POKER GAMES WHERE PLAYERS HAVE NON-UNIFORM HAND DISTRIBUTIONS

Minoru Sakaguchi*

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Abstract. The most familiar parlor games High-Hand-Wins poker, Hi-Lo poker and La Relance poker are discussed, under the situation where the players' hands are delivered by extremely non-uniform distributions. It is shown that if players' hands are distantly (closely) distributed from (to) the opponent's one, then they behave cautiously (boldly), when the amount of bet is not so small.

1 Introduction. We discuss about (1) High-Hand-Wins poker, (2) Hi-Lo poker, and (3) La Relance poker under the situation that players' hands are delivered by non-uniform distributions. For these poker games the solutions are given in Ref.[1, 2].

We use the following notations throughout this paper :
$u =$ p.d.f. $u(x) = 1, \ \forall \ x \in [0, 1]$,
$f =$ p.d.f. $f(x) = 4(x \wedge \overline{x}), \ \forall x \in [0, 1]$,
$g =$ p.d.f. $g(x) = 4|x - \frac{1}{2}|, \ \forall x \in [0, 1]$,
u-u = Players' hands are delivered by *i.i.d.* random variables with p.d.f.$u(x)$.
f-f and g-g are interpreted analogously.

Let denote

$$E_{f\text{-}f}|x-y| = \int_0^1 \int_0^1 |x-y| f(x)f(y)dxdy,$$

and $E_{u\text{-}u}|x - y|$ and $E_{g\text{-}g}|x - y|$, similarly.

Then we have

Lemma $E_{u\text{-}u}|x-y| = \dfrac{1}{3}, E_{f\text{-}f}|x-y| = \dfrac{7}{30}$ *and* $E_{g\text{-}g}|x-y| = \dfrac{2}{5}.$

It is intuitively supposed that, when playing poker games, if the amount of bet is not so small, players will behave more cautious (boldly), as $E|x - y|$ becomes larger (smaller). We call this intuition the "Monotonicity Property".

The object of the present paper is to solve some familiar poker games where the hand distributions are extremely non-uniform. We show that there exist a positive border value m, such that if the amount of bet is larger (smaller) than m, then the Monotonicity Property (its reverse property) holds true. Three familiar pokers are discussed in Sections 2, 3 and 4, and the proofs of the theorems obtained there are given in Section 5. Final remark is given in Section 6.

2 High-Hand-Wins Poker. Each player I and II receives a hand x and y, respectively in $[0, 1]$ according to $U_{[0,1]}$-distribution, and chooses one of the two alternatives : Fold or Bet the amount $A(> 0)$, paying the ante 1 to the game. If both players fold, then the game is a draw and no payoffs return. If both players bet, the showdown is made, and the player with higher hand wins the pot. If one player bets and the other folds, then the player who

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made the bet wins the pot. Then the game, denoted by $\Gamma^{HHW,u\text{-}u}$ is described by the payoff matrix

$$\mathbf{M}^{HHW}(x,y) = \begin{array}{c} \\ \text{Fold} \\ \text{Bet} \end{array} \begin{array}{c} \begin{array}{cc} \text{Fold} & \text{Bet} \end{array} \\ \left[\begin{array}{cc} 0 & -1 \\ 1 & (1+A)\text{sgn}(x-y) \end{array}\right] \end{array}. \tag{2.1}$$

It is well-known that players have the same optimal strategy

$$\text{Bet (Fold), if his hand is } > (<) A/(1+A)$$

and the value of the game is zero.

We obtain the following results.

Theorem 1 *For the game $\Gamma^{HHW,f\text{-}f}$ players' common optimal strategy is*

$$\textit{Bet (Fold), if hand is } > (<)\ 1 - (2(1+A))^{-\frac{1}{2}}$$

when $A \geq 1$; and

$$\textit{Bet (Fold), if hand is } > (<) \sqrt{A/2(1+A)}$$

when $0 \leq A < 1$. The value of the game is zero.

It is clear that

$$A > 1\ (\Rightarrow)\ \frac{A}{1+A} > 1 - (2(1+A))^{-1/2} > \frac{1}{2},$$

and

$$0 < A < 1\ (\Rightarrow)\ \frac{A}{1+A} < \sqrt{A/(2(1+A))} > \frac{1}{2}.$$

Thus the monotonicity property mentioned in Section 1 holds true when $A > 1$.

Theorem 2 *For the game $\Gamma^{HHW,g\text{-}g}$ players' common optimal strategy is*

$$\textit{Bet (Fold), if hand is } > (<)\ \frac{1}{2}\left(1 + \sqrt{\frac{A-1}{A+1}}\right)$$

when $A > 1$; and

$$\textit{Bet (Fold), if hand is } > (<)\ \frac{1}{2}\left(1 - \sqrt{\frac{1-A}{1+A}}\right)$$

when $0 < A < 1$. The value of the game is zero.

It is easy to show that

$$A > 1\ (\Rightarrow)\ \left(\frac{1}{2} <\right) 1 - (2(1+A))^{-1/2} < \frac{A}{1+A} < \frac{1}{2}\left(1 + \sqrt{\frac{A-1}{A+1}}\right),$$

and

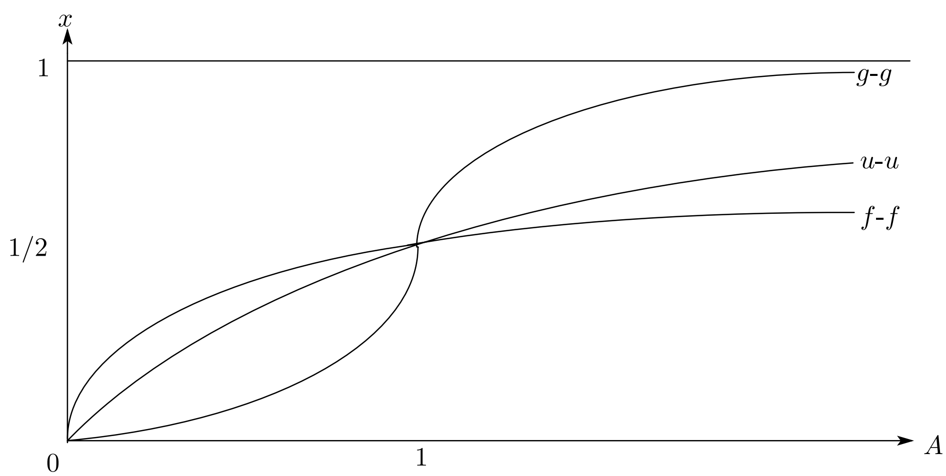
$$0 < A < 1\ (\Rightarrow)\ \left(\frac{1}{2} >\right) \sqrt{A/(2(1+A))} > \frac{A}{1+A} > \frac{1}{2}\left(1 - \sqrt{\frac{1-A}{1+A}}\right).$$

Thus the Monotonicity Property holds true when $A > 1$.

The decision-thresholds as the function of A in the three games $\Gamma^{HHW,u\text{-}u}$, $\Gamma^{HHW,f\text{-}f}$ and $\Gamma^{HHW,g\text{-}g}$ are compared as shown by Table 1 and Figure 1. For each curve in Figure 1, the upper (lower) part is the Bet (Fold) region.

Table 1 and Figure 1. Decision-threshold in Γ^{HHW}

	$0 < A < 1$	$A > 1$
g-g (Th.2)	$\frac{1}{2}\left(1-\sqrt{\frac{1-A}{1+A}}\right)$	$\frac{1}{2}\left(1+\sqrt{\frac{A-1}{A+1}}\right)$
u-u	$A/(1+A)$	A/(1+A)
f-f (Th.1)	$\sqrt{A/2(1+A)}$	$1-(2(1+A))^{-1/2}$



3 Hi-Lo Poker. Each player I and II receives a hand x and y, respectively, in according to $U_{[0,1]}$-distribution, and chooses one of the two alternatives High and Low. Choices are made simultaneously and independently of his opponent's choice. Then players make show-down, and one with the higher (lower) hand than the opponent wins, if the players' choices are Hi-Hi (Lo-Lo). If the players' choices are Hi-Lo or Lo-Hi the game is a draw. Thus the game, $\Gamma^{HL,u\text{-}u}$ is described by the payoff matrix

$$\mathbf{M}^{HL}(x,y) = \begin{array}{c} \\ \text{Hi} \\ \text{Lo} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hi} & \text{Lo} \end{array} \\ \begin{bmatrix} Bsgn(x-y) & 0 \\ 0 & sgn(y-x) \end{bmatrix} \end{array}, \tag{3.1}$$

where $B(>0)$ is the prize given to the winner from the loser, when the hand-pair is Hi-Hi.

It is well-known that for the game $\Gamma^{HL,u\text{-}u}$ players have the same optimal strategies.
Choose Hi (Lo), if hand is $>$ $(<)$ $B/(1+B)$,
and the value of the game is zero.

It is interesting that if $B = \nu^{-1} = \frac{1}{2}\left(\sqrt{5}+1\right) \approx 1.61804$, then $\frac{B}{1+B} = \nu = \frac{1}{2}\left(\sqrt{5}-1\right) \approx 0.61804$, the golden bisection number. (Note that $\nu^{-1} = 1+\nu$).

Theorem 3 *For the game $\Gamma^{HL,f\text{-}f}$, the value of the game is zero. Players' common optimal strategy is*
Choose Hi (Lo), if hand is $> (<)\ 1 - (2(B+1))^{-1/2}$
when $B > 1$ *; and*
Choose Hi (Lo), if hand is $> (<)\ \sqrt{B/2(B+1)}$
when $0 < B < 1$.

Theorem 4 *For the game $\Gamma^{HL,g\text{-}g}$, the value of the game is zero. Players' common optimal strategy is*
Choose Hi (Lo), if hand is $> (<)\ \frac{1}{2}\left(1 + \sqrt{\frac{B-1}{B+1}}\right)$
when $B > 1$ *; and*
Choose Hi (Lo), if hand is $> (<)\ \frac{1}{2}\left(1 - \sqrt{\frac{1-B}{1+B}}\right)$
when $0 < B < 1$.

Theorem 1∼4 state that, under each of hand distributions u-u, f-f and g-g the games Γ^{HHW} and Γ^{HL} have the same solution with A (in Theorems 1-2) and B (in Theorems 3-4) interchanged. This result may be a surprise since the two payoff matrices (2.1) and (3.1) seem quite unrelated.

4 La Relance Poker. Player I (II) receives a hand x (y) according to $U_{[0,1]}$. Player I acts first. He either bets the amount $1 + A$ or folds, losing his ante 1. If he bets, then II may either folds, losing his ante, or bet yielding the showdown. The player with the higher hand wins the pot. So, the payoff matrix of the game, denoted by $\Gamma^{LR,u\text{-}u}$, is

$$\mathbf{M}^{LR}(x,y) = \begin{array}{c} \\ \text{Fold} \\ \text{Bet} \end{array} \begin{array}{c} \begin{array}{cc} \text{Fold} & \text{Bet} \end{array} \\ \left[\begin{array}{cc} -1 & -1 \\ 1 & (1+A)\text{sgn}(x-y) \end{array}\right] \end{array}. \tag{4.1}$$

It is well-known that players' optimal strategies are

$$\phi^*(x) = \begin{cases} \text{a function } \alpha(x) : [0,c] \to [0,1] \text{ satisfying} & \\ \qquad \text{a restriction } \int_0^c \alpha(x)dx = c\overline{c}, & \text{if } x \le c \\ 1, & \text{if } x > c, \end{cases} \tag{4.2}$$

for I ; and $\psi^*(y) = I(y > c)$, for II, where $c = A/(A+2)$. The value of the game is $-c^2$.
We obtain the following results.

Theorem 5 *The solution of the game $\Gamma^{LR,f\text{-}f}$ is as follows ;*
(a) *Case* $A > 2$.
The value of the game is $-\left(\frac{A}{A+2}\right)^2$. *Players' optimal strategies are*

$$\begin{aligned} \phi^*(x) &= \alpha(x)I(x \le c) + I(x > c) \\ \psi^*(y) &= I(y > c), \end{aligned}$$

where $c = 1 - (2+A)^{-1/2}$, *and* $\alpha(x)$ *is a function* $[0,c] \to [0,1]$ *satisfying a restriction*

$$\int_0^{1/2} \alpha(x)xdx + \int_{1/2}^c \alpha(x)\overline{x}dx = \frac{A}{2(2+A)^2}. \tag{4.3}$$

This restriction is satisfied, for example, by $\alpha(x) = \frac{2}{2+A} I(x \le c)$.
(b) *Case* $0 < A < 2$.

The value of the game is $-\left(\frac{A}{2+A}\right)^2$. *Players' optimal strategies are*

$$\begin{aligned} \phi^*(x) &= \beta(x) I(x \le b) + I(x > b) \\ \psi^*(y) &= I(y > b), \end{aligned}$$

where $b = (A/2(2+A))^{1/2}$ *and* $\beta(x)$ *is a function* $[0, b] \to [0, 1]$ *satisfying a restriction*

$$\int_0^b \beta(x) x dx = \frac{A}{2(2+A)^2}. \tag{4.4}$$

This restriction is satisfied, for example, by $\beta(x) = \frac{2}{2+A} I(x \le b)$.

Theorem 6 *The solution of the game* $\Gamma^{LR,g\text{-}g}$ *is as follows ;*
(a) *Case* $A > 2$.
The value of the game is $-\left(\frac{A}{A+2}\right)^2$. *Players' optimal strategies are*

$$\begin{aligned} \phi^*(x) &= \alpha(x) I(x \le c) + I(x > c) \\ \psi^*(y) &= I(y > c), \end{aligned}$$

where $c = \frac{1}{2}\left(1 + \sqrt{\frac{A-2}{A+2}}\right)$, *and* $\alpha(x)$ *is a function* $[0, c] \to [0, 1]$ *satisfying a restriction*

$$\int_0^{1/2} \alpha(x) \left(\frac{1}{2} - x\right) dx + \int_{1/2}^c \alpha(x) \left(x - \frac{1}{2}\right) dx = \frac{A}{2(2+A)^2}. \tag{4.5}$$

This restriction is satisfied, for example, by $\alpha(x) = \frac{2}{2+A} I(x \le c)$.
(b) *Case* $0 < A < 2$.

The value of the game is $-\left(\frac{A}{2+A}\right)^2$. *Players' optimal strategies are*

$$\begin{aligned} \phi^*(x) &= \beta(x) I(x \le b) + I(x > b) \\ \psi^*(y) &= I(y > b), \end{aligned}$$

where $b = \frac{1}{2}\left(1 - \sqrt{\frac{2-A}{2+A}}\right)$, *and* $\beta(x)$ *is a function* $[0, b] \to [0, 1]$ *satisfying a restriction*

$$\int_0^b \beta(x) \left(\frac{1}{2} - x\right) dx = \frac{A}{2(2+A)^2}. \tag{4.6}$$

This restriction is satisfied, for example, by $\beta(x) = \frac{2}{2+A} I(x \le b)$.

It is easily shown that

$$A > 2 \quad (\Rightarrow) \quad \left(\frac{1}{2} <\right) 1 - (2+A)^{-1/2} < \frac{A}{A+2} < \frac{1}{2}\left(1 + \sqrt{\frac{A-2}{A+2}}\right),$$

$$0 < A < 2 \quad (\Rightarrow) \quad \left(\frac{1}{2} >\right) \sqrt{\frac{A}{2(A+2)}} > \frac{A}{A+2} > \frac{1}{2}\left(1 - \sqrt{\frac{2-A}{2+A}}\right),$$

and therefore the Monotonicity Property mentioned in Section 1 holds true when $A > 2$.

The decision-thresholds as the function of A in the three games $\Gamma^{LR,u\text{-}u}, \Gamma^{LR,f\text{-}f}$ and $\Gamma^{LR,g\text{-}g}$ are compared as shown by Table 2. The curves of these functions are almost the same as in Figure 1, with the point of concentration $(A, x) = \left(1, \frac{1}{2}\right)$, replaced by $\left(2, \frac{1}{2}\right)$.

Table 2. Decision-threshold in Γ^{LR}

	$0 < A < 2$	$A > 2$
g-g (Th.6)	$\frac{1}{2}\left(1-\sqrt{\frac{2-A}{2+A}}\right)$	$\frac{1}{2}\left(1+\sqrt{\frac{A-2}{A+2}}\right)$
u-u	$A/(A+2)$	A/(A+2)
f-f (Th.5)	$\sqrt{A/2(2+A)}$	$1-(2+A)^{-1/2}$

It is a surprise that the values of these three games in La Relance are identical, *i.e.*,

$$V\left(\Gamma^{LR,g\text{-}g}\right) = V\left(\Gamma^{LR,u\text{-}u}\right) = V\left(\Gamma^{LR,f\text{-}f}\right) = -\left(\frac{A}{A+2}\right)^2. \tag{4.7}$$

Also "the amount of bluff" in these games are again identical,

$$\begin{aligned} E_{u\text{-}u}\alpha(x) &= \frac{2A}{(A+2)^2}, && \text{by (4.2)} \\ E_{f\text{-}f}\alpha(x) &= E_{f\text{-}f}\beta(x) = \frac{2A}{(A+2)^2}, && \text{by (4.3)-(4.4)} \\ E_{g\text{-}g}\alpha(x) &= E_{g\text{-}g}\beta(x) = \frac{2A}{(A+2)^2}, && \text{by (4.5)-(4.6).} \end{aligned} \tag{4.8}$$

The hand distributions, considered in the present paper, are symmetric (*i.e.* $f(x) = f(\overline{x})$, in $[0,1]$, *etc*) and linear. It may be thought that the equalities (4.7) and (4.8) come from this setting. If p.d.f.s are not so, for example, if they are

$$h(x) = \frac{6}{5}\left(3x-2x^2\right), -4x\log x, -4\overline{x}\log\overline{x}, etc.$$

then $V\left(\Gamma^{LR,h\text{-}h}\right) = V\left(\Gamma^{LR,u\text{-}u}\right)$ is questionable.

5 Proofs.

Lemma :

$$\begin{aligned} E_{u\text{-}u}|x-y| &= 2\int_0^1 dx\int_0^x (x-y)dy = \frac{1}{3}. \\ E_{f\text{-}f}|x-y| &= 32\left[2\int_0^{\frac{1}{2}} x dx\int_0^x y(x-y)dy + \int_{\frac{1}{2}}^1 \overline{x}dx\int_{\frac{1}{2}}^x (x-y)\overline{y}dy\right] = \frac{7}{30}. \\ E_{g\text{-}g}|x-y| &= \int_0^1\int_0^1 |x-y|\left\{4-2f(x)-2f(y)+f(x)f(y)\right\}dxdy = \frac{2}{5}. \end{aligned}$$

The last one is obtained by using $g(x) = 2-f(x), E_{f\text{-}f}|x-y| = \frac{7}{30}$ and

$$\int_0^1\int_0^1 |x-y|f(x)dxdy = \int_0^1\left(\frac{1}{2}-x+x^2\right)f(x)dx = \frac{7}{24}.$$

Theorem 1 : Let $\phi(x)$ ($\psi(y)$) be the probability that I (II) bets on the hand x (y). The expected payoff is

$$M(\phi,\psi) = \int_0^1\int_0^1 \left(\overline{\phi}(x),\phi(x)\right)\mathbf{M}^{HHW}(x,y)\left(\overline{\psi}(y),\psi(y)\right)^T f(x)f(y)dxdy \tag{5.1}$$

where $\mathbf{M}^{HHW}(x,y)$ is given by (2.1). By symmetry the value of the game is zero. It follows that

$$M^{HHW}(\phi,\psi) = E\left[\phi-\psi+(1+A)sgn(x-y)\phi\psi\right] = \int_0^1 K(x|\psi)\phi(x)f(x)dx + (\text{an expression not-involving } \phi) \tag{5.2}$$

where

$$K(x|\psi) = 1+(1+A)\int_0^1 sgn(x-y)\psi(y)f(y)dy. \tag{5.3}$$

Suppose temporally that

$$\psi(y) = I(y>c), \text{ for some appropriate } c\in\left(\frac{1}{2},\ 1\right), \tag{5.4}$$

where $I(e)$ is the indicator of the event e. Considering the three cases $x<\frac{1}{2}<c, \frac{1}{2}<x<c$ and $x>c$, we obtain from (5.3)-(5.4),

$$K(x|\psi) = \begin{cases} 1-2(1+A)\ \overline{c}^2, & \text{if } x\le c \\ 1+2(1+A)\left(\overline{c}^2-2\overline{x}^2\right), & \text{if } x>c \end{cases} \tag{5.5}$$

which is continuous and non-decreasing in $x\in[0,1]$. We choose $c=1-(2(1+A))^{-1/2}$, which satisfies $c\in(\frac{1}{2},1)$, if $A>1$. Then $K(x|\psi)=(>)0$, if $x\le(>)c$.
Hence, $\phi^*(x)$, that maximizes (5.2) for $\psi(y)$ given by (5.4), is $\phi^*(x)=I(x>c)$. This proves the first part of Theorem 1.

Now, suppose this time that

$$\psi(y) = I(y>b) \quad \text{for some } b\in\left(0,\ \frac{1}{2}\right). \tag{5.6}$$

Then (5.3) becomes

$$K(x|\psi) = \begin{cases} 1-(1+A)(1-2b^2), & \text{if } x\le b<\frac{1}{2} \\ 1+(1+A)(4x^2-2b^2-1), & \text{if } b<x<\frac{1}{2} \\ 1+(1+A)(-4\overline{x}^2-2b^2+1), & \text{if } b<\frac{1}{2}<x \end{cases} \tag{5.7}$$

which is continuous and non-decreasing in $x\in[0,1]$. We choose $b=\sqrt{A/2(1+A)}$ which satisfies $b\in(0,\frac{1}{2})$ if $0<A<1$. Then $K(x|\psi)=(>)\ 0$ if $x\le(>)\ b$.
Hence $\phi^*(x)$ that maximized (5.2) for $\psi(y)$ given by (5.6), is $\phi^*(x)=I(x>b)$. This proves the secound half of Theorem 1. □

Theorem 2 : For the game $\Gamma^{HHW,g\text{-}g}$ the expected payoffs is

$$M(\phi,\psi) = \int_0^1 K(x|\psi)\phi(x)g(x)dx + (\text{an expression not-involving } \phi) \tag{5.2$'$}$$

and

$$K(x|\psi) = 1+(1+A)\int_0^1 sgn(x-y)\psi(y)g(y)dy. \tag{5.3$'$}$$

Hence for (5.4), we get

$$K(x|\psi) = \begin{cases} 1-2(1+A)c\overline{c}, & \text{if } x\le c \\ 1+2(1+A)(-x\overline{x}+c\overline{c}), & \text{if } x>c\ (>\frac{1}{2}), \end{cases} \tag{5.5$'$}$$

which is continuous and non-decreasing in $x \in [0,1]$. We choose $2c\overline{c} = (1+A)^{-1}$ *i.e.*, $c = \frac{1}{2}\left(1+\sqrt{\frac{A-1}{A+1}}\right)$ if $A > 1$. Then $\phi^*(x) = I(x > c)$ maximizes (5.2′).

Meanwhile, for (5.6), we obtain

$$K(x|\psi) = \begin{cases} 1-(1+A)(1-2b+2b^2), & \text{if } 0 < x \le b \\ 1-(1+A)\left\{(2x-1)^2+2b\overline{b}\right\}, & \text{if } b < x < \frac{1}{2} \\ 1+(1+A)\left\{(2x-1)^2-2b\overline{b}\right\}, & \text{if } b < \frac{1}{2} < x \end{cases} \tag{5.7′}$$

which is continuous and non-decreasing in $x \in [0,1]$. We choose b such that $1-2b+2b^2 = (1+A)^{-1}$, *i.e.*, $b = \frac{1}{2}\left(1-\sqrt{\frac{1-A}{1+A}}\right) \in (0, \frac{1}{2})$, if $0 < A < 1$. The $\phi^*(x) = I(x > b)$ maximizes (5.2′). □

Theorem 3 : For the game $\Gamma^{HL,f\text{-}f}$ the expected payoff is

$$M(\phi,\psi) = \int_0^1\int_0^1 \left(\phi(x),\overline{\phi}(x)\right)\mathbf{M}^{HL}(x,y)\left(\psi(y),\overline{\psi}(y)\right)^T f(x)f(y)dxdy \tag{5.8}$$

where $\mathbf{M}^{HL}(x,y)$ is given by (3.1). By symmetry the value of the game is zero.

It follows that, by (5.8),

$$\begin{aligned} M^{HL}(\phi,\psi) &= E\left[\{\phi+\psi+(B-1)\phi\psi\}\, sgn(x-y)\right] \\ &= \int_0^1 K(x|\psi)\phi(x)f(x)dx + (\text{an expression non-involving } \phi) \end{aligned} \tag{5.9}$$

where

$$K(x|\psi) = \int_0^1 \left(B\psi(y)+\overline{\psi}(y)\right) sgn(x-y)f(y)dy. \tag{5.10}$$

Suppose, temporally that

$$\psi(y) = I(y > c), \quad \text{for some } c \in (1/2,\ 1)\,. \tag{5.11}$$

Then, from (5.10) and (5.11), we have

$$K(x|\psi) = \begin{cases} 4x^2-1-2(B-1)\overline{c}^2, & \text{if } 0 < x < \frac{1}{2} < c \\ -4\overline{x}^2+1+2(1-B)\overline{c}^2, & \text{if } \frac{1}{2} < x < c \\ B(-4\overline{x}^2+2\overline{c}^2)+1-2\overline{c}^2, & \text{if } c < x < 1 \end{cases} \tag{5.12}$$

which is continuous and increasing in $x \in [0,1]$. We choose $c = 1-(2(B+1))^{-1/2}$ which satisfies $c \in (\frac{1}{2},1)$ if $B > 1$. Then $K(x|\psi) < (>)\ 0$, if $x < (>)\ c$. Hence $\phi^*(x)$ that maximizes (5.9), for $\psi(y)$ given by (5.11), is $\phi^*(x) = I(x > c)$. This proves the first half of the Theorem 3.

Now, next, we let

$$\psi(y) = I(y > b), \quad \text{for some } b \in (0,\ 1/2)\,. \tag{5.13}$$

Then (5.10) gives

$$K(x|\psi) = \begin{cases} 4x^2-2b^2-B(1-2b^2), & \text{if } 0 < x < b \\ 2b^2+B(4x^2-2b^2-1), & \text{if } b < x < \frac{1}{2} \\ 2b^2+B(-4\overline{x}^2-2b^2+1), & \text{if } b < \frac{1}{2} < x < 1 \end{cases} \tag{5.14}$$

which is continuous and increasing in $x \in [0,1]$. Choose $b = \sqrt{B/2(B+1)}$ which satisfies $b \in (0,\frac{1}{2})$ if $0 < B < 1$. Thus $K(x|\psi) < (>)\ 0$ if $x < (>)\ b$. Hence $\phi^*(x)$ that maximizes

(5.9) for $\psi(y)$ given by (5.13) is $\phi^*(x) = I(x > b)$. This proves the second half of Theorem 3. $\square$

Theorem 4 : The proof is much similar as in Theorem 3. We have (5.9′) and (5.10′) which are (5.9) and (5.10) with $f(x)$ replaced by $g(x)$.

First let

$$\psi(y) = I(y > c), \quad \text{for some } c \in (1/2,\ 1)\,. \tag{5.11′}$$

Then from (5.10′) and (5.11′) we obtain

$$K(x|\psi) = \begin{cases} -(2x-1)^2 + 2c\overline{c} - 2Bc\overline{c}, & \text{if } 0 < x < \frac{1}{2} < c \\ 4x^2 - 2c^2 - 2c + 1 - 2Bc\overline{c}, & \text{if } \frac{1}{2} < x < c \\ B(4x^2 - 4x + 2c\overline{c}) + 2c^2 - 2c + 1, & \text{if } c < x < 1 \end{cases} \tag{5.12′}$$

which is continuous and increasing in $x \in [0,1]$. We choose $c\overline{c} = \frac{1}{2(B+1)}$ *i.e.*, $c = \frac{1}{2}\left(1 + \sqrt{\frac{B-1}{B+1}}\right)$ if $B > 1$. Then $K(x|\psi) < (>)\ 0$, if $x < (>)\ c$. Hence $\phi^*(x) = I(x > c)$ maximaizes (5.9′), for $\psi(y)$ given by (5.11′).

Now next let

$$\psi(y) = I(y > b), \quad \text{for some } b \in (0,\ 1/2)\,. \tag{5.13′}$$

Then (5.10′) becomes

$$K(x|\psi) = \begin{cases} -4x^2 + 4x - 2b\overline{b} - B(2b^2 - 2b + 1), & \text{if } 0 < x < b \\ B(-4x^2 + 4x + 2b^2 - 2b - 1) + 2b\overline{b}, & \text{if } b < x < \frac{1}{2} \\ B(4x^2 - 4x + 2b^2 - 2b + 1) + 2b\overline{b}, & \text{if } b < \frac{1}{2} < x \end{cases} \tag{5.14′}$$

which is continuous and increasing in $x \in [0,1]$. Choose b such that $b\overline{b} = \frac{B}{2(1+B)}$, *i.e.*, $b = \frac{1}{2}\left(1 - \sqrt{\frac{1-B}{1+B}}\right) \in (0, \frac{1}{2})$, if $0 < B < 1$. Then $K(x|\psi) < (>)\ 0$, if $x < (>)\ b$.
Hence $\phi^*(x)$ that maximizes (5.9′) for $\psi(y)$ given by (5.13′) is $\phi^*(x) = I(x > b)$. $\square$

Theorem 5 : For the game $\Gamma^{LR,f\text{-}f}$ the expected payoff is

$$\begin{aligned} M(\phi,\psi) &= \int_0^1 \int_0^1 \left(\overline{\phi}(x), \phi(x)\right) \mathbf{M}^{LR}(x,y) \left(\overline{\psi}(y), \psi(y)\right)^T f(x)f(y)dxdy \\ &= \int_0^1 \int_0^1 \left\{-\overline{\phi}(x) + \phi(x)\overline{\psi}(y) + \phi(x)\psi(y)(1+A)sgn(x-y)\right\} f(x)f(y)dxdy \end{aligned} \tag{5.15}$$

which can be rewritten by

$$M(\phi,\psi) = \int_0^1 K(x|\psi)\phi(x)f(x)dx - 1, \tag{5.16}$$

where

$$K(x|\psi) = 2 + \int_0^1 \{-1 + (1+A)sgn(x-y)\}\,\psi(y)f(y)dy\ ; \tag{5.16a}$$

or,

$$M(\phi,\psi) = \int_0^1 L(y|\phi)\psi(y)f(y)dy + (\text{an expression not-involving } \psi) \tag{5.17}$$

where

$$L(y|\phi) = \int_0^1 \{-1 + (1+A)sgn(x-y)\}\,\phi(x)f(x)dx. \tag{5.17a}$$

Proof of (a). Suppose temporally that

$$\psi^*(y) = I(y > c) \quad \text{for some } c \in (1/2,\ 1)\,. \tag{5.18}$$

Then (5.16a) becomes

$$K(x|\psi^*) = \begin{cases} 2 - 2(2+A)\overline{c}^2, & \text{if } x \le c \\ 2 - 4(1+A)\overline{x}^2 + 2A\overline{c}^2, & \text{if } x > c \end{cases} \tag{5.19}$$

which is continuous and non-decreasing in $x \in [0,1]$. We choose c such that $\overline{c}^2 = \frac{1}{2+A}$, *i.e.*, $c = 1 - (2+A)^{-1/2} \in (\frac{1}{2}, 1)$ if $A > 2$. Thus $K(x|\psi^*) = (>)\ 0$ if $x \le (>)\ c$. Hence by (5.16),

$$\phi^*(x) = \begin{cases} \text{a function } \alpha(x) : [0,c] \to [0,1], & \text{if } c \le c \\ 1, & \text{if } x > c \end{cases}$$

maximizes $M(\phi, \psi^*)$.

On the other hand by (5.17a) we have

$$\frac{1}{4}L(y|\phi^*) = \begin{cases} \left[-(2+A)\int_0^y + A\int_y^{1/2}\right]\alpha(x)xdx \\ \qquad + A\left\{\int_{1/2}^c \alpha(x)\overline{x}dx + \frac{1}{2}\overline{c}^2\right\}, & \text{if } y < \frac{1}{2} < c \\ -(2+A)\int_0^{1/2}\alpha(x)xdx \\ + \left[-(2+A)\int_{1/2}^y + A\int_y^c\right]\alpha(x)\overline{x}dx + \frac{1}{2}A\overline{c}^2, & \text{if } \frac{1}{2} < y < c \\ -(2+A)\int_0^{1/2}\alpha(x)xdx - (2+A)\left[\int_{1/2}^c \alpha(x)\overline{x}dx \right. \\ \qquad \left. + \frac{1}{2}\left(\overline{c}^2 - \overline{y}^2\right)\right] + \frac{1}{2}A\overline{c}^2, & \text{if } \frac{1}{2} < c < y \end{cases} \tag{5.20}$$

which is continuous and decreasing in $y \in [0,1]$. Choosing $\alpha(x)$ such taht $L(c|\phi^*) = 0$, *i.e.*,

$$\int_0^{1/2}\alpha(x)xdx + \int_{1/2}^c \alpha(x)\overline{x}dx = \frac{A}{2(2+A)}\overline{c}^2 = \frac{A}{2(2+A)^2} \tag{5.21}$$

we have $L(y|\phi^*) > (=,<)\ 0$, for $y < (=,>)\ c$. The condition (5.21) is possible, if, for example, $\alpha(x) = I\left(x < \sqrt{\frac{A}{A+2}}\ \overline{c}\right)$. Therefore $\psi^*(y) = I(y > c)$ minimizes $M(\phi^*, \psi)$. Thus we have shown that (ϕ^*, ψ^*) is the optimal strategy-pair.

Finally we compute the value of the game. By (5.16), (5.19) and $\overline{c}^2 = \frac{1}{2+A}$, we have

$$\begin{aligned} M(\phi^*, \psi^*) &= \int_0^1 K(x|\psi^*)\phi^*(x)f(x)dx - 1 = 4\int_0^{\overline{c}} \left\{2 - 4(1+A)x^2 + 2A\overline{c}^2\right\}xdx - 1 \\ &= 4\left\{\left(1 + A\overline{c}^2\right)\overline{c}^2 - (1+A)\overline{c}^4\right\} - 1 = 4\left(\overline{c}^2 - \overline{c}^4\right) - 1 = -\left(\frac{A}{2+A}\right)^2. \end{aligned}$$

Proof of (b). Let $0 < A < 2$. Now let

$$\psi^*(y) = I(y > c), \quad \text{for some } c \in (0,\ 1/2)\,. \tag{5.22}$$

Then (5.16a) becomes

$$K(x|\psi^*) = \begin{cases} -A + 2(2+A)c^2, & \text{if } 0 < x < c \\ 4x^2 + A(4x^2 - 2c^2 - 1), & \text{if } c < x < \frac{1}{2} \\ 2 + A(1-2c^2) - 4(A+1)\overline{x}^2, & \text{if } c < \frac{1}{2} < x < 1 \end{cases} \tag{5.23}$$

which is continuous and non-decreasing in $x \in [0,1]$. Choose c such that $K(c|\psi^*) = 0$, *i.e.*, $c = \sqrt{A/2(2+A)} \in (0, \frac{1}{2})$. That is, $\phi^*(x) = \alpha(x)I(x \le c) + I(x > c)$, for some $\alpha(x) : [0,c] \to [0,1]$, maximizes $M(\phi, \psi^*)$.

Meanwhile, by (5.17a), we have

$$\frac{1}{4}L(y|\phi^*) = \begin{cases} \left[-(2+A)\int_0^y + A\int_y^c\right]\alpha(x)xdx \\ \qquad\qquad + A\left\{\frac{1}{2}\left(\frac{1}{4} - c^2\right) + \frac{1}{8}\right\}, & \text{if } y < c < \frac{1}{2} \\ -(2+A)\int_0^c \alpha(x)xdx \\ \qquad -(2+A)\frac{1}{2}\left(y^2 - c^2\right) + A\left(\frac{1}{4} - \frac{1}{2}y^2\right), & \text{if } c < y < \frac{1}{2} \\ -(2+A)\int_0^c \alpha(x)xdx \\ \qquad -(2+A)\left\{\frac{1}{4} - \frac{1}{2}c^2 - \frac{1}{2}\overline{y}^2\right\} + A\frac{1}{2}\overline{y}^2, & \text{if } c < \frac{1}{2} < y \end{cases} \tag{5.24}$$

which is continuous and decreasing in $y \in [0,1]$. By using $c^2 = A/2(2+A)$ and choosing $\alpha(x)$ such that $L(c|\phi^*) = -(2+A)\int_0^c \alpha(x)xdx + A\left(\frac{1}{4} - \frac{1}{2}c^2\right) = 0$ we obtain

$$\int_0^c \alpha(x)xdx = c^2\left(\frac{1}{2} - c^2\right). \tag{5.25}$$

Hence we find that $L(y|\phi^*) < (=, >)\ 0$, for $y < (=, >)\ c$ and, by (5.17), $\psi^*(y) = I(y > c)$ minimizes $M(\phi^*, \psi)$. This condition for $\alpha(x)$ is possible, for example, $\alpha(x) = \frac{2}{2+A}I(x \le c)$.

Finally we must derive the value of the game when $0 < A < 2$. From (5.16), (5.23) and (5.25) we find

$$\begin{aligned} \frac{1}{4}M(\phi^*, \psi^*) &= \int_c^{1/2}\left\{4x^2 + A\left(4x^2 - 2c^2 - 1\right)\right\}xdx + \int_{1/2}^1\left\{2 + A(1-2c^2) - 4(A+1)\overline{x}^2\right\}\overline{x}dx - \frac{1}{4} \\ &= -(1+A)c^4 - \frac{1}{8}(1+2c^2) + \frac{1}{8}A(1-2c^2) + \frac{1}{2}A(c^2 + 2c^4) = -c^4. \end{aligned}$$

Therefore $M(\phi^*, \psi^*) = -4c^4 = -\frac{A^2}{(2+A)^2}$. □

Theorem 6 : All computations made in the proof of Theorem 5 are made again with $f(x) = 4(x \wedge \overline{x})$, replaced by $g(x) = 4\left|\frac{1}{2} - x\right|$.

Proof of (a). For $\psi^*(y)$ defined by (5.18), Eq.(5.16a) becomes

$$\begin{aligned} K(x|\psi^*) &= 2 + \int_0^1 \{-1 + (1+A)sgn(x-y)\}\,\psi^*(y)g(y)dy \\ &= \begin{cases} 2 - 2(2+A)c\overline{c}, & \text{if } x \le c \\ 2 + 4(1+A)(x^2 - x) + 2Ac\overline{c}, & \text{if } x > c \end{cases} \end{aligned} \tag{5.19$'$}$$

which is continuous and non-decreasing in $x \in [0,1]$. Choose c such that $c\overline{c} = \frac{1}{2+A}$, *i.e.*, $c = \frac{1}{2}\left(1 + \sqrt{\frac{A-2}{A+2}}\right) \in (\frac{1}{2}, 1)$, if $A > 2$. Thus $K(x|\psi^*) = (>)\ 0$ if $x < (>)\ c$.

Hence by (5.16),

$$\phi^*(x) = \begin{cases} \text{a function } \alpha(x) : [0,c] \to [0,1], & \text{if } x < c \\ 1, & \text{if } x > c \end{cases}$$

maximizes $M(\phi, \psi^*)$.

On the other hand, we have by (5.17′)

$$
(5.20') \quad \frac{1}{4}L(y|\phi^*) = \begin{cases} \left[-(2+A)\int_0^y + A\int_y^{1/2}\right]\alpha(x)\left(\frac{1}{2}-x\right)dx \\ \qquad + A\int_{1/2}^{c}\alpha(x)\left(x-\frac{1}{2}\right)dx + \frac{1}{2}Ac\overline{c}, & \text{if } y < \frac{1}{2} < c \\ -(2+A)\int_0^{1/2}\alpha(x)\left(\frac{1}{2}-x\right)dx \\ + \left[-(2+A)\int_{1/2}^{y} + A\int_y^c\right]\alpha(x)\left(x-\frac{1}{2}\right)dx + \frac{1}{2}Ac\overline{c}, & \text{if } \frac{1}{2} < y < c \\ -(2+A)\int_0^{1/2}\alpha(x)\left(\frac{1}{2}-x\right)dx \\ \qquad -(2+A)\int_{1/2}^{c}\alpha(x)\left(x-\frac{1}{2}\right)dx \\ \qquad -(2+A)\frac{1}{2}\left(y^2-y+c\overline{c}\right)+\frac{1}{2}A(y-y^2), & \text{if } \frac{1}{2} < c < y \end{cases}
$$

which is continuous and decreasing in $y \in [0, 1]$.

Choosing c such that $L(c|\phi^*) = 0$, we obtain

$$
(5.21') \qquad \int_0^{1/2}\alpha(x)\left(\frac{1}{2}-x\right)dx + \int_{1/2}^{c}\alpha(x)\left(x-\frac{1}{2}\right)dx = \frac{Ac\overline{c}}{2(2+A)} = \frac{A}{2(2+A)^2}.
$$

This restriction on $\alpha(x)$ is satisfied, for example, by $\alpha(x) = 2c\overline{c}I(x < c)$.

We want to derive the value of the game,

$$
\begin{aligned}
M(\phi^*, \psi^*) &= \int_0^1 K(x|\psi^*)\phi^*(x)g(x)dx - 1 \\
&= 4\int_0^1 \left\{2 + 4(1+A)(x^2-x) + 2Ac\overline{c}\right\}\left(x-\frac{1}{2}\right)dx - 1 \\
&= 4\left\{(1+A)\left(-c^4 + 2c^3 - \frac{4+A}{2+A}c^2 + \frac{2}{2+A}c\right)\right\} - 1 \\
&= \frac{1}{4}(1+A)\left\{-(c\overline{c})^2 + \frac{2}{2+A}c\overline{c}\right\} - 1 = -\left(\frac{A}{2+A}\right)^2.
\end{aligned}
$$

Proof of (b). For $\psi^*(y) = I(y > b)$, for some $b \in (0, \frac{1}{2})$, we derive, by (5.16a).

$$
K(x|\psi^*) = \begin{cases} 2 - 2(2+A)\left(b^2 - b + \frac{1}{2}\right), & \text{if } x < b < \frac{1}{2} \\ 2 + 4(1+A)(x-x^2) - 2b\overline{b}A - (2+A), & \text{if } b < x < \frac{1}{2} \\ 2 + 4(1+A)(x^2-x) + 2A\left(b^2-b+\frac{1}{2}\right), & \text{if } b < \frac{1}{2} < x. \end{cases}
$$

Choose b such that $K(b|\psi^*) = 0$, *i.e.*, $b = \frac{1}{2}\left(1 - \sqrt{\frac{2-A}{2+A}}\right)$ if $0 < A < 2$.

Hence, by (5.16a), we get the optimal $\phi^*(x) = \beta(x)I(x < b) + I(x > b)$, where $\beta(x)$ is a function : $[0, b] \to [0, 1]$ which will be more clarified later.

Now, by (5.17a), we compute

$$
\frac{1}{4}L(y|\phi^*) = \begin{cases} \left[-(2+A)\int_0^y + A\int_y^b\right]\beta(x)\left(\frac{1}{2}-x\right)dx \\ \qquad + A\left(\frac{1}{4} - \frac{1}{2}b\overline{b}\right), & \text{if } y < b < \frac{1}{2} \\ -(2+A)\int_0^b \beta(x)\left(\frac{1}{2}-x\right)dx \\ \qquad -(2+A)\frac{1}{2}(y\overline{y} - b\overline{b}) + A\left(\frac{1}{4} - \frac{1}{2}y\overline{y}\right), & \text{if } b < y < \frac{1}{2} \\ -(2+A)\int_0^b \beta(x)\left(\frac{1}{2}-x\right)dx \\ \qquad + \frac{1}{4} - \frac{1}{2}(b\overline{b} + y\overline{y}) + A\frac{1}{2}y\overline{y}, & \text{if } b < \frac{1}{2} < y. \end{cases}
$$

By the condition $L(b|\phi^*) = 0$, we find that

$$\int_0^b \beta(x)\left(\frac{1}{2}-x\right)dx = \frac{A}{2+A}\left(\frac{1}{4}-\frac{1}{2}b\overline{b}\right) = \frac{A}{2(2+A)^2}.$$

We want to find the value of the game.

$$\begin{aligned}
M(\phi^*,\psi^*) &= \int_0^1 K(x|\psi^*)\phi^*(x)g(x)dx - 1 \\
&= 4\left[\int_b^{1/2}\left[2+4x\overline{x}-2b\overline{b}A-(2+A)\right]\left(\frac{1}{2}-x\right)dx\right. \\
&\qquad \left. + \int_{\frac{1}{2}}^1 \left[2+4(1+A)(-x\overline{x})+2A\left(b^2-b+\frac{1}{2}\right)\right]\left(x-\frac{1}{2}\right)dx\right] - 1 \\
&= 4\left[\int_0^{\frac{1}{2}-b}\left\{(1+A)(1-4t^2)-A(1+2b\overline{b})\right\}tdt\right. \\
&\qquad \left. + \int_0^{\frac{1}{2}}\left\{(1+A)(4s^2-1)+2+A(1-2b\overline{b})\right\}sds\right] - 1 \\
&= \left(\frac{1}{2}-b\right)^2(1-A+4b\overline{b}) + \frac{3}{4} + A\left(\frac{1}{4}-b\overline{b}\right) - 1 \\
&= -\left[\frac{A^2-A-2}{4(2+A)} + \frac{2A^2-A^3}{4(2+A)^2} + \frac{1}{4}\right] = -\left(\frac{A}{2+A}\right)^2 \qquad \square
\end{aligned}$$

6 Remark. Newman's real poker (denoted by $\Gamma^{NRP,u\text{-}u}$) is different from the usual kinds of poker. It brings the following two changes into the game $\Gamma^{LR,u\text{-}u}$ in that (1) Player I is not permitted to fold, and (2) he must choose and announce the amount of bet which can be arbitrary high. In Ref.[3] Newman gave the interesting solution of the game and noted that the value of the game is 1/7 and the integer 7 is mysteriously present in the solution. What change will appear, instead of the integer 7 in the solutions of the games $\Gamma^{NRP,f\text{-}f}$ and $\Gamma^{NRP,g\text{-}g}$? It is an interesting question. See also Ref.[4].

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*3-26-4 Midorigaoka, Toyonaka, Osaka, 560-0002, Japan,
Fax: +81-6-6856-2314 E-mail: minorus@tcct.zaq.ne.jp