EXTREME AND NICE OPERATORS ON CERTAIN FUNCTION SPACES

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ABSTRACT. Let X be a Banach space, and $L(X)$ be the space of bounded linear operators from X into X. $B_1[X]$ denotes the closed unit ball of X, and $S_1[X]$ is unit sphere of X. An element $T \in S_1[L(X)]$ is called extreme operator if there is no $A \in L(X)$ such that $\|T \pm A\| \leq 1$. The set of extreme points of $S_1[X]$ will be denoted by $ext(S_1[X])$. $T \in S_1[L(X)]$ is called nice if $T^*(ext(S_1[L(X^*)]) \subseteq ext(S_1[L(X^*)])$. The object of this paper is to give simpler proofs of old results and present new results on extreme operators of $S_1[L(\ell^p)]$. We introduce the concept of k-extreme points. Further, we characterize the nice operators on most of the classical function and sequence spaces. Nice compact operators on ℓ^p-spaces are characterized.

I. **Introduction**. Let X be a Banach space. The closed unit ball of X will be denoted by $B_1[X]$, and the unit sphere by $S_1[X]$. An element $x \in S_1[X]$ is called an extreme point of $B_1[X]$ if whenever $x = \frac{1}{2}(y+z)$, with y and z in $S_1[X]$, then $x = y = z$. The space of bounded linear operators on X will be denoted by $L(X)$, and the compact ones by $K(X)$. Extreme elements of $S_1[L(X)]$ are called extreme operators. An operator $T \in L(X)$ is called nice if the set of extreme points of $B_1[X^*]$ is an invariant set for T^*.

$L^p(I), 1 \leq p < \infty$, denotes the Banach space of p-Bochner integrable functions (equivalence classes) defined on the unit interval, with the usual classical norm. Similarly, ℓ^p denotes the Banach space of p-summable sequences, with the usual classical norm. The space of continuous functions on a compact set Ω, with the uniform norm is denoted by $C(\Omega)$.

The problem of characterizing the extreme operators of $L(X)$ is an old and deep one.

The spaces $L(L^p(I))$, $L(\ell^p)$ and $L(C(\Omega))$ did have the major part of the study of extreme operators. Extreme operators of $L(\ell^2)$ and $L(L^2))$ were characterized by Kadison [8]. Blumenthal, Lindenstrauss, and Phelps [1], studied the extreme operators of $L(C(\Omega))$. Extreme operators of $L(\ell^1)$ were characterized by Sharir [13]. Characterizing the extreme operators of $L(\ell^p))$ and $L(L^p(I))$ for $p \neq 2$ turned out to be a difficult one, and still an problem. Many papers have been written on the problem. We refer to Grzaslewicz [4] , Kan [9], and Khalil [10], for results on extreme operators of $L(\ell^p))$ $p \neq 2$. For results on positive extreme operators we refer to Grzaslewicz [5], and Drury [3]. Extreme operators of $K(\ell^p)$ were discussed by Grzaslewicz [6], Hennefeld [7], and Rues and Stegall [12]. In this paper, we give a new class of extreme operators of $L(L^p(I))$, $p \neq 2$ and give simple proof of the result in [4].

Nice operators were introduced by Sharir [13]. Werener [14], and Choy [2] investigated nice operators on certain function spaces. In this paper, we characterize nice operators on some function spaces.

Through out this paper, $\ell^p = \{(x_n) : \sum |x_n|^p < \infty\}, 1 \leq p < \infty$, and $\ell^\infty = \{(x_n) : \sup |x_n| < \infty\}$ are the classical sequence spaces with respective norms: $\|x\|_p = (\sum |x_n|^p)^{\frac{1}{p}}$,

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and $\|x\|_{\infty} = \sup |x_n|$. I denotes the unit interval with the Lebesgue measure, and $L^p(I)$ is the space of Lebesgue measurable functions(equivalence classes) f such that $\int |f(t)|^p \, dt < \infty$, $1 \leq p < \infty$. The p-norm of f is $(\int |f(t)|^p \, dt)^{\frac{1}{p}}$. The space of continuous functions on I is denoted by $C(I)$, with $\|f\|_{\infty} = \sup |f(t)|$. N denotes the set of natural numbers.

II. Nice Operators.

Let X and Y be Banach spaces and $L(X,Y)$ be the space of bounded linear operators from X into Y. X^* is the dual of X. The closed unit ball of X is denoted by $B_1[X]$. By the Alaoglu Theorem and the Krein-Milman Theorem, $B_1[X^*]$ is the closed convex hull of its extreme points. An operator $T \in L(X,Y)$ is called nice if T^* maps the extreme points of $B_1[Y^*]$ into the extreme points of $B_1[X^*]$. Nice operators were introduced by Sharir[13] and were investigated by many authors. We refer to [2] and [14] for some results on nice operators. In this section we present some results on nice operators in certain function spaces. It easy to prove [13], that the class of nice operators form a subclass of extreme operators. For convenience We divide this section into two subsections.

(A) Nice Operators On Function Spaces

In this subsection we characterize nice operators and nice compact operators on the function space $C(I)$, the space of continuous functions on a compact interval I. Further, we give a necessary condition for an operator on $C(I,X)$, the space of all continuous functions on the compact interval I with values in the Banach space X , to be nice. We Let $M(I\ )$ denote the space of all regular Borel measures on I, which equals the dual of $C(I)$

Theorem 2.1. Let $T \in S_1[C(I)]$. Then the following are equivalent.

(i) T is nice.

(ii) There exists a continuous surjective function $\varphi : I \longrightarrow I$, such that $Tf = f \circ \varphi$ for all $f \in C(I)$.

Proof. $(i) \longrightarrow (ii)$. Let T be nice. Then $T^* : C(I)^* \longrightarrow C(I)^*$ preserves extreme points of the unit ball of $C(I)^* = M(I)$. But the extreme points of the unit ball of $M(I)$ are the unit point mass measures. That is $ext(S_1[M(I)]) = \{\delta_t : t \in I\}$, where $\delta_t(f) = f(t)$. Hence $T^*(\delta_t) = \delta_s$ for some $s \in I$. Define $\varphi : I \longrightarrow I$, $\phi(t) = s$. Now

$f(s) =< \delta_s, f >=< T^*(\delta_t), f >=< \delta_t, Tf >= Tf(t)$.

But $f(s) = f \circ \phi(t)$. Hence $Tf(t) = f \circ \varphi(t)$.

The continuity of φ follows from the continuity of $f \circ \varphi$ for all continuous functions f on I.

$(ii) \longrightarrow (i)$. Let $T(f) = f \circ \varphi$. Then $< T^*(\delta_t), f >=< \delta_t, Tf >= f \circ \varphi(t) = f(s)$, where $s = \varphi(t)$. Hence

$$< T^*(\delta_t), f >= f(s) =< \delta_s, f >$$

Since this is true for all $f \in C(I)$, it follows that $T^*(\delta_t) = \delta_s$.

Now, let us study the nice operators on $C(I,X)$, the space of all continuous functions on the compact interval I with values in the Banach space X. For $f \in C(I,X)$, $\|f\| = \sup\{\|f(t)\| : t \in I\}$.

Theorem 2.2. Let $T \in S_1[L(C(I,X))]$, and $A \in S_1[L(X,X)]$ be nice. If There exists a continuous surjective function $\varphi : I \longrightarrow I$, such that $(\ Tf)(t) = A(f(\varphi(t))$ for all $f \in C(I,X)$, then T is nice.

Proof. It is known that $C(I,X) = C(I) \overset{\vee}{\otimes} X$, the completed injective tensor product of $C(I)$ with X. We refer to [11] for the basic properties of tensor products of Banach spaces. However, for any two Banach spaces X and Y, the projective tensor product of the duals, $X^* \overset{\wedge}{\otimes} Y^*$, is contained in the dual of $(X \overset{\vee}{\otimes} Y)$ [12]. So $M(I) \overset{\wedge}{\otimes} X^* \subseteq (C(I) \overset{\vee}{\otimes} X)^*$. Further, $ext(S_1[(C(I) \overset{\vee}{\otimes} X)^*] = ext(S_1[M(I)) \otimes ext(S_1[X])$ [12]. So the extreme points of

$S_1[(C(I) \overset{\vee}{\otimes} X)^*]$ has the form $\delta_t \otimes x^*$, with x^* extreme in $S_1[X]$.

Now,

$< T^*(\delta_t \otimes x^*), f >=< \delta_t \otimes x^*, Tf >=< A(f(\varphi(t)), x^* >=< f(\varphi(t), A^*x^* >=< \delta_{\varphi(t)} \otimes A^*x^*, f >$. Since this is true for all $f \in C(I,X)$, it follows that $T^*(\delta_t \otimes x^*) = \delta_{\varphi(t)} \otimes A^*x^*$. But by assumption, A is nice. Hence A^*x^* is extreme. Consequently, T^* preserves extreme points, and T is nice.

As for compact nice operators on $C(I)$ we have the following.

Theorem 2.3. Let $T \in K(C(I), C(I))$, with $\|T\| = 1$. The following are equivalent.

(i) T is nice

(ii) $T^* = 1 \otimes \delta_t$ for some $t \in I$.

Proof. Clearly if $T^* = 1 \otimes \delta_t$, then $T^*(\delta_s) =< \delta_s, 1 > \delta_t = \delta_t$, and T is nice.

Now, Assume that T is nice. Since T is compact, T^* is compact, and so T^* has separable range. Thus range of T^* has at most countable number of $\delta_t^{;s}$. So one can write T^* in the form: $T^* = \sum_{n=1}^{\infty} g_i \otimes \delta_{t_i}$ with $g_i \in C(I)$. So $T^*(\delta_t)$ is some δ_{t_i}. So, for any $t \in I$, there exists j such that $g_j(t) = 1$ and $g_k(t) = 0$ for any $k \neq j$. Since $g_i \in C(I)$, it follows that there exists g_{t_m} such that $g_{t_m}(t) = 1$ for all $t \in I$ and $g_s = 0$ for all $s \neq t_m$. Hence $T^* = 1 \otimes \delta_t$.This ends the proof.

(B) Nice Operators On Sequence spaces.

In this subsection we study nice operators and nice compact operators on the classical sequence spaces including ℓ^1, ℓ^p and c_0.

Theorem 2.4. Let $F \in K(\ell^p, \ell^p)$, the following are equivalent.

(i) F is nice.

(ii) There exists two isometries $F_1 \in L(\ell^p)$ and $F_2 \in L(\ell^{p^*})$ such that $F^* = F_1 \otimes F_2$ or $F^* = F_2 \otimes F_1$.

Proof. $(ii) \Longrightarrow (i)$. The dual of the compact operators on ℓ^p is the nuclear operators on ℓ^{p^*}. But the extreme points of the nuclear operators on ℓ^{p^*} are the atoms $x \otimes y$, [12], with $\|x\| = \|y\| = 1$. Clearly $F^*(\ x \otimes y) = F_1(x) \otimes F_2(y)$ which is extreme. Similarly if $F^* = F_2 \otimes F_1$.

$(i) \Longrightarrow (ii)$. Since F is nice, then F^* preserves the extreme points of $\ell^p \overset{\wedge}{\otimes} \ell^{p^*}$. Hence , [12], F^* preserves the atoms: $x \otimes y$. Consequently, [15], for each $x \in \ell^p$ and $y \in \ell^{p^*}$ there exists $z \in \ell^p$ and $w \in \ell^{p^*}$ such that $F^*(x \otimes y) = z \otimes w($ or $= w \otimes z)$. Further, $\|F^*(x \otimes y)\| = \|z \otimes w\|$.

Define $F_1 \in L(\ell^p)$ as $F_1(x) = z$, and $F_2 \in L(\ell^{p^*})$ as $F_2(y) = w$. Using a similar argument as in [15], we get F_i are well defined linear isometric operators, and $F^* = F_1 \otimes F_2$ or $F^* = F_2 \otimes F_1$.

Another nice result on nice compact operators is the following.

Theorem 2.5. There are no nice operators in $K(X, c_0)$ for any Banach space X.

Proof. Every $T \in K(X, c_0)$ has a representation $T = \sum_{n=1}^{\infty} x_n^* \otimes \delta_n$, with $x_n^* \in X^*$ and $\|x_n^*\| \longrightarrow 0$ [16]. If T is nice, then $T^*(ext(B_1(\ell^1)) \subseteq ext(B_1(\ell^1)$. But $T^* = \sum_{n=1}^{\infty} \delta_n \otimes x_n^*$ and the extreme points of the unit ball of ℓ^1 are the $\delta_n^{;s}$. Hence, $T^*(\delta_n) = x_n^*$, which is not extreme for large n. Hence T is not nice.

A similar type result is the following.

Theorem 2.6. There are no nice operators in $L(\ell^1, \ell^p)$.

Proof. It is known [11], that $L(\ell^1, \ell^p) = (\ell^1 \overset{\wedge}{\otimes} \ell^{p^*})^* = (\ell^1(\ell^{p^*}))^* = \ell^\infty(\ell^p)$. Hence

$L(\ell^1, \ell^p) = \{T = (f_n) : f_n \in \ell^p, \forall n \in N, \sup \|f_n\| < \infty\}$. In such a case, $\|T\| = \sup \|f_n\|$. Further, for $x = (x_n) \in \ell^1$, $Tx = \sum x_n f_n$, and for $g = (y_n) \in \ell^{p^*}$, $T^*g = (z_n)$, where $z_n =$

$< g, f_n >$.

Now, if T is nice, then T^* maps $extB_1(\ell^{p^*})$ into $extB_1(\ell^\infty)$. But in such a case, $T^*(\delta_n)$ must be an extreme point of the unit ball of ℓ^∞. This means $|< T^*(\delta_n), \delta_m >| = 1$ for all m. Consequently, $|< \delta_m, f_n >| = 1$ for all n and m. But that contradicts the fact that $f_n \in \ell^p$. So T cant be extreme.

As for operators on ℓ^1, we have the following result.

Theorem 2.7. Let $T \in S_1(L(\ell^1, \ell^1))$ with $T = (f_n)$, $f_n \in \ell^1$ $\forall n \in N$ and $\|T\| = \sup \|f_n\|$. Then the following are equivalent.

(i) T is nice.

(ii) $T = (\pm\delta_{\varphi(n)})$, where $\varphi : N \longrightarrow N$.

Proof. $(i) \longrightarrow (ii)$. Assume T is nice, and $f_1 = (a_{11}, a_{12}, a_{13}, ...)$. Choose $x^1 = (1, 1, 1, ...) \in extB_1(\ell^\infty)$ and set $T^*x^1 = (z_1, z_2, z_3, ...)$, with $z_n =< x^1, f_n >$. Since T is nice, $T^*x^1 \in extB_1(\ell^\infty)$. Hence $|z_n| = \left|< x^1, f_n >\right| = 1$. In particular $\left|< x^1, f_1 >\right| = 1$. Hence

$|\sum a_{1m}| = 1$..........................$(*)$

We claim that for all n, either $a_{1n} = 1$, or $a_{1n} = 0$ and $\sum\limits_{m \neq n} a_{1m} = 0$. Indeed, let us choose n_0 and $x^2 = (1, 1, 1, ..., -1, 1, 1, ...)$, where -1 appears in the n_0^{th}-coordinate. Clearly, $x^2 \in ext(B_1(\ell^\infty))$. Thus $T^*(x^2) \in ext(B_1(\ell^\infty))$. Let $T^*(x^2) = (w_1, w_2, w_3, ...)$. So $w_n =< x^2, f_n >$ for all n. Further, being an extreme point of $B_1(\ell^\infty)$, we have $|w_n| = 1$ for all n. In particular $|w_1| = 1$, so $\left|< x^2, f_1 >\right| = 1$. Hence

$\left| -a_{1n_0} + \sum\limits_{m \neq n_0} a_{1m} \right| = 1$..............................$(**)$.

It follows from $(*)$ and $(**)$, that

$|\sum a_{1m}| = 1 = \left| -a_{1n_0} + \sum\limits_{m \neq n_0} a_{1m} \right|$.

Consequently, either

$\sum a_{1m} = -a_{1n_0} + \sum\limits_{m \neq n_0} a_{1m}$

or

$\sum a_{1m} = a_{1n_0} - \sum\limits_{m \neq n_0} a_{1m}$.

In the first case, $a_{1n_0} = 0$, while in the second case, $\sum\limits_{m \neq n_0} a_{1m} = 0$ and $|a_{1n_0}| = 1$. So we have proved that for $f_1 = (a_{11}, a_{12}, a_{13}, ...)$, and for all n, either $a_{1n} = 0$, or $|a_{1n}| = 1$ and $\sum\limits_{m \neq n_0} a_{1m} = 0$.

Now, we prove that there is a unique k such that $a_{1k} \neq 0$. Notice that if $a_{1m} = 0$ for all m, then $T^*x^1 \notin ext(B_1(\ell^\infty))$, where $x^1 = (1, 1, 1, ...) \in ext(B_1(\ell^\infty))$. So there is at least one k with $a_{1k} \neq 0$. Thus by the above argument, $|a_{1k}| = 1$, and $\sum\limits_{m \neq k} a_{1m} = 0$. Such k is unique. Indeed, if $a_{1s} \neq 0$,with $s \neq k$, then again $|a_{1s}| = 1$, and $\sum\limits_{m \neq s} a_{1m} = 0$. But then $\|f_1\| = \sum |a_{1m}| \geq |a_{1k}| + |a_{1s}| = 2$. This cant be true since $\|f_1\| \leq \|T\| = 1$. Hence k is unique, and $f_1 \in ext(B_1(\ell^1))$. Similarly, we can prove $f_n \in ext(B_1(\ell^1))$ for all $n > 1$.

$(ii) \longrightarrow (i)$. Clear.

The following theorem is an immediate corollary to Theorem 2.7.

Theorem 2.8. $T \in B_1(L(\ell^1))$ is nice if and only if T is extreme.

III. Extreme Operators In $L(\ell^p)$.

Let $T \in S[L(\ell^p)]$. T is an extreme operator if whenever $T = \frac{1}{2}(A + B)$, with A and B in $B_1[L(\ell^p)]$, then $T = A = B$. This is equivalent to saying there is no $S \in B_1[L(\ell^p)]$ such that $\|T \pm S\| \leq 1$. While extreme operators in $L(\ell^2)$ are isometries and co-isometries [8], this is not the case for $L(\ell^p)$ $1 \leq p < \infty$. In [4], a class of operators which are not isometries and not co-isometries were introduced. Such operators were of the form $T = \delta_i \otimes y$, with $\|y\| = 1$ and $\text{supp}(y) = N$. A different class of extreme operators in $L(\ell^p)$ were introduced in [10]. Such operators were of the form $T = \sum \delta_{i_k} \otimes y_k$, with $\underset{k}{\cup}\text{supp}(y_k) = N$. The first result in this section is a simple proof for the above mentioned results.

Theorem 3.1([4]). Let $T = \delta_i \otimes y$, with $\text{supp}(y) = N$. If $\|y\|_p = 1$, for $1 < p < \infty$, then T is an extreme operator in $L(\ell^p)$

Proof. If possible assume T is not extreme. So there exists S such that $\|T \pm S\| \leq 1$. Put $J = T + S$. Since $\|T(\delta_i)\| = \|y\| = 1$, it follows from the uniform convexity of ℓ^p that $S\delta_i = 0$, and consequently $J(\delta_i) = y$. Now, if there exists δ_k with $k \neq i$ such that $S\delta_k \neq 0$, then $J\delta_k \neq 0$. Since $\text{supp}(y) = N$, then $\text{supp}(J\delta_i) \cap (J\delta_k) \neq \phi$. It follows from Lemma 2.1 of [7], that $\|J\| > \max\{1, \|J\delta_k\|\} \geq 1$. This contradicts the assumption that $\|J\| = 1$. Hence $S\delta_j = 0$ for all j, and T is extreme.

For normed spaces X and Y, we write $X \underset{p}{\oplus} Y$ to denote the set $\{x + y : x \in X, y \in Y$, and $\|x + y\|^p = \|x\|^p + \|y\|^p$.

Now, we prove the following result.

Theorem 3.2. Let $T \in S[L(\ell^p)]$, and $E = \{i : \|T\delta_i\| = 1\}$. If there exists $j \in N \setminus E$ such that $\|T\delta_j\| < 1$ with $\text{supp}(T\delta_i) \cap \text{supp}(T\delta_j) = \phi$ for all $i \in E^c$, then T is not extreme.

Proof. $N = E \,\dot{\cup}\, (N \setminus E)$. Being a disjoint union, this gives a decomposition of $\ell^p = \ell^p(E) \underset{p}{\oplus} \ell^p(N \setminus E)$. By Lemma 2.1 of [7] we have $\text{supp}(T\delta_i) \cap \text{supp}(T\delta_j) = \phi$ for all $i \neq j$, whenever i or j is in E. Thus $\{T\delta_i : i \in E\}$ is a p-orthonormal set in ℓ^p. Let $H = \overline{span\{T\delta_i : i \in E\}}$. Let $x \in \ell^p(E)$. Then $x = \sum\limits_{i \in E} a_i\delta_i$, and $\|Tx\|^p = \|\sum a_i T\delta_i\|^p = \sum |a_i|^p = \|x\|^p$. Hence $T : \ell^p(E) \longrightarrow H$ is an isometry, and so $T\,|_{\ell^p(E)}$ is an extreme operator.

Now, let $Y = \overline{span\{T\delta_i : i \in N \setminus E\}}$. Then $Range(T) = H \underset{p}{\oplus} Y$, noting that by Lemma 2.1 of [7],

$supp(T\delta_i) \cap supp(T\delta_j) = \phi$, for $i \in E$, and $j \in E^c$. Thus T has the decomposition

$$T = T_1 + T_2 : \ell^p(E) + \ell^p(E^c) \longrightarrow H + Y,$$

and T is extreme if and only if both T_1 and T_2 are extreme. But T_2 can be decomposed into:

$T_2 = T_{21} + T_{22} : [\delta_j] \underset{p}{\oplus} \ell^p(E^c \setminus \{j\}) \longrightarrow [T\delta_j] \underset{p}{\oplus} W$, which follows from the assumption that $supp(T\delta_i) \cap supp(T\delta_j) = \phi$ for all $i \in E^c$.

Once again, T_2 is extreme if and only if both T_{21} and T_{22} are extreme . But $T_{21} : [\delta_j] \longrightarrow [T\delta_i]$ is not extreme since $\|T_{21}\| < 1$. Thus T can't be extreme. This ends the proof.

Now we prove a positive result for extreme operators. Here $\ell_2^p = (R^2, \|(x, y\| = (|x|^p + |y|^p)^{\frac{1}{p}}$. For $u = (x, y)$, we assume $|x| \neq |y|$

Theorem 3.3. Let $T = u^* \otimes u : \ell_2^p \longrightarrow \ell_2^p$, $2 < p < \infty$, with $u^*(u) = \|u\| = \|u^*\| = 1$. Then T is an extreme operator if and only if $supp(u) = \{1, 2\}$.

Proof. Assume $supp(u) = \{1, 2\}$, but if possible assume T is not extreme. Then there exists $S \in L(\ell_2^p)$, with $\|S\| = 1$ such that $\|S \pm T\| \leq 1$. Now ℓ_2^p can be decomposed as $\ell_2^p = [u] \oplus \ker(u^*)$. Since ℓ_2^p is uniformly convex, then $S(u) = 0$ and $S^*u^* = 0$, so $[v] = \ker(u^*)$ is an invariant subspace of S. Hence $Sv = rv$. With no loss of generality, we can assume $r = 1$. Note that $Tw = 0$ for all $w \in \ker(u^*)$. Let $z = au + bv \in B[\ell_2^p]$. Then $(S \pm T)z = Tau + Sbv = au + bSv$. Hence

$\|au + bSv\|^p + \|au - bSv\|^p \leq 2\|z\|^p$.

Since $2 < p < \infty$, we can use Clarkson inequalities to get

$2\|au\|^p + 2\|bSv\|^p \leq 2\|z\|$.

Consequently, $\|au\| \leq \|z\|$. Further, since $Sv = v$, we get $\|bv\| \leq \|z\|$. So both of the subspaces $[u]$ and $[v]$ are 1-complemented. Hence, using Lemma 2.2 of [7], we must have $supp(u) \cap supp(v) = \phi$. But by assumption on support of u, this is impossible. Hence T is extreme.

Assume $supp(u) = \{1\}$. Then $supp(u^*) = \{1\}$. But then $S = \delta_2 \otimes \delta_2$ satisfies $\|S \mp T\| \leq 1$, and T is not extreme.

IV. Partially Extreme Points.

Let X be a Banach space and k be any natural number. An element $x \in S_1[X]$ is called $k-$extreme point if there is no $y \in X$ such that $\|y\| = \frac{1}{k}$, and $\|x \pm y\| = 1$. Clearly x is extreme if and only if x is $k-$extreme for all $k \in N$. In $S_1[\ell^\infty]$, the point $(\frac{9}{10}, 1)$ is $1-$extreme, but it is not $\frac{1}{10}-$extreme point. The number $\frac{1}{k}$ measures how far the point from being extreme. In fact, if $x \in S_1[X]$ is some $k-$extreme point, then $\frac{1}{k} = d(x, E) = \inf\{\|x - e\| : e \in E\}$, where E is the set of extreme points of $S_1[X]$.

Theorem 4.1. Let $T \in S_1[L(\ell_n^p)], 2 < p < \infty$, such that $T = \sum\limits_{i=1}^{n} \delta_i \otimes w_i$, with $\{w_1, w_2, w_3, ..., w_n\}$ be linearly independent. Then T is $1-$extreme.

Proof. If possible assume that T is not $1-$extreme. Then there exists $S \in S_1[L(\ell_n^p)]$ such that $\|T \pm S\| = 1$. Being operators on a finite dimensional Banach space, both T and S attain their norms. Hence there exists $x, y \in S_1[\ell_n^p]$ such that $\|Tx\| = \|Sy\| = 1$. But then $\|Ty \pm Sy\| \leq 1$. Since ℓ_n^p is uniformly convex, $Ty = 0$. So $\sum\limits_{i=1}^{n} < \delta_i, y > w_i = 0$. Since the $w_i^{,s}$ are independent, it follows that $< \delta_i, y >= 0$ for all $i = 1, 2, 3, ...n$. But this implies that $y = 0$. This contradicts the assumption on y. Hence T is $1-$extreme.

In fact we prove a stronger result.

Theorem 4.2. Let $T = \sum\limits_{i=1}^{k} \delta_i \otimes u_i$, with $\|T\| = 1$, $\{u_1, u_2, ...u_k\}$ independent, $k \leq n$ and $\bigcup\limits_i \text{supp}(u_i) = \{1, 2, 3, ...n\}$. Then T is $1-$ extreme operator in $S_1[L(\ell_n^p)]$.

Proof. Assume if possible that T is not $1-$ extreme. Then there exists $S \in S_1[L(\ell_n^p)]$ such that $\|T \pm S\| = 1$. Being operators on finite dimensional normed space, there exists $x, y \in S_1[\ell_n^p]$ such that $\|Tx\| = \|Sy\| = 1$. But then $Ty = 0 = Sx$. Since the $u_i^{,s}$ are independent, it follows that $\text{supp}(y) \subset \{k+1, ..., n\}$, and so x and y have disjoint support. Now, consider $\|(T+S)(x+y)\|^p + \|(T-S)(x+y)\|^p \leq 2\|x+y\|^p = 2(\|x\|^p + \|y\|^p)$.Hence, $\|Tx + Sy\|^p + \|Tx - Sy\|^p \leq 2\|x+y\|^p = 2(\|x\|^p + \|y\|^p) = 4$.Since $p > 2$, we can use Clarkson^{,s} inequalities to get $2(\|Tx\|^p + \|Sx\|^p) \leq \|Tx + Sy\|^p + \|Tx - Sy\|^p \leq 4 = 2(\|Tx\|^p + \|Sx\|^p)$,since $\|x\| = \|Tx\| = \|Ty\| = \|y\|$. This implies that $\|Tx + Sy\|^p + \|Tx - Sy\|^p = 2(\|Tx\|^p + \|Sx\|^p)$.This can happen only if Tx, and Sy have disjoint support. which is not true since $\bigcup\limits_i \text{supp}(u_i) = \{1, 2, 3, ...n\}$. Hence there is no such S, and so T is $1-$extreme.

We end This section by the following question.

Question1. Let $T \in S_1[L(\ell_n^p)]$, such that $T = u^* \otimes u$, with $u^*(u) = \|u\| = 1$. Is T an extreme operator if $\text{supp}(u) = \{1, 2, 3, ...n\}$?. Is T $1-$extreme?.

V. Further Results.

Let X be any Banach space. We say X is extremal if every extreme operator $T \in L(X, Y^*)$ maps the extreme elements of X into the extreme elements of Y^*. We say X is 1-

decomposable if $X^* = X_1 \oplus X_2$, with X_1 extremal and $ext(B_1(X^*) = ext(B_1(X_1)$.Examples of 1-decomposable spaces are ℓ^∞ and $C(K)$ [17].

We give an example of an extremal Banach space.

Theorem 5.1. ℓ^1 is extremal.

Proof. Let Y be any Banach space, and $T \in L(\ell^1, Y^*)$ be an extreme operator. Now, $L(\ell^1, Y^*) = (\ell^1 \overset{\wedge}{\otimes} Y)^* = (\ell^1(Y))^* = \ell^\infty(Y^*)$. So any $T \in L(\ell^1, Y^*)$ has the representation $T = (f_n)$, $f \in Y^*$, and $\|T\| = \sup_n \|f_n\|$. Since T is extreme, each f_n is extreme in $B_1(Y^*)$. But the extreme points of ℓ^1 are the $\delta_n^{;s}$, and $T(\delta_n) = f_n$. Hence ℓ^1 is extremal.

Now we prove the following result.

Theorem 5.2. Let X be a 1-decomposable Banach space, and Y be any other Banach space. The following are equivalent:

(i) $T \in B_1(L(X, Y)$ is nice.

(ii) $T^* \in B_1(L(Y^*, X^*)$ is extreme.

Proof. Since (i) gives (ii) easily, we need only to show (ii) implies (i).

So, suppose T^* is extreme. Since X is 1-decomposable, $X^* = X_1 \oplus X_2$, with X_1 extremal and $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$ for all $x_1 \in X_1$ and $x_2 \in X_2$. Further, $extB_1(X) = extB_1(X_1)$. We claim that T_1^*, the restriction of T^* to X_1, is extreme. Indeed, if this is not true, then there exists $S \in L(X_1, Y^*)$ such that $S \neq 0$, and $\|T_1^* \pm S\| \leq 1$. Define $S \in L(X^*, Y^*)$ as follows:

For $x^* = x_1 + x_2$, $S(x^*) = S_1(x_1)$. Now,

$$\begin{aligned}\|(T^* \pm S)x^*\| &= \|T_1^* x_1 \pm S_1 x_1 + T^* x_2\| \\ &\leq \|T_1^* x_1 \pm S_1 x_1\| + \|T^* x_2\| \\ &\leq \|x_1\| + \|x_2\| = \|x^*\|.\end{aligned}$$

Hence, T^* is not extreme, which contradicts (ii), and T_1^* is extreme. Since X_1 is extremal, then $T^*(ext(B_1(X_1)) \subseteq ext(B_1(Y^*))$. But $ext(B_1(X_1) = ext(B_1(X)$. Consequently T is nice.

A nice consequence of the above theorem is

Corollary 5.3. Let X be any Banach space and $T \in L(X.C(I))$. Then T is nice if and only if T^* is extreme.

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