MODULAR GROUP ALGEBRAS WITH ALMOST MAXIMAL LIE NILPOTENCY INDICES, II

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Abstract. Let K be a field of positive characteristic p and KG the group algebra of a group G. It is known that, if KG is Lie nilpotent, then its upper (or lower) Lie nilpotency index is at most $|G'|+1$, where $|G'|$ is the order of the commutator subgroup. Previously we determined the groups G for which the upper/lower nilpotency index is maximal or the upper nilpotency index is 'almost maximal' (that is, of the next highest possible value, namely $|G'|-p+2$). Here we determine the groups for which the lower nilpotency index is 'almost maximal'.

Let R be an associative algebra with identity. The algebra R can be regarded as a Lie algebra, called the associated Lie algebra of R, via the Lie commutator $[x, y] = xy - yx$, for every $x, y \in R$. Set $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$, where $x_1, \dots, x_n \in R$. The *n-th lower Lie power* $R^{[n]}$ of R is the associative ideal generated by all the Lie commutators $[x_1, \dots, x_n]$, where $R^{[1]} = R$ and $x_1, \dots, x_n \in R$. By induction, we define the *n-th upper Lie power* $R^{(n)}$ of R as the associative ideal generated by all the Lie commutators $[x, y]$, where $R^{(1)} = R$ and $x \in R^{(n-1)}$, $y \in R$.

The algebra R is called *Lie nilpotent* (respectively *upper Lie nilpotent*) if there exists m such that $R^{[m]} = 0$ $(R^{(m)} = 0)$. The algebra R is called *Lie hypercentral* if for each sequence $\{a_i\}$ of elements of R there exists some n such that $[a_1, \dots, a_n] = 0$. The minimal integers m, n such that $R^{[m]} = 0$ and $R^{(n)} = 0$ are called *the lower Lie nilpotency index* and *the upper Lie nilpotency index* of R and they are denoted by $t_L(R)$ and $t^L(R)$, respectively.

Let $U(KG)$ be the group of units of a group algebra KG. For the noncommutative modular group algebra KG the following Theorem due to A.A. Bovdi, I.I. Khripta, I.B.S. Passi, D.S. Passman and etc. (see [3, 10]) is well known: The following statements are equivalent: (a) KG is Lie nilpotent; (b) KG is Lie hypercentral; (c) KG is upper Lie nilpotent; (d) $U(KG)$ is nilpotent; (e) $\mathrm{char}(K) = p > 0$, G is nilpotent and its commutator subgroup G' is a finite p-group.

It is well known (see [12, 14]) that, if KG is Lie nilpotent, then

$$t_L(KG) \le t^L(KG) \le |G'| + 1.$$

Moreover, according to [1], if $\mathrm{char}(K) > 3$, then $t_L(KG) = t^L(KG)$. But the question of when does $t_L(KG) = t^L(KG)$ hold for $\mathrm{char}(K) = 2, 3$ is in general still open. Using the program packages GAP and LAGUNA (see [5, 9]), A.Konovalov in [11] verified that $t_L(KG) = t^L(KG)$ for all 2-groups of order at most 256 and $\mathrm{char}(K) = 2$. Several important results on this topic were obtained in [4].

We say that a Lie nilpotent group algebra KG has

- *upper maximal* Lie nilpotency index, if $t^L(KG) = |G'| + 1$;
- *lower maximal* Lie nilpotency index, if $t_L(KG) = |G'| + 1$;

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- *upper almost maximal* Lie nilpotency index, if $t^L(KG) = |G'| - p + 2$;
- *lower almost maximal* Lie nilpotency index, if $t_L(KG) = |G'| - p + 2$.

A. Shalev in [13] began to study the question when do the Lie nilpotent group algebras KG have lower maximal Lie nilpotency index. In [6, 13] there was given the complete description of the Lie nilpotent group algebras KG with lower/upper maximal Lie nilpotency index. In [7] the characterization of such KG with upper almost maximal Lie nilpotency index was obtained. In the present paper we prove the following

0.1. Theorem. *Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then KG has lower almost maximal Lie nilpotency index if and only if one of the following conditions holds:*

(i) $p = 2$, $cl(G) = 2$ *and* $\gamma_2(G)$ *is noncyclic of order* 4*;*
(ii) $p = 2$, $cl(G) = 4$, $\gamma_2(G) \cong C_4 \times C_2$ *and* $\gamma_3(G) \cong C_2 \times C_2$*;*
(iii) $p = 2$, $cl(G) = 4$ *and* $\gamma_2(G)$ *is elementary abelian of order* 8*;*
(iv) $p = 3$, $cl(G) = 3$ *and* $\gamma_2(G)$ *is elementary abelian of order* 9.

Now using results of [6, 7, 13] we obtain

0.2. Corollary. *Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. The group algebra KG has lower almost maximal Lie nilpotency index if and only if it has upper almost maximal Lie nilpotency index.*

According to Du's and Khripta's Theorems (see [8, 10]) we have

0.3. Corollary. *Let KG be the group algebra of a finite p-group G over a field K of positive characteristic p and $U(KG)$ its group of units. Then the nilpotency class of $U(KG)$ is equal to $|G'| - p + 1$ if and only if G and K satisfy one of the conditions (i)–(iv) of Theorem 1.*

As a consequence, we obtain that the Theorem 3.9 of [13] can not be extent for $p = 2$ and $p = 3$:

0.4. Corollary. *Let K be a field of positive characteristic p and G a nilpotent group such that $|G'| = p^n$.*

(i) *If $p = 2$ and $t_L(KG) < 2^n + 1$, then $t_L(KG) \leq 2^n$.*
(ii) *If $p = 3$ and $t_L(KG) < 3^n + 1$, then $t_L(KG) \leq 3^n - 1$.*

We shall use the following results:

0.5. Proposition. *([6, 13]) Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then $t^L(KG) = |G'| + 1$ if and only if either G' is cyclic or $p = 2$ and G' is noncyclic of order 4 such that $\gamma_3(G) \neq 1$. Moreover, if $t^L(KG) = |G'| + 1$ then $t_L(KG) = t^L(KG)$.*

0.6. Proposition. *([7]) Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then KG has upper almost maximal Lie nilpotency index if and only if one of the conditions of Theorem 1 holds. Moreover, if $t^L(KG) < |G'| + 1$ then $t^L(KG) \leq |G'| - p + 2$.*

Let KG be a Lie nilpotent group algebra over a field K of $char(K) = p$ and $t_L(KG) = |G'| - p + 2$. Obviously $t_L(KG) \leq t^L(KG) \leq |G'| + 1$. If $t^L(KG) > |G'| - p + 2$, then by Propositions 0.5 and 0.6 we get $t^L(KG) = |G'| + 1$ and also $t_L(KG) = t^L(KG) = |G'| + 1$, a contradiction. Thus by Proposition 0.6 we obtain that $t^L(KG) = |G'| - p + 2$ and G satisfies one of the conditions of our Theorem.

Conversely, let condition (i) of the Theorem holds. Since $cl(G)=2$, by Theorem 3.2 of [4] we obtain that $t_L(KG)\geq 4$ and $t_L(KG)=t^L(KG)$.

First, let G be a nilpotent group of class $cl(G)=4$, such that either $\gamma_2(G)\cong C_4\times C_2$ or $\gamma_2(G)\cong C_2\times C_2\times C_2$. For $g_1,\dots,g_n\in G$ we set $(g_1,g_2)=g_1^{-1}g_2^{-1}g_1g_2$ and $(g_1,\dots,g_n)=((g_1,\dots,g_{n-1}),g_n)$. If G is finite, then by [2] there exist $g,h\in G$ with the properties

$$a=(g,h),\qquad b=(g,h,h),\qquad c=(g,h,h,h), \tag{1}$$

such that $\gamma_2(G)=\langle a,b,c\rangle$, $\gamma_3(G)=\langle b,c\rangle$, $\gamma_4(G)=\langle c\rangle$, where for the case $\gamma_2(G)\cong C_4\times C_2$ we put $c=a^2$.

Some finitely generated subgroup will have the same lower central series, so there is no harm in assuming that G itself is finitely generated and therefore residually finite. Let N be maximal among the normal subgroups of finite index which avoid $\gamma_2(G)$: then G/N is a finite 2-group, and so there exist $g,h\in G$ such that the commutator (g,h) lies in the coset aN, $(g,h,h)\in bN$, and $(g,h,h,h)\in cN$. Now $a^{-1}(g,h)\in\gamma_2(G)\cap N=1$ shows that in fact $(g,h)=a$ and similar arguments show that also $(g,h,h)=b$ and $(g,h,h,h)=c$.

Let G be a finitely generated nilpotent group of class $cl(G)=4$, such that $\gamma_2(G)=\langle a\rangle\times\langle b\rangle\cong C_4\times C_2$. Therefore we have

$$(a,g)=f,\qquad (a,h)=b,\qquad (b,g)=t,\qquad (f,g)=z_1,\qquad (f,h)=z_2,$$

where $f,t\in\gamma_3(G)$, $z_1,z_2\in\gamma_4(G)$. Since $a^{gh}=a^{hga}$, we get $t=z_2$, so

$$a^g=af,\qquad f^g=fz_1,\qquad f^h=fz_2,\qquad b^h=a^2b,\qquad b^g=bz_2. \tag{2}$$

We consider the following two cases:

Case 1. Let $f\in\{b,a^2b\}$. By (1) and (2), using the well known equality $(ab,c)=(a,c)(a,c,b)(b,c)$, we get $(g^2,h)=a^2f$ so $(g^2h^2)^{-1}gh^2g=a^2b$ and $(hg^2h^2)^{-1}g^2h^3=a^2f$. It follows that

$$\begin{aligned}[h,gh,g]&=[g^2h^2(a^3b+1),g]=gh^2(a^2b(a^3fbz_2+1)+a^3b+1)\\ &\in\{\ g^2h^2(1+a^2),\quad g^2h^2(1+(a+a^2+a^3)b)\ \};\\ [h,gh,g,h]&=hg^2h^2(a^2fh^{-1}[h,gh,g]h+[h,gh,g])\\ &\in\{\ hg^2h^2(1+a^2),\quad hg^2h^2(1+a^2)b\ \}.\end{aligned}$$

Now, since $(ghg^2h^2)^{-1}hg^2h^2g=a^3b$ and $(hghg^2h^2)^{-1}ghg^2h^3=abfz_1$, we obtain that

$$\begin{aligned}[h,gh,g,h,g]&=(hg^2h^2g+hghg^2h^2)(1+a^2)\\ &=ghg^2h^2(1+ab)(1+a^2),\\ [h,gh,g,h,g,h]&=(ghg^2h^3(1+a^3)+hghg^2h^2(1+ab))(1+a^2)\\ &=hghg^2h^2a(1+b)(1+a^2).\end{aligned}$$

Finally, by $(h^2ghg^2h^2)^{-1}hghg^2h^3=abfz_1$ we get

$$\begin{aligned}[h,gh,g,h,g,h,h]&=(hghg^2h^3+h^2ghg^2h^2)a(1+a^2)(1+b)\\ &=\eta a(1+a)(1+a^2)(1+b)=\eta\cdot\widehat{a}\cdot\widehat{b}\neq 0,\end{aligned}$$

where $\eta=h^2ghg^2h^2$ and $\widehat{g}=\sum_{h\in\langle g\rangle}h$.

Case 2. Let $f\in\{1,a^2\}$. By (1) and (1) it yields that

$$a^g=af,\qquad a^h=ab,\qquad b^g=b,\qquad b^h=a^2b.$$

Clearly, that $[gh, g, gh] = [g^2h(a^3 + 1), gh] = g^2hgh(abf + a^3)$. Since $(ghg^2hgh)^{-1}g^2hghgh = a^3$ and $(g^2hg^2hgh)^{-1}ghg^2hghg = a^3b$, this yields

$$\begin{aligned}[gh, g, gh, gh] &= ghg^2hgh(a^3(a^3 + a^3bf) + abf + a^3)\\ &= ghg^2hgh(a^2 + a^2bf + abf + a^3);\\ [gh, g, gh, gh, g] &= \alpha(a^3b(a^2 + a^2bf + ab + a^3f) + a^2 + a^2bf + abf + a^3)\\ &\in \{\ \alpha(1 + a + a^2 + a^3),\quad \alpha(1 + a^2)(1 + ab)\ \},\end{aligned}$$

where $\alpha = g^2hg^2hgh$. Obviously, $(hg^2hg^2hgh)^{-1}g^2hg^2hgh^2 = ab$ and $(h^2g^2hg^2hgh)^{-1}hg^2hg^2hgh^2 = ab$, so it follows that

$$\begin{aligned}[gh, g, gh, gh, g, h] &\in \{\ \beta(1 + a^2)(1 + b),\quad \beta a(1 + a^2)(1 + b)\ \};\\ [gh, g, gh, gh, g, h, h] &= \gamma(1 + a)(1 + a^2)(1 + b) = \gamma \cdot \widehat{a} \cdot \widehat{b} \neq 0,\end{aligned}$$

where $\beta = hg^2hg^2hgh$, $\gamma = h^2g^2hg^2hgh$ and $\widehat{g} = \sum_{h\in\langle g\rangle} h$.

Therefore in both cases, the lower Lie nilpotent index is at least 8. Since $t^L(KG) = 8$, we obtain that $t_L(KG) = t^L(KG) = 8$.

Let G be a finitely generated nilpotent group of class $cl(G) = 4$, such that $\gamma_2(G) = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong C_2 \times C_2 \times C_2$. The proof is similar to the previous case, using the same commutators.

Let condition (iv) of the Theorem holds. Obviously, similarly to the previous cases, we can assume that G is finitely generated and, according to [2], there exist $g, h \in G$ such that

$$(g, h) = a, \qquad (g, h, h) = (a, h) = b, \qquad (a, g) = t \in \langle b\rangle. \tag{3}$$

Therefore, $a^h = ab$, $a^g = at$ and we consider the following cases:
Case 1. Let $t = 1$. By (3), using a simple computation we obtain that

$$\begin{aligned}[gh, g, g] = g^3h \cdot \widehat{a};\qquad [gh, g, g, h] &= hg^3h(a^2b^2 + ab - a^2 - a);\\ [gh, g, g, h, g] &= gh^2g^3h(a + b + a^2b^2 - b^2 - a^2 - ab);\\ [gh, g, g, h, gh, h] &= hgh^2g^3h(1 - a^2)(1 + b + b^2);\\ [gh, g, g, h, gh, gh, h] &= h^2gh^2g^3h \cdot \widehat{a} \cdot \widehat{b} \neq 0.\end{aligned}$$

Case 2. Let $t = b$. By (3) it is easy to check that

$$\begin{aligned}[h, g, gh] = ghgha^2(b - 1);\qquad [h, g, gh, g] &= g^2hgh(1 - a^2)(b - 1);\\ [h, g, gh, g, gh] &= ghg^2hgh(a^2 + ab - 1 - b^2)(b - 1);\\ [h, g, gh, g, gh, h] &= hghg^2hgh(a^2b + ab - ab^2 - a^2)(b - 1);\\ [h, g, gh, g, gh, h, g] &= -ghghg^2hgh \cdot \widehat{a} \cdot \widehat{b} \neq 0.\end{aligned}$$

Case 3. Let $t = b^2$. Similarly to the previous two cases we have

$$\begin{aligned}[g, gh, g] &= g^2hg(-1 - a - a^2b);\\ [g, gh, g, h] &= hg^2hg(a^2b + 1 - b^2 - a^2b^2);\\ [g, gh, g, h, gh] &= gh^2g^2hg(a + b + a^2b^2 - ab^2 - a^2b - 1);\\ [g, gh, g, h, gh, h] &= hgh^2g^2hg(a - a^2)(1 + b + b^2);\\ [g, gh, g, h, gh, h, h] &= -h^2gh^2g^2hg \cdot \widehat{a} \cdot \widehat{b} \neq 0.\end{aligned}$$

Therefore the lower Lie nilpotency index is at least 8. Since $t^L(KG) = 8$ we obtain that $t_L(KG) = t^L(KG) = 8$ and the proof is complete.

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