [0, 1]$. Define binary relations $\sim_\mu$ and $\sim_\gamma$ on $\Omega(X)$ as follows:

$$A \sim_\mu B \Leftrightarrow U(\mu_A,\, t) = U(\mu_B,\, t) \text{ and } A \sim_\gamma B \Leftrightarrow L(\gamma_A,\, t) = L(\gamma_B,\, t),$$

respectively, for $A = \langle x, \mu_A, \gamma_A\rangle$ and $B = \langle x, \mu_B, \gamma_B\rangle$ in $\Omega(X)$. Then clearly $\sim_\mu$ and $\sim_\gamma$ are equivalence relations on $\Omega(X)$. For any $A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)$, let $[A]_\mu$ (resp. $[A]_\gamma$) denote the equivalence class of $A = \langle x, \mu_A, \gamma_A\rangle$ modulo $\sim_\mu$ (resp. $\sim_\gamma$), and denote by $\Omega(X)/\sim_\mu$ (resp. $\Omega(X)/\sim_\gamma$) the collection of all equivalence classes of A modulo $\sim_\mu$ (resp. $\sim_\gamma$), i.e.,

$$\Omega(X)/\sim_\mu := \{[A]_\mu \mid A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)\}$$
$$(\text{resp. } \Omega(X)/\sim_\gamma := \{[A]_\gamma \mid A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)\}).$$

Now let $S(X)$ denote the family of all d-subalgebras of X and let $t \in [0, 1]$. Define maps α_t and β_t from $\Omega(X)$ to $S(X) \cup \{\emptyset\}$ by $\alpha_t(A) = U(\mu_A, t)$ and $\beta_t(A) = L(\gamma_A, t)$, respectively, for all $A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)$. Then α_t and β_t are clearly well-defined.

Theorem 3.13. *For any $t \in (0, 1)$ the maps α_t and β_t are surjective from $\Omega(X)$ to $S(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0, 1)$. Note that $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1}\rangle$ is in $\Omega(X)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$. Obviously $\alpha_t(\mathbf{0}_\sim) = U(\mathbf{0}, t) = \emptyset = L(\mathbf{1}, t) = \beta_t(\mathbf{0}_\sim)$. Let $G(\neq \emptyset) \in S(X)$. For $G_\sim = \langle x, \chi_G, \bar{\chi}_G\rangle \in \Omega(X)$, we have $\alpha_t(G_\sim) = U(\chi_G, t) = G$ and $\beta_t(G_\sim) = L(\bar{\chi}_G, t) = G$. Hence α_t and β_t are surjective. □

Theorem 3.14. *The quotient sets $\Omega(X)/\sim_\mu$ and $\Omega(X)/\sim_\gamma$ are equipotent to $S(X) \cup \{\emptyset\}$ for every $t \in (0, 1)$.*

Proof. For $t \in (0, 1)$ let α_t^* (resp. β_t^*) be a map from $\Omega(X)/\sim_\mu$ (resp. $\Omega(X)/\sim_\gamma$) to $S(X) \cup \{\emptyset\}$ defined by $\alpha_t^*([A]_\mu) = \alpha_t(A)$ (resp. $\beta_t^*([A]_\gamma) = \beta_t(A)$) for all $A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)$. If $U(\mu_A, t) = U(\mu_B, t)$ and $L(\gamma_A, t) = L(\gamma_B, t)$ for $A = \langle x, \mu_A, \gamma_A\rangle$ and $B = \langle x, \mu_B, \gamma_B\rangle$ in $\Omega(X)$, then $A \sim_\mu B$ and $A \sim_\gamma B$; hence $[A]_\mu = [B]_\mu$ and $[A]_\gamma = [B]_\gamma$. Therefore the maps α_t^* and β_t^* are injective. Now let $G(\neq \emptyset) \in S(X)$. For $G_\sim = \langle x, \chi_G, \bar{\chi}_G\rangle \in \Omega(X)$, we have

$$\alpha_t^*([G_\sim]_\mu) = \alpha_t(G_\sim) = U(\chi_G, t) = G$$

and

$$\beta_t^*([G_\sim]_\gamma) = \beta_t(G_\sim) = L(\bar{\chi}_G, t) = G.$$

Finally, for $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1}\rangle \in \Omega(X)$ we get

$$\alpha_t^*([\mathbf{0}_\sim]_\mu) = \alpha_t(\mathbf{0}_\sim) = U(\mathbf{0}, t) = \emptyset$$

and

$$\beta_t^*([\mathbf{0}_\sim]_\gamma) = \beta_t(\mathbf{0}_\sim) = L(\mathbf{1}, t) = \emptyset.$$

This shows that α_t^* and β_t^* are surjective, and we are done. □

For any $t \in [0, 1]$, we define another relation $\mathfrak{R}^t$ on $\Omega(X)$ as follows:

$$(A, B) \in \mathfrak{R}^t \Leftrightarrow U(\mu_A, t) \cap L(\gamma_A, t) = U(\mu_B, t) \cap L(\gamma_B, t)$$

for any $A = \langle x, \mu_A, \gamma_A\rangle$, $B = \langle x, \mu_B, \gamma_B\rangle \in \Omega(X)$. Then the relation $\mathfrak{R}^t$ is also an equivalence relation on $\Omega(X)$.

Theorem 3.15. *For any $t \in (0, 1)$, the map $\phi_t : \Omega(X) \to S(X) \cup \{\emptyset\}$ defined by $\phi_t(A) = \alpha_t(A) \cap \beta_t(A)$ for each $A = \langle x, \mu_A, \gamma_A\rangle \in \Omega(X)$ is surjective.*

Proof. Let $t \in (0, 1)$. For $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1}\rangle \in \Omega(X)$, we get

$$\phi_t(\mathbf{0}_\sim) = \alpha_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U(\mathbf{0}, t) \cap L(\mathbf{1}, t) = \emptyset.$$

For any $H \in \Omega(X)$, there exists $H_\sim = \langle x, \chi_H, \bar{\chi}_H\rangle \in \Omega(X)$ such that

$$\phi_t(H_\sim) = \alpha_t(H_\sim) \cap \beta_t(H_\sim) = U(\chi_H, t) \cap L(\bar{\chi}_H, t) = H.$$

This completes the proof. □

Theorem 3.16. *For any $t \in (0,1)$, the quotient set $\Omega(X)/\mathfrak{R}^t$ is equipotent to $S(X) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$ and let $\phi_t^* : \Omega(X)/\mathfrak{R}^t \to S(X) \cup \{\emptyset\}$ be a map defined by $\phi_t^*([A]_{\mathfrak{R}^t}) = \phi_t(A)$ for all $[A]_{\mathfrak{R}^t} \in \Omega(X)/\mathfrak{R}^t$. Assume that $\phi_t^*([A]_{\mathfrak{R}^t}) = \phi_t^*([B]_{\mathfrak{R}^t})$ for any $[A]_{\mathfrak{R}^t}, [B]_{\mathfrak{R}^t} \in \Omega(X)/\mathfrak{R}^t$. Then $\alpha_t(A) \cap \beta_t(A) = \alpha_t(B) \cap \beta_t(B)$, i.e., $U(\mu_A, t) \cap L(\gamma_A, t) = U(\mu_B, t) \cap L(\gamma_B, t)$. Hence $(A, B) \in \mathfrak{R}^t$, and so $[A]_{\mathfrak{R}^t} = [B]_{\mathfrak{R}^t}$. Therefore ϕ_t^* is injective. Now for $\mathbf{0}_\sim = \langle x, \mathbf{0}, \mathbf{1} \rangle \in \Omega(X)$ we have

$$\phi_t^*([\mathbf{0}_\sim]_{\mathfrak{R}^t}) = \phi_t(\mathbf{0}_\sim) = \alpha_t(\mathbf{0}_\sim) \cap \beta_t(\mathbf{0}_\sim) = U(\mathbf{0}, t) \cap L(\mathbf{1}, t) = \emptyset.$$

For $H_\sim = \langle x, \chi_H, \bar{\chi}_H \rangle \in \Omega(X)$ we get

$$\phi_t^*([H_\sim]_{\mathfrak{R}^t}) = \phi_t(H_\sim) = \alpha_t(H_\sim) \cap \beta_t(H_\sim) = U(\chi_H, t) \cap L(\bar{\chi}_H, t) = H.$$

Thus ϕ_t^* is surjective. This completes the proof. □

4. Intuitionistic Fuzzy Topological d-algebras

In [3], Çoker generalized the concept of fuzzy topological space, first initiated by Chang [2], to the case of intuitionistic fuzzy sets as follows.

Definition 4.1 (see [3])**.** An *intuitionistic fuzzy topology* (IFT for short) on a nonempty set X is a family Φ of IFSs in X satisfying the following axioms:

(T1) $\mathbf{0}_\sim, \mathbf{1}_\sim \in \Phi$,

(T2) $G_1 \cap G_2 \in \Phi$ for any $G_1, G_2 \in \Phi$,

(T3) $\bigcup_{i \in J} G_i \in \Phi$ for any family $\{G_i : i \in J\} \subseteq \Phi$.

In this case the pair (X, Φ) is called an *intuitionistic fuzzy topological space* (IFTS for short) and any IFS in Φ is called an *intuitionistic fuzzy open set* (IFOS for short) in X.

Definition 4.2 (see [3])**.** Let (X, Φ) and (Y, Ψ) be two IFTSs. A mapping $f : X \to Y$ is said to be *intuitionistic fuzzy coninuous* if the preimage of each IFS in Ψ is an IFS in Φ; and f is said to be *intuitionistic fuzzy open* if the image of each IFS in Φ is an IFS in Ψ.

Definition 4.3. Let D be an IFS in an IFTS (X, Φ). Then the *induced intuitionistic fuzzy topology* (IIFT for short) on D is the family of IFSs in D which are the intersection with D of IFOSs in X. The IIFT is denoted by Φ_D, and the pair (D, Φ_D) is called an *intuitionistic fuzzy subspace* of (X, Φ).

Definition 4.4. Let (D, Φ_D) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ), respectively, and let $f : X \to Y$ be a mapping. Then f is a mapping of D into B if $f(D) \subset B$. Furthermore, f is said to be *relatively intuitionistic fuzzy continuous* if for each IFS V_B in Ψ_B, the intersection $f^{-1}(V_B) \cap D$ is an IFS in Φ_D; and f is said to be *relatively intuitionistic fuzzy open* if for each IFS U_D in Φ_D, the image $f(U_D)$ is an IFS in Ψ_B.

Proposition 4.5. *Let (D, Φ_D) and (B, Ψ_B) be intuitionistic fuzzy subspaces of IFTSs (X, Φ) and (Y, Ψ) respectively, and let f be an intuitionistic fuzzy continuous mapping of X into Y such that $f(D) \subset B$. Then f is relatively intuitionistic fuzzy continuous mapping of D into B.*

Proof. Let V_B be an IFS in Ψ_B. Then there exists $V \in \Psi$ such that $V_B = V \cap B$. Since f is intuitionistic fuzzy continuous, it follows that $f^{-1}(V)$ is an IFS in Φ. Hence

$$f^{-1}(V_B) \cap D = f^{-1}(V \cap B) \cap D = f^{-1}(V) \cap f^{-1}(B) \cap D = f^{-1}(V) \cap D$$

is an IFS in Φ_D. This completes the proof. □

For any d-algebra X and any element $a \in X$ we use a_r denote the selfmap of X defined by $a_r(x) = x * a$ for all $x \in X$.

Definition 4.6. Let X be a d-algebra, Φ an IFT on X and D an intuitionistic fuzzy d-algebra with IIFT Φ_D. Then D is called an *intuitionistic fuzzy topological d-algebra* if for each $a \in X$ the mapping $a_r : (D, \Phi_D) \to (D, \Phi_D)$, $x \mapsto x * a$, is relatively intuitionistic fuzzy continuous.

Theorem 4.7. *Given d-algebras X and Y, and a d-homomorphism $\alpha : X \to Y$, let Φ and Ψ be the IFTs on X and Y respectively such that $\Phi = \alpha^{-1}(\Psi)$. If B is an intuitionistic fuzzy topological d-algebra in Y, then $\alpha^{-1}(B)$ is an intuitionistic fuzzy topological d-algebra in X.*

Proof. Let $a \in X$ and let U be an IFS in $\Phi_{\alpha^{-1}(B)}$. Since α is an intuitionistic fuzzy continuous mapping of (X, Φ) into (Y, Ψ), it follows from Proposition 4.5 that α is a relatively intuitionistic fuzzy continuous mapping of $(\alpha^{-1}(B), \Phi_{\alpha^{-1}(B)})$ into (B, Ψ_B). Note that there exists an IFS V in Ψ_B such that $\alpha^{-1}(V) = U$. Then

$$\begin{aligned} \mu_{a_r^{-1}(U)}(x) &= \mu_U(a_r(x)) = \mu_U(x * a) = \mu_{\alpha^{-1}(V)}(x * a) \\ &= \mu_V(\alpha(x * a)) = \mu_V(\alpha(x) * \alpha(a)) \end{aligned}$$

and

$$\begin{aligned} \gamma_{a_r^{-1}(U)}(x) &= \gamma_U(a_r(x)) = \gamma_U(x * a) = \gamma_{\alpha^{-1}(V)}(x * a) \\ &= \gamma_V(\alpha(x * a)) = \gamma_V(\alpha(x) * \alpha(a)). \end{aligned}$$

Since B is an intuitionistic fuzzy topological d-algebra in Y, the mapping

$$b_r : (B, \Psi_B) \to (B, \Psi_B),\ y \mapsto y * b$$

is relatively intuitionistic fuzzy continuous for every $b \in Y$. Hence

$$\begin{aligned} \mu_{a_r^{-1}(U)}(x) &= \mu_V(\alpha(x) * \alpha(a)) = \mu_V(\alpha(a)_r(\alpha(x))) \\ &= \mu_{\alpha(a)_r^{-1}(V)}(\alpha(x)) = \mu_{\alpha^{-1}(\alpha(a)_r^{-1}(V))}(x) \end{aligned}$$

and

$$\begin{aligned} \gamma_{a_r^{-1}(U)}(x) &= \gamma_V(\alpha(x) * \alpha(a)) = \gamma_V(\alpha(a)_r(\alpha(x))) \\ &= \gamma_{\alpha(a)_r^{-1}(V)}(\alpha(x)) = \gamma_{\alpha^{-1}(\alpha(a)_r^{-1}(V))}(x). \end{aligned}$$

Therefore $a_r^{-1}(U) = \alpha^{-1}(\alpha(a)_r^{-1}(V))$, and so

$$a_r^{-1}(U) \cap \alpha^{-1}(B) = \alpha^{-1}(\alpha(a)_r^{-1}(V)) \cap \alpha^{-1}(B)$$

is an IFS in $\Phi_{\alpha^{-1}(B)}$. This completes the proof. □

Theorem 4.8. *Given d-algebras X and Y, and a d-isomorphism α of X to Y, let Φ and Ψ be the IFTs on X and Y respectively such that $\alpha(\Phi) = \Psi$. If D is an intuitionistic fuzzy topological d-algebra in X, then $\alpha(D)$ is an intuitionistic fuzzy topological d-algebra in Y.*

Proof. It is sufficient to show that the mapping

$$b_r : (\alpha(D), \Psi_{\alpha(D)}) \to (\alpha(D), \Psi_{\alpha(D)}),\ y \mapsto y * b$$

is relatively intuitionistic fuzzy continuous for each $b \in Y$. Let U_D be an IFS in Φ_D. Then there exists an IFS U in Φ such that $U_D = U \cap D$. Since α is one-one, it follows that

$$\alpha(U_D) = \alpha(U \cap D) = \alpha(U) \cap \alpha(D)$$

which is an IFS in $\Psi_{\alpha(D)}$. This shows that α is relatively intuitionistic fuzzy open. Let $V_{\alpha(D)}$ be an IFS in $\Psi_{\alpha(D)}$. The surjectivity of α implies that for each $b \in Y$ there exists

$a \in X$ such that $b = \alpha(a)$. Hence

$$\begin{aligned}\mu_{\alpha^{-1}(b_r^{-1}(V_{\alpha(D)}))}(x) &= \mu_{\alpha^{-1}(\alpha(a)_r^{-1}(V_{\alpha(D)}))}(x) = \mu_{\alpha(a)_r^{-1}(V_{\alpha(D)})}(\alpha(x)) \\ &= \mu_{V_{\alpha(D)}}(\alpha(a)_r(\alpha(x))) = \mu_{V_{\alpha(D)}}(\alpha(x) * \alpha(a)) \\ &= \mu_{V_{\alpha(D)}}(\alpha(x * a)) = \mu_{\alpha^{-1}(V_{\alpha(D)})}(x * a) \\ &= \mu_{\alpha^{-1}(V_{\alpha(D)})}(a_r(x)) = \mu_{a_r^{-1}(\alpha^{-1}(V_{\alpha(D)}))}(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_{\alpha^{-1}(b_r^{-1}(V_{\alpha(D)}))}(x) &= \gamma_{\alpha^{-1}(\alpha(a)_r^{-1}(V_{\alpha(D)}))}(x) = \gamma_{\alpha(a)_r^{-1}(V_{\alpha(D)})}(\alpha(x)) \\ &= \gamma_{V_{\alpha(D)}}(\alpha(a)_r(\alpha(x))) = \gamma_{V_{\alpha(D)}}(\alpha(x) * \alpha(a)) \\ &= \gamma_{V_{\alpha(D)}}(\alpha(x * a)) = \gamma_{\alpha^{-1}(V_{\alpha(D)})}(x * a) \\ &= \gamma_{\alpha^{-1}(V_{\alpha(D)})}(a_r(x)) = \gamma_{a_r^{-1}(\alpha^{-1}(V_{\alpha(D)}))}(x).\end{aligned}$$

Therefore $\alpha^{-1}(b_r^{-1}(V_{\alpha(D)}) = a_r^{-1}(\alpha^{-1}(V_{\alpha(D)}))$. By hypothesis, the mapping

$$a_r : (D, \Phi_D) \to (D, \Phi_D),\ x \mapsto x * a$$

is relatively intuitionistic fuzzy continuous and α is a relatively intuitionistic fuzzy continuous map: $(D, \Phi_D) \to (\alpha(D), \Psi_{\alpha(D)})$. Thus

$$\alpha^{-1}(b_r^{-1}(V_{\alpha(D)})) \cap D = a_r^{-1}(\alpha^{-1}(V_{\alpha(D)})) \cap D$$

is an IFS in Φ_D. Since α is relatively intuitionistic fuzzy open,

$$\alpha(\alpha^{-1}(b_r^{-1}(V_{\alpha(D)})) \cap D) = b_r^{-1}(V_{\alpha(D)}) \cap \alpha(D)$$

is an IFS in $\Psi_{\alpha(D)}$. This completes the proof. □

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