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CORRECTION AND ADDITION TO "L^2-BOUNDEDNESS OF MARCINKIEWICZ INTEGRALS ALONG SURFACES WITH VARIABLE KERNELS"

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Abstract. In our paper "L^2-boundedness of Marcinkiewicz integrals along surfaces with variable kernels", we have found that Theorems 3 and 5 are not correct. Here, we reformulate Theorems 3 and 5, so that L^2-boundedness holds as in Theorems 1 and 4.

1 Introduction We became aware of an essential defect in the proof of Theorem 3 of our paper [8, p. 380]. We have missed to apply a change of variable to the integral defining $N_t(\xi)$. When $\Phi(t) = t^{-\alpha}$ $(\alpha > 0)$, for the change of variable $\rho = \Phi(r)|\xi|$ we have

$$N_t(\xi) = \frac{1}{t}\int_0^t \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}}dr = -C_2\frac{1}{\Phi^{-1}(\frac{s}{|\xi|})}\int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho,$$

where $s = \Phi(t)|\xi|$ and $\frac{\Phi(t)}{t} = C_2\Phi'(t)\varphi(t)$. In our paper [8, p. 380], we have taken the integration interval as $(0, s)$ in place of the correct one $(s, +\infty)$. Because of this error we have led false claims in Theorems 3 and 5.

So, in our paper [8], we delete the sentence "We also show some sharp difference betweeen properties of singular integrals and the Marcinkiewicz integral with rough variable kernels." in the abstract. We delete the lines 26 through 34 in the page 371, and Theorem 5 in the page 372. We also delete the section 4.

Other corrections: In the assumption of Lemma 2.3 we add "$\nu - \lambda > -1$". In the line 9 of the page 371, "exists" should be "exist". In the line 3 from the bottom in the page 372, "we only give" should be "we have only to give". In the line 2 of the page 375, "$\frac{1}{\Phi'(\Phi^{-1}(\frac{\rho}{|\xi|}))}$" should be "$\frac{1}{|\xi|\Phi'(\Phi^{-1}(\frac{\rho}{|\xi|}))}$". In the line 6 from the bottom of the same page, we should multiply the first and second terms in the brace by C_2. In the last line in the page 379, "$\sum_{j=1}^{D_k}$" after the integral sign should be deleted. In the line 5 in the page 380, "$\frac{1}{t^\sigma}$" shoud be "$\frac{C}{t^\sigma}$", and the term "$e^{-i\Phi(r)\xi\cdot y'}$" should be deleted. In the line 8 in the page 380, "$\frac{[g^{-1}(t)]^\sigma}{t^\varepsilon}$" should be "$\frac{[\Phi^{-1}(t)]^\sigma}{t^\varepsilon}$".

We have noticed that also in the case of Theorems 3 and 5, the same L^2-boundedness holds as in Theorems 1 and 4, respectively. So, we shall give new Theorems 3 and 5 in this paper.

First, we recall some definitions (see also in [8]).

Definition 1. Let S^{n-1} be the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with Lebesgue measure $d\sigma = d\sigma(x')$. A function $\Omega(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ $(q \geq 1)$ if

(e) $\Omega(x, \lambda y) = \Omega(x, y)$, for any $x, y \in \mathbb{R}^n$ and $\lambda > 0$;

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(f) $\|\Omega\|_{L^\infty \times L^q(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x,y')|^q d\sigma(y') \right)^{1/q} < \infty$, where $y' = y/|y|$ for any $y \in \mathbb{R}^n \backslash \{0\}$.

We have defined the Marcinkiewicz integral with rough variable kernels associated with surfaces of the form $\{x = \Phi(|y|)y'\}$ by

$$\mu_\Omega^\Phi(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-1}} f(x - \Phi(|y|)y') dy.$$

Then, new Theorem 3 can be formulated as follows:

Theorem 3. *Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ $(q > 2(n-1)/n)$ and satisfies $\int_{S^{n-1}} \Omega(x,y')\, d\sigma(y') = 0$. Let Φ be a positive and strictly decreasing (or negative and strictly increasing) C^1 function and satisfy $\frac{\Phi(t)}{t} = C_2 \Phi'(t)\varphi(t)$ for all $t \in (0,\infty)$, where φ is a function defined on $(0,\infty)$ and there exist two constants δ, M such that $0 < \delta \le \varphi(t) \le M$. Suppose moreover φ satisfies one of the following conditions:*

(i) *$t\varphi'(t)$ is bounded;*

(ii) *φ is a monotonic function.*

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \le C\|f\|_2$, where constant C is independent of f.

Remark 1. There is no including relationship between condition (i) and condition (ii), this can be seen from the examples given in Section 2 of [8]. The case where Φ is a positive and strictly increasing (or negative and strictly decreasing) function has already been studied in [8] (Theorem 1 there). We also must point out that C_2 is negative under the condition of Φ and φ in Theorem 3.

To state new Theorem 5, we recall the following parametric Marcinkiewicz integral, parametric area integral and parametric μ_λ^* function, which are defined by

$$\mu_\Omega^{\Phi,\sigma}(f)(x) = \left(\int_0^\infty \left| \int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-\sigma}} f(x-\Phi(|y|)y')dy \right|^2 \frac{dt}{t^{1+2\sigma}} \right)^{1/2},$$

$$\mu_S^{\Phi,\sigma}(f)(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^\sigma} \int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

$$\mu_{\lambda,\Phi}^{*,\sigma}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \frac{1}{t^\sigma} \int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and $\lambda > 1$.

Now new Theorem 5 is:

Theorem 5. *Let $\sigma > 0$. Then Theorem 3 still holds for the parametric operator $\mu_\Omega^{\Phi,\sigma}$, $\mu_S^{\Phi,\sigma}$ and $\mu_{\lambda,\Phi}^{*,\sigma}$.*

2 Proof of Theorem 3 We begin with a lemma playing the role of Lemma 2.2 in [8].

Lemma 2.1. *Let $g(t)$ be a nonnegative (positive) and non-increasing (strictly decreasing) function on $(0,\infty)$ such that there exists a bounded function $\varphi(t)$ satisfying*

$$\frac{g(t)}{t} = -g'(t)\varphi(t).$$

Let $\sigma > 0$. If there exists $\delta > 0$ such that $0 < \delta \leq \varphi(t)$ on $(0,\infty)$, then $t^\varepsilon[g^{-1}(t)]^\sigma$ is non-increasing (strictly increasing) on $(0,\infty)$ for $0 < \varepsilon \leq \sigma\delta$ $(0 < \varepsilon < \sigma\delta)$. Conversely, if $\frac{[g^{-1}(t)]^\sigma}{t^\varepsilon}$ is non-increasing (strictly decreasing) for some $\varepsilon > 0$, then $\varphi(t) \geq \frac{\varepsilon}{\sigma}$ $(\varphi(t) > \frac{\varepsilon}{\sigma})$.

Proof. It is easily seen that we have only to give the proof for $\sigma = 1$. Set $f(t) = t^\varepsilon g^{-1}(t)$. Then

$$\begin{aligned} f'(t) &= \varepsilon t^{\varepsilon-1}g^{-1}(t) + t^\varepsilon\frac{1}{g'(g^{-1}(t))} = \varepsilon t^{\varepsilon-1}g^{-1}(t) - t^\varepsilon\frac{g^{-1}(t)\varphi(g^{-1}(t))}{t} \\ &= t^{\varepsilon-1}g^{-1}(t)\big(\varepsilon - \varphi(g^{-1}(t))\big). \end{aligned}$$

Thus we have $t^\varepsilon g^{-1}(t)$ is non-increasing (strictly decreasing) if and only if $\varphi(t) \geq \varepsilon$ $(\varphi(t) > \varepsilon)$. This implies the desired conclusion. $\square$

Remark 2. We note the following: If $g(t) \in C^1(0,\infty)$ is positive and decreasing (increasing) on $(0,\infty)$ and $g(t)/(tg'(t))$ is bounded on $(0,\infty)$, then it follows that $\lim_{t\to 0} g(t) = +\infty$ $(\lim_{t\to 0} g(t) = 0)$ and $\lim_{t\to+\infty} g(t) = 0$ $(\lim_{t\to+\infty} g(t) = +\infty)$, respectively.

We give a proof in the decreasing case, since in the increasing case we can show similarly. Since $g(t)/(tg'(t))$ is bounded on $(0,\infty)$, we take $M > 0$ so that $g(t)/|tg'(t)| \leq M$ on $(0,\infty)$. Then, since $g(t)$ is positive and decreasing, we have

$$-g'(t) \geq \frac{g(t)}{Mt} \geq \frac{g(1)}{Mt} \text{ for } 0 < t < 1.$$

Hence for $0 < t < 1$

$$g(t) = g(1) - \int_t^1 g'(s)\,ds \geq g(1) + \frac{g(1)}{M}\int_t^1 \frac{ds}{s} = g(1) + \frac{g(1)}{M}\log\frac{1}{t},$$

which implies $\lim_{t\to 0} g(t) = +\infty$. Next, since $g(t)$ is positive and decreasing, $\lim_{t\to+\infty} g(t)$ exists. So,

$$\lim_{t\to+\infty}\big(g(1) - g(t)\big) = \lim_{t\to+\infty}\int_1^t -g'(s)\,ds = \int_1^\infty -g'(s)\,ds,$$

which means $g'(t) \in L^1(1,\infty)$. On the other hand,

$$-g'(t) \geq \frac{g(t)}{Mt} \geq \frac{\lim_{s\to+\infty} g(s)}{Mt} \text{ for } 1 < t < \infty.$$

Hence, $g'(t) \in L^1(1,\infty)$ implies $\lim_{t\to+\infty} g(t) = 0$, since otherwise $g'(t) \notin L^1(1,\infty)$. $\square$

We recall two lemmas in [8].

Lemma 2.2 ([5]). *Suppose ν and λ satisfy $\nu - \lambda > -1$, and $|\nu| > 1/2, \lambda \geq -1/2$ or $\nu > -1$, $\lambda \geq 0$. Then*

$$\left|\int_0^r \frac{J_\nu(t)}{t^\lambda}dt\right| \leq \frac{C}{|\nu|^\lambda}, \qquad \textit{for} \quad 0 < r < \infty,$$

where $J_\nu(t)$ is the Bessel function of order ν.

Lemma 2.3 ([1]). *Suppose $m \geq 1$ and $\lambda > 0$. Then*

$$\left|\frac{1}{r}\int_0^r \frac{J_{m+\lambda}}{t^\lambda}dt\right| \leq \frac{C}{m^{\lambda+1}}, \qquad \textit{for} \quad 0 < r < \infty.$$

Now we turn to the proof of Theorem 3. As is shown in [8, p. 374], to prove the L^2 boundedness of μ_Ω^Φ, we only need to show

$$\sum_{j=1}^{D_k}\int_0^\infty \left|\frac{1}{t}\int_0^t \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}}drY_{k,j}(\xi')\right|^2\frac{dt}{t} \leq Ck^{-2}, \tag{2.3}$$

where D_k is the dimension of the space of surface spherical harmonics of degree k on S^{n-1}, and $\{Y_{k,j}\}$ $(k \geq 1, j = 1, 2, ..., D_k)$ denotes the complete system of normalized surface spherical harmonics. Denote $N_t(\xi) = \frac{1}{t}\int_0^t \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}}dr$. Note that $\frac{\Phi(t)}{t} = C_2\Phi'(t)\varphi(t)$. Then, letting $\rho = \Phi(r)|\xi|$, and then $s = \Phi(t)|\xi|$, and noting Remark 2, we have

$$\int_0^\infty |N_t(\xi)|^2\frac{dt}{t} = -C_2^3\int_0^\infty \frac{\varphi(\Phi^{-1}(\frac{s}{|\xi|}))}{\Phi^{-1}(\frac{s}{|\xi|})^2}\left|\int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right|^2\frac{ds}{s}. \tag{2.4}$$

We only need to treat two cases (i) and (ii) in Theorem 3, where $\Phi(t)$ is positive and decreasing on $(0,\infty)$. We fix $\varepsilon > 0$ with $\varepsilon < -C_2\delta$. Then, by Lemma 2.1 we see that $t^\varepsilon\Phi^{-1}(t)$ is decreasing on $(0,\infty)$.

First we consider Case (i): Let $\nu = \frac{n}{2} + k - 1$.

(1) For $s \geq \nu$, integrating by parts and noting $\left(\Phi^{-1}(t)\right)' = C_2\Phi^{-1}(t)\varphi(\Phi^{-1}(t))/t$, we obtain

$$\begin{aligned}
I &:= \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \\
&= \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-1}}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \\
&= -\left(\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-1}}d\rho\right)\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s}\varphi(\Phi^{-1}(\frac{s}{|\xi|})) - \int_s^\infty \left(\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-1}}du\right) \\
&\times \left\{C_2\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho}\varphi^2(\Phi^{-1}(\frac{\rho}{|\xi|}))\frac{1}{\rho} + \frac{1}{\rho}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi'(\Phi^{-1}(\frac{\rho}{|\xi|}))C_2\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))\right. \\
&\left. - \frac{1}{\rho^2}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))\right\}d\rho.
\end{aligned}$$

Taking $\eta > 0$ with $0 < \eta < 1$, we see that $\Phi^{-1}(\frac{t}{|\xi|})/t^{1-\eta}$ is decreasing on $(0,\infty)$. Therefore, using Lemma 2.2, we get the following estimate:

$$\begin{aligned}
|I| &\leq \left|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-1}}d\rho\right|\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s}\|\varphi\|_\infty + \int_s^\infty \left|\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-1}}du\right|\frac{d\rho}{\rho^{1+\eta}}\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^{1-\eta}} \\
&\times \left\{|C_2|\|\varphi\|_\infty^2 + |C_2|\|t\varphi'(t)\|_\infty\|\varphi\|_\infty + \|\varphi\|_\infty\right\} \\
&\leq \frac{C}{(\frac{n}{2}+k-1)^{\frac{n}{2}-1}}\Phi^{-1}(\frac{s}{|\xi|})\frac{1}{s}.
\end{aligned}$$

(2) For $0 < s < \nu$, we have

$$\begin{aligned} I &= \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \\ &= \int_s^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\varepsilon}} \rho^\varepsilon \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho + \int_\nu^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

By (1) and the decreasingness of $t^\varepsilon \Phi^{-1}(\frac{t}{|\xi|})$, we have

$$|I_2| \le \frac{C}{(\frac{n}{2}+k-1)^{\frac{n}{2}-1}} \Phi^{-1}(\frac{\nu}{|\xi|})\frac{1}{\nu} = \frac{C}{\nu^{\frac{n}{2}+\varepsilon}} \nu^\varepsilon \Phi^{-1}(\frac{\nu}{|\xi|}) \le \frac{C}{\nu^{\frac{n}{2}+\varepsilon}} s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}).$$

As for I_1, since $J_{\frac{n}{2}+k-1}(\rho) > 0$ for $0 < \rho < n/2+k-1$, by the decreasingness of $t^\varepsilon \Phi^{-1}(\frac{t}{|\xi|})$, together with Lemma 2.2, we have for $0 < s < \nu$

$$|I_1| \le \int_s^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\varepsilon}} d\rho\, s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|})\|\varphi\|_\infty \le \frac{C}{\nu^{\frac{n}{2}+\varepsilon}} s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}).$$

Thus by (1) and (2), we get

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \le C \int_0^\nu \frac{s^{2\varepsilon}}{\nu^{n+2\varepsilon}} \frac{ds}{s} + C \int_\nu^\infty \frac{1}{s^2} \frac{ds}{s} \frac{1}{(\frac{n}{2}+k-1)^{n-2}} \le \frac{C}{k^n}.$$

Case (ii). We may assume φ is increasing since the proof is similar for the case φ is decreasing. Letting $\nu = \frac{n}{2}+k-1$ as before, we will consider the following two cases :

(1) For $s \ge \nu$, let $h > s$. Then, since $\Phi^{-1}(\frac{\rho}{|\xi|})/\rho$ and $\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))$ are positive and decreasing, we have by using the second mean value theorem for some $s \le h' \le h$ and by Lemma 2.2

$$\begin{aligned} \Big| \int_s^h \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \Big| &= \Big| \int_s^{h'} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-1}} d\rho \Big| \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s} \varphi(\Phi^{-1}(\frac{s}{|\xi|})) \\ &\le \frac{C\|\varphi\|_\infty}{\nu^{\frac{n}{2}-1}} \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s}. \end{aligned}$$

Letting $h \to \infty$, we have

$$\Big| \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \Big| \le \frac{C\|\varphi\|_\infty}{\nu^{\frac{n}{2}-1}} \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s} \le \frac{C\|\varphi\|_\infty}{k^{\frac{n}{2}-1}} \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s}.$$

(2) For $0 < s < \nu$, we have

$$\begin{aligned} \Big| \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \Big| &\le \Big| \int_s^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \Big| \\ &\quad + \Big| \int_\nu^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \Big| \\ &= I_1 + I_2, \text{say.} \end{aligned}$$

By (1) and the decreasingness of $t^\varepsilon \Phi^{-1}(t)$, we see that

$$I_2 \le \frac{C\|\varphi\|_\infty}{\nu^{\frac{n}{2}-1}} \frac{\Phi^{-1}(\frac{\nu}{|\xi|})}{\nu} = \frac{C\|\varphi\|_\infty}{\nu^{\frac{n}{2}+\varepsilon}} \nu^\varepsilon \Phi^{-1}(\frac{\nu}{|\xi|}) \le \frac{C\|\varphi\|_\infty}{k^{\frac{n}{2}+\varepsilon}} s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}).$$

As for I_1, since $J_{\frac{n}{2}+k-1}(\rho) > 0$ for $0 < \rho < n/2 + k - 1$ and $t^\varepsilon \Phi^{-1}(t)$ is positive and decreasing on $(0, \infty)$, together with Lemma 2.2, we have for $0 < s < \nu$

$$\begin{aligned} I_1 &= \left| \int_s^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\varepsilon}} \rho^\varepsilon \Phi^{-1}(\frac{\rho}{|\xi|}) \varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) d\rho \right| \\ &\leq \left| \int_s^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\varepsilon}} d\rho \right| s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}) \|\varphi\|_\infty \\ &\leq \frac{C\|\varphi\|_\infty}{(\frac{n}{2}+k-1)^{\frac{n}{2}+\varepsilon}} s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}) \leq \frac{C\|\varphi\|_\infty}{k^{\frac{n}{2}+\varepsilon}} s^\varepsilon \Phi^{-1}(\frac{s}{|\xi|}). \end{aligned}$$

By (2.4), and (1), (2) above, we have

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \leq C \int_0^\nu \frac{s^{2\varepsilon}}{k^{n+2\varepsilon}} \frac{ds}{s} + C \int_\nu^\infty \frac{1}{k^{n-2} s^2} \frac{ds}{s} \leq C \frac{1}{k^n}.$$

Thus, in both cases (i) and (ii), we have

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \leq \frac{C}{k^n}.$$

Therefore, by the fact $\sum_{j=1}^{D_k} |Y_{k,j}(\xi')|^2 = w^{-1} D_m \sim k^{n-2}$ (see [3, p. 255, (2.6)]), where w denotes the area of S^{n-1}, we get

$$\sum_{j=1}^{D_k} \int_0^\infty |N_t(\xi)(Y_{k,j})(\xi')|^2 \frac{dt}{t} \leq Ck^{-2}.$$

Thus, inequality (2.3) holds and the proof of Theorem 3 is finished. □

3 Proof of Theorems 5 For any $\sigma > 0$, if we take $0 < \varepsilon < -C_2\sigma\delta$, then we see by Lemma 2.1 that $t^\varepsilon [\Phi^{-1}(t)]^\sigma$ is strictly decreasing on $(0, \infty)$. Thus, as is pointed out in the section 3 in [8], Theorem 5 follows from repeating the steps in the proof of Theorem 3. □

4 Final comment Finally, we note that the proof of Theorem 1 in [8] can be simplified in the case $\varphi(t)$ is monotonic. We give here a simple one. In this case $C_2 > 0$.

We assume first φ is increasing. Fix $\varepsilon > 0$ with $0 < \varepsilon < \min\{1, C_2\delta\}$. Then, $\Phi^{-1}(\rho/|\xi|)/\rho^\varepsilon$ is increasing on $(0, \infty)$. Letting $\nu = \frac{n}{2} + k - 1$, we will consider the following two cases :

(1) The case $0 < s \leq \nu$. Since $J_{\frac{n}{2}+k-1}(\rho) > 0$ for $0 < \rho < n/2 + k - 1$ and $t\Phi^{-1}(t)$ is positive and increasing on $(0, \infty)$, together with Lemma 2.3, we have for $0 < s < \nu$

$$\begin{aligned} \left| \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|}) \varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) d\rho \right| &\leq \left(\frac{1}{s} \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} d\rho \right) s\Phi^{-1}(\frac{s}{|\xi|}) \|\varphi\|_\infty \\ &\leq \frac{Cs\Phi^{-1}(\frac{s}{|\xi|})}{(k-1)^{n/2+1}}. \end{aligned}$$

(2) For $\nu < s$, let $0 < h < s$. Then, since $\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))$ and $\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}$ are positive and increasing, by using the second mean value theorem, and Lemma 2.2, we get for some

$h \le s' \le s$

$$\begin{aligned}\left|\int_h^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right| &= \left|\int_h^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^{\varepsilon}}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right| \\ &= \left|\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^{\varepsilon}}\varphi(\Phi^{-1}(\frac{s}{|\xi|}))\int_{s'}^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho\right| \\ &\le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^{\varepsilon}}\|\varphi\|_\infty \frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.\end{aligned}$$

Letting $h \to 0$, we get

$$\left|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right| \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^{\varepsilon}}\|\varphi\|_\infty \frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.$$

Using (1) and (2) above, we obtain

$$\int_0^\infty |N_t(\xi)|^2\frac{dt}{t} \le C\int_0^\nu \frac{s^2}{(k-1)^{n+2}}\frac{ds}{s} + C\int_\nu^\infty \frac{1}{s^{2\varepsilon}}\frac{ds}{s}\frac{1}{(\frac{n}{2}+k-1)^{n-2\varepsilon}} \le \frac{C}{k^n}.$$

In the case φ is decreasing, since $\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^{\varepsilon}}$ is increasing and $\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))$ is decreasing, we use the second mean value theorem twice in the step (2), and obtain the same estimate. Therefore, by the fact $\sum_{j=1}^{D_k}|Y_{k,j}(\xi')|^2 = w^{-1}D_m \sim k^{n-2}$ (see [3, p. 255, (2.6)]), where w denotes the area of S^{n-1}, we get

$$\sum_{j=1}^{D_k}\int_0^\infty |N_t(\xi)(Y_{k,j})(\xi')|^2\frac{dt}{t} \le Ck^{-2}.$$

Thus, inequality (2.3) holds and the proof of Theorem 1 in [8] is finished in the case $\varphi(t)$ is monotonic. $\square$

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