ON QUADRATIC BF-ALGEBRAS

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Abstract. In this paper we introduce the notion of a quadratic BF-algebra, and obtain that quadratic BF-algebras, quadratic Q-algebras, BG -algebras and B-algebras are equivalent notions on a field X with $|X| \geq 3$, and hence every quadratic BF-algebra is a BCI-algebra.

1. Introduction

The concept of B-algebras was introduced by J. Neggers and H. S. Kim. We refer to [1, 6, 7, 8, 10, 11, 12] for details. They defined a *B-algebra* as an algebra $(X; *, 0)$ of type $(2, 0)$ (i.e., a non-empty set X with a binary operation $*$ and a constant 0) satisfying $(B1)$, $(B2)$ and (B) $(x * y) * z = x * [z * (0 * y)]$, for any $x, y, z \in X$. In [2], Y. B. Jun, R. H. Roh and H. S. Kim introduced BH-algebras, which are generalization of $BCK/BCH/B$-algebras. An algebra $(X; *, 0)$ of type $(2, 0)$ is a *BH-algebra* if it satisfies $(B1)$, $(B2)$ and (BH) $x * y = 0 = y * x$ implies $x = y$. Recently, C. B. Kim and H. S. Kim ([3]) defined a *BG-algebra* as an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying $(B1)$, $(B2)$ and (BG) $x = (x * y) * (0 * y)$, for any $x, y \in Z$. J. Neggers, S. S. Ahn and H. S. Kim ([8]) introduced the notion of a Q-algebra, i.e., $(B1)$, $(B2)$ and (Q) $(x * y) * z = (x * z) * y$, which is a generalization of the idea of $BCH/BCI/BCK$-algebras and generalized some theorems discussed in BCI-algebras. A. Walendziak ([13]) introduced the notion of BF-algebras, which is a generalization of B-algebras, and investigated some properties of (normal) ideals in BF-algebras. For another generalization of B-algebras we refer to [3, 8, 13]. Recently, the present authors ([5]) studied some properties of (normal, closed) ideals in BF-algebras, especially they showed that any ideal of BF-algebra can be decomposed into the union of some sets, and obtained the greatest closed ideal I^0 of an ideal I of a BF-algebra X contained in I. In this paper we introduce the notion of a quadratic BF-algebra, and obtain that quadratic BF-algebras, quadratic Q-algebras, BG -algebras and B-algebras are equivalent notions on a field X with $|X| \geq 3$, and hence every quadratic BF-algebra is a BCI-algebra.

2. Preliminaries

Definition 2.1. A *BF- algebra* is an algebras $(X; *, e)$ of type (2,0) satisfying the following axioms:

$(B1)$ $x * x = e$,

$(B2)$ $x * e = x$,

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(BF) $e * (x * y) = y * x$, for any $x, y \in X$.

Example 2.2. ([13] Let $\mathbf{R}$ be the set of all real numbers and let "$*$" be defined by

$$x * y = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then $(\mathbf{R}; *, 0)$ is a BF-algebra.

Example 2.3. ([13]) Let $A := [0, \infty)$. Define a binary operation "$*$" on A as follows; $x * y := |\, x - y \,|$, $x, y \in A$. Then $(A; *, 0)$ is a BF-algebra.

Example 2.4. Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following table:

$*$	0	1	2	3	4
0	0	1	1	1	1
1	1	0	1	2	2
2	2	1	0	2	2
3	3	1	1	0	3
4	4	1	1	1	0

Then $(X; *, 0)$ is a BF-algebra.

3. The Quadratic BF-algebras

Let X be a field with $|\, X \,| \geq 3$. An algebra $(X; *)$ is said to be *quadratic* if $x * y$ is defined by $x * y = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6$, where $a_1, a_2, a_3, a_4, a_5, a_6 \in X$, for any $x, y \in X$. A quadratic algebra $(X; *)$ is said to be a *quadratic BF-algebra* if it satisfies the condition $(B1)$, $(B2)$ and (BF).

Theorem 3.1. *Let X be a field with $|\, X \,| \geq 3$. Then every quadratic BF-algebra $(X; *, e)$ has of the form $x * y = x - y + e$ where $x, y \in X$.* *Proof.* Define $x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, where $A, B, C, D, E, F \in X$ and $x, y, \in X$. Consider $(B1)$.

$$\begin{aligned} e &= x * x \\ &= (A + B + C)x^2 + (D + E)x + F \end{aligned}$$

It follows that $F = e$, $A + B + C = 0 = D + E$, i.e., $D = -E$.
Consider $(B2)$.

$$x = x * e \quad = \quad Ax^2 + Bxe + Ce^2 + Dx + Ee + e$$

It follows that $A = 0$, $Be + D = 1$ and $Ce^2 + Ee + e = 0$.
Thus $B + C = 0, D = 1 - Be$. Since $D = -E$, we have $E = -1 + Be$.
From this information, we have the following more simpler form:

$$\begin{aligned} x * y &= Bxy + Cy^2 + Dx + Ey + e \\ &= Bxy + (-B)y^2 + (1 - Be)x + (Be - 1)y + e \\ &= B(xy - y^2 - ex + ey) + (x - y + e) \\ &= B(x - y)(y - e) + (x - y + e) \end{aligned}$$

By applying (BF) we obtain:

$$\begin{aligned} e * (x * y) &= B * (e - x * y)(x * y - e) + (e - x * y + e) \\ &= B[e - B(x - y)(y - e) - (x - y + e)][B(x - y)(y - e) + (x - y + e) - e] \\ &\quad +[2e - B(x - y)(y - e) - (x - y + e)] \\ &= B[-B(x - y)(y - e) - (x - y)][B(x - y)(y - e) + (x - y)] \\ &\quad +[e - B(x - y)(y - e) - (x - y)] \\ &= -B[B(x - y)(y - e) + (x - y)]^2 - [B(x - y)(y - e) + (x - y)] + e \\ &= -B(x - y)^2[B(y - e) + 1]^2 - (x - y)[B(y - e) + 1] + e. \end{aligned}$$

Since $y * x = B(y - x)(x - e) + (y - x + e) = B(y - x)(x - e) + (y - x) + e$, in order to satisfy (BF), it should be

$$-B(x - y)^2[B(y - e) + 1]^2 - (x - y)[B(y - e) + 1] + e = B(y - x)(x - e) + (y - x) + e.$$

If we let $x := e$ in the above identity, then $-B(e-y)^2[B(y-e)+1]^2 - (e-y)[B(y-e)+1] = y - e$. It follows that $B = 0$. Hence, $C = 0, E = -1$ and $D = 1$. Thus $x * y = x - y + e$, proving the theorem. □

Example 3.2. Let $\mathbf{R}$ be the set of all real numbers. If we define $x * y := x - y$, the usual subtraction on $\mathbf{R}$, then $(\mathbf{R}; *, 0)$ be a BF-algebra.

Example 3.3. Let $\kappa := GF(p^n)$ be a Galois field. If we define $x * y := x - y + e$, then $(\kappa; *, e)$ is a BF-algebra.

Proposition 3.4. *Let X be a field with $|X| \geq 3$. If X is a quadratic BF-algebra, then $(x * z) * (y * z) = x * y$ for any $x, y, z \in X$.*

Proof. Straightforward. □

H. K. Park and H. S. Kim ([10]) proved that every quadratic B-algebra $(X; *, e)$, $e \in X$, has the form $x * y = x - y + e$, where X is a field with $|X| \geq 3$. J. Neggers, S. S. Ahn and H. S. Kim ([9]) introduced the notion of $Q-$algebras, and obtained that every quadratic Q-algebra $(X; *, e)$, $e \in X$, has the form $x * y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. Also, H. S. Kim and H. D. Lee ([4]) showed that every quadratic BG-algebra $(X; *, e)$, $e \in X$, has the form $x * y = x - y + e$, $x, y \in X$, where X is a field with $|X| \geq 3$. We summarize:

Theorem 3.5. *Let X be a field with $|X| \geq 3$. Then the following are equivalent:*
(1) $(X; *, e)$ is a quadratic BF-algebra,
(2) $(X; *, e)$ is a quadratic BG-algebra,
(3) $(X; *, e)$ is a quadratic Q-algebra,
(4) $(X; *, e)$ is a quadratic B-algebra.

Theorem 3.6. ([10]) *Let X be a field with $|X| \geq 3$. Then every quadratic B-algebra on X is a BCI-algebra.* **Theorem 3.7.** *Let X be a field with $|X| \geq 3$. Then every quadratic BF-algebra on X is a BCI-algebra.*

Proof. It is an immediate consequence of Theorem 3.5 and Theorem 3.6. □

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