$\delta\theta$-REFINABILITY OF PRODUCT SPACES

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Abstract. In this paper we shall show: (1) Let X be a zero-dimensional metric space and Y be a $\delta\theta$-refinable P-space. Then $X \times Y$ is $\delta\theta$-refinable.

(2) Let X be an almost expandable strong Σ-space and Y be a strong $\delta\theta$-refinable P-space. Then $X \times Y$ is $\delta\theta$-refinable.

(3) Let X be a metrizable space and Y be a w-$\delta\theta$-refinable P-space. Then $X \times Y$ is w-$\delta\theta$-refinable.

Similar results of (3) for analogous properties also hold.

1. Introduction

Throughout this paper we assume that each space is a T_1-space. Each map is assumed to be continuous.

It is known the following.

(I) Suppose X is a Σ-space and Y is a P-space.

(i) ([10]). If X and Y are both paracompact, then $X \times Y$ is paracompact.

(ii) ([1]). If X and Y are both submetacompact (θ-refinable), then $X \times Y$ is submetacompact (θ-refinable).

A space X is called a Σ-space if X has a Σ-net.

(II) ([3]). Suppose X is a separable metric space and Y is a normal $P(\omega)$-space. If Y is $\delta\theta$-refinable, then $X \times Y$ is $\delta\theta$-refinable.

In this paper we shall investigate the conditions for the product space $X \times Y$ has $\delta\theta$-refinability and other $\delta\theta$-refinability-like properties.

Let Ω be a set. Denote $\Omega^n = \{(\alpha_0, \alpha_1, ..., \alpha_{n-1}) | \alpha_i \in \Omega, i = 0, ..., n-1\}$ for each $n \in \omega, \Omega^{<\omega} = \bigcup_{n\in\omega} \Omega^n$ and $\Omega^\omega = \{(\alpha_0, \alpha_1, ..., \alpha_n, ...,) | \alpha_n \in \Omega$ for each $n \in \omega\}$. For each $\sigma = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in \Omega^n$ and $\alpha \in \Omega$, we denote $\sigma \vee \alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}, \alpha)$. For each $\sigma = (\alpha_0, \alpha_1, ..., \alpha_n, ...) \in \Omega^\omega$, we denote $\sigma \restriction n = (\alpha_0, \alpha_1, ..., \alpha_{n-1})$. It is obvious that $\sigma \restriction n \in \Omega^n$.

A space X is said to be a P-space (resp. $P(\omega)$-space) ([9]) if for any open cover $\{U(\sigma) | \sigma \in \Omega^{<\omega}\}$ (resp. with $|\Omega| \le \omega$) of X where $U(\sigma) \subset U(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$ and $\alpha \in \Omega$, then there is a closed cover $\{K(\sigma) | \sigma \in \Omega^{<\omega}\}$ of X such that

(i) $K(\sigma) \subset U(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,

(ii) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} U(\sigma \restriction n) = X$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = X$.

For a space X, $\dim X$ denotes the covering dimension of X and X is a zero-dimensional space means $\dim X = 0$.

A subset A of X is called "*clopen*" set if A is both an open set and a closed set of X.

The following lemmas $1 \sim 3$ are well known.

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Lemma 1. *If X is a zero-dimensional metric space, then X has a base $\mathcal{B}$ satisfying the following conditions:*

(i) $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, $\mathcal{B}_n$ is a discrete cover of X by clopen sets,

(ii) $\mathcal{B}_n = \{B(\sigma)|\sigma \in \Omega^n\}, B(\sigma) = \bigcup_{\alpha\in\Omega} B(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{B(\sigma \restriction n)|n \in \omega\}$ is a local base of x in X.

A collection $\mathcal{F}$ of subsets of X is called a *net* of X if for each $x \in X$ and each open set U such that $x \in U$, there is an $F \in \mathcal{F}$ such that $x \in F \subset U$.

A space X is called a *σ-space* if σ-locally finite net.

Lemma 2. *If X is a σ-space, then X has a net $\mathcal{F}$ satisfying the following conditions:*

(i) $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{F}_n$ is a locally finite closed cover of X,

(ii) $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}, F(\sigma) = \bigcup_{\alpha\in\Omega} F(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{F(\sigma \restriction n)|n \in \omega\}$ is a net of x.

Lemma 3. *If X is a Σ-space, then X has a spectral Σ-net $\mathcal{F}$, i. e., satisfying the following conditions:*

(i) $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\mathcal{F}_n$ is a locally finite closed cover of X,

(ii) $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}, F(\sigma) = \bigcup_{\alpha\in\Omega} F(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(iii) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{F(\sigma \restriction n)|n \in \omega\}$ is a K-net of $C(x)$, i. e., if U is an open set in X such that $C(x) \subset U$, then $F(\sigma \restriction n) \subset U$ for some n. Here $C(x) = \bigcap_{n\in\omega} F(\sigma \restriction n)$.

A space X is called a *strong Σ-space* if X has a Σ-net such that $C(x)$ is compact for each $x \in X$.

It is well known that each metrizable space is a regular σ-space and each regular σ-space is a strong Σ-space.

2. $\delta\theta$-refinability

Definition 1. *([1]) A space X is called "θ-refinable" (resp. "$\delta\theta$-refinable") if every open cover $\mathcal{G}$ of X has a θ-sequence (resp. $\delta\theta$-sequence) $\langle\mathcal{H}_n\rangle_{n\in\omega}$ of X such that each $\mathcal{H}_n$ is an open cover of X and $\mathcal{H}_n \prec \mathcal{G}$. A sequence $\langle\mathcal{H}_n\rangle_{n\in\omega}$ of X is called a "θ-sequence" (resp. "$\delta\theta$-sequence") of X if for any $x \in X$ there is an $n_x \in \omega$ such that ord$(x, \mathcal{H}_{n_x}) < \omega$ (resp. ord$(x, \mathcal{H}_{n_x}) \le \omega$) where ord$(x, \mathcal{H}_{n_x}) = |\{H|x \in H \in \mathcal{H}_{n_x}\}|$. Here $|A|$ denotes the cardinal number of a set A.*

For collections $\mathcal{G}$ and $\mathcal{H}$ of subsets in X, $\mathcal{H} \prec \mathcal{G}$ denotes $\mathcal{H}$ is a refinement or a partial refinement of $\mathcal{G}$.

Theorem 1. *Let X be a zero-dimensional metric space and Y be a $\delta\theta$-refinable P-space. Then $X \times Y$ is $\delta\theta$-refinable.*

Proof. Let $\mathcal{B}$ be a base of X satisfying the conditions in Lemma 1. Let $\mathcal{G} = \{G_\xi|\xi \in \Xi\}$ be an open cover of $X \times Y$. For each $\sigma \in \Omega^{<\omega}$ and each $\xi \in \Xi$, let us define
$V_{\sigma,\xi} = \bigcup\{V|V \text{ is an open set in } Y, B(\sigma) \times V \subset G_\xi\}$. Then
(1) $V_{\sigma,\xi}$ is an open set in Y,
(2) $B(\sigma) \times V_{\sigma,\xi} \subset G_\xi$.

For each $\sigma \in \Omega^{<\omega}$, put $V(\sigma) = \bigcup_{\xi\in\Xi} V_{\sigma,\xi}$. Then
(3) Let $\sigma \in \Omega^\omega$. If $\{B(\sigma \restriction n)|n \in \omega\}$ is a local base of a point x, then $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$.
(4) $V(\sigma) \subset V(\sigma \vee \alpha)$ for each $\sigma \in \Omega^{<\omega}$ and each $\alpha \in \Omega$.

Since Y is a P-space, there is a closed cover $\{K(\sigma)|\sigma \in \Omega^{<\omega}\}$ of Y such that
(5) $K(\sigma) \subset V(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,

(6) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} V(\sigma \upharpoonright n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \upharpoonright n) = Y$.

Put $M_n = \bigcup\{B(\sigma) \times K(\sigma)|\sigma \in \Omega^n\}$. Then we have
(7) $X \times Y = \bigcup_{n\in\omega} M_n$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{V}_\sigma = \{V_{\lambda,\xi}|\xi \in \Xi\}$ is a collection of open sets of Y, cover $K(\sigma)$ and $\mathcal{V}_\sigma' = \mathcal{V}_\sigma \cup \{X \setminus K(\sigma)\}$ is an open cover of Y.

Since Y is $\delta\theta$-refinable, there is a sequence $\langle \mathcal{O}_{\sigma,m}' \rangle_{m\in\omega}$ of open refinements of $\mathcal{V}_\sigma'$ such that for each $y \in Y$, there is an m_y with $\mathrm{ord}(y, \mathcal{O}_{\sigma,m_y}') \leq \omega$. Put $\mathcal{O}_{\sigma,m} = \{O \in \mathcal{O}_{\sigma,m}'|O \cap K(\sigma) \neq \emptyset\}$. Then $\mathcal{O}_{\sigma,m}$ are collections of open sets in Y, covers $K(\sigma)$, $\mathcal{O}_{\sigma,m} \prec \mathcal{V}_\sigma$ and

(8) for each $y \in Y$, there is an m_y with $\mathrm{ord}(y, \mathcal{O}_{\sigma,m_y}) \leq \omega$.

Put $\mathcal{L}(\sigma;m) = \{B(\sigma) \times O|O \in \mathcal{O}_{\sigma,m}\}$ and

$\mathcal{L}_{n,m} = \bigcup_{\sigma\in\Omega^n} \mathcal{L}(\sigma;m) \cup \{(X \times Y \setminus M_n) \cap G_\xi|\xi \in \Xi\}$. Then

(9) $\mathcal{L}_{n,m}$ is an open cover of $X \times Y$ and a refinement of $\mathcal{G}$.
(10) $\langle \mathcal{L}_{n,m} \rangle_{n,m\in\omega}$ is a $\delta\theta$-sequence.

Proof. Let $(x,y) \in X \times Y$. Then $(x,y) \in M_n$ for some n. Since $\mathcal{B}_n$ is discrete, there is the only element $\sigma \in \Omega^n$ such that $x \in B(\sigma)$. Then, there is an m such that $\mathrm{ord}(y, \mathcal{O}_{\sigma,m}') \leq \omega$. It is easy to see that $\mathrm{ord}((x,y), \mathcal{L}_{n,m}) \leq \omega$. □

Lemma 4. *Suppose $X = \bigcup_{n\in\omega} F_n$, each F_n is closed in X and is $\delta\theta$-refinable. Then X is $\delta\theta$-refinable.*

Theorem 2. *Let X be a union of countable number of zero-dimensional metrizable closed subspaces and Y be a $\delta\theta$-refinable P-space. Then $X \times Y$ is $\delta\theta$-refinable.*

Proof. This theorem follows from Theorem 2 by using Lemma 4. □

A space X is said to be *strongly $\delta\theta$-refinable* if for any open cover $\mathcal{G}$ of X there is a sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ of open refinements of $\mathcal{G}$ such that for each $x \in X$, there is an n_x with $\mathrm{ord}(x, \mathcal{H}_n) \leq \omega$ for every $n \geq n_x$. Such a sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ is said to be a *strong $\delta\theta$-sequence.*

A space X is called *almost expandable* if for every locally finite collection $\{F_\xi|\xi \in \Xi\}$ of closed subsets of X, there exists a point finite collection $\{G_\xi|\xi \in \Xi\}$ of open subsets of X such that $F_\xi \subset G_\xi$ for each ξ.

Theorem 3. *Let X be an almost expandable strong Σ-space and Y be a strong $\delta\theta$-refinable P-space. Then $X \times Y$ is $\delta\theta$-refinable.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a spectral Σ-net of X, i.e., for some set Ω, $\mathcal{F}_n = \{F(\sigma)|\sigma \in \Omega^n\}$ be a locally finite closed cover of X for each $n \in \omega$ satisfying the conditions in Lemma 3.

Since X is almost expandable, there is a point finite open cover $\mathcal{H}_n = \{H(\sigma)|\sigma \in \Omega^n\}$ of X such that $H(\sigma) \supset F(\sigma)$ for each $\sigma \in \Omega^n$.

Let $\mathcal{G} = \{G_\xi|\xi \in \Xi\}$ be an open cover of $X \times Y$. For each $\sigma \in \Omega^{<\omega}$, let $\mathcal{W}_\sigma$ be the maximal family of $U_\lambda \times V_\lambda$ satisfying the following conditions:
(1) U_λ is an open set in X, $H(\sigma) \supset U_\lambda \supset F(\sigma)$,
(2) V_λ is an open set in Y,
(3) there is a finite open cover $\mathcal{U}_{\sigma,\lambda}$ of U_λ such that $\{U \times V_\lambda|U \in \mathcal{U}_{\sigma,\lambda}\} \prec \mathcal{G}$.

Put $\mathcal{W}_\sigma = \{U_\lambda \times V_\lambda|\lambda \in \Lambda_\sigma\}$.
For each $\sigma \in \Omega^{<\omega}$, put $V(\sigma) = \bigcup_{\lambda\in\Lambda_\sigma} V_\lambda$. Then

(4) Let $\sigma \in \Omega^\omega$. If $\{F(\sigma \upharpoonright n)|n \in \omega\}$ is a K-net of $C(x)$, then $\bigcup_{n\in\omega} V(\sigma \upharpoonright n) = Y$.

Proof. Let y be an arbitrary element of Y. Then, since $C(x)$ is compact, there is a finite set $\{U_i^{'}|i = 1, 2, ..., k\}$ of open sets in X and an open set V of Y such that $C(x) \subset \cup_{i=1}^k U_i^{'}, y \in V, \{U_i^{'} \times V|i = 1, 2, ..., k\} \prec \mathcal{G}$. Then there is an n such that $C(x) \subset F(\sigma \restriction n) \subset \cup_{i=1}^k U_i^{'}$. Put $U_i = U_i^{'} \cap H(\sigma \restriction n)$. Then $F(\sigma \restriction n) \subset \cup_{i=1}^k U_i$ and $\{U_i \times V|i = 1, 2, ..., k\} \prec \mathcal{G}$. By the definition of $V(\sigma \restriction n), V \subset V(\sigma \restriction n)$. Thus $y \in V(\sigma \restriction n)$.

(5) $V(\sigma) \subset V(\sigma \vee \alpha)$ for each $\sigma \in \Omega^{<\omega}$ and each $\alpha \in \Omega$.

Proof. Let $y \in V(\sigma)$. Then there are an open set U_λ in X, an open set V_λ in Y and a finite open cover $\mathcal{U}_{\sigma,\lambda}$ of U_λ such that $y \in V_\lambda$, $F(\sigma) \subset U_\lambda \subset H(\sigma)$ and $\{U \times V_\lambda|U \in \mathcal{U}_{\sigma,\lambda}\} \prec \mathcal{G}$. Put $U_\lambda^{'} = U_\lambda \cap H(\sigma \vee \alpha)$ and $\mathcal{U}_{\sigma,\lambda}^{'} = \{U \cap H(\sigma \vee \alpha)|U \in \mathcal{U}_{\sigma,\lambda}\}$. Then $U_\lambda^{'}$ is an open set in X, V_λ is an open set in Y, $\mathcal{U}_{\sigma,\lambda}^{'}$ is a finite open cover of $U_\lambda^{'}$, $F(\sigma \vee \alpha) \subset U_\lambda^{'} \subset H(\sigma \vee \alpha)$ and $\mathcal{U}_{\sigma,\lambda}^{'} \prec \mathcal{G}$. By the maximality of $\mathcal{W}_{\sigma\vee\alpha}$, we have $U_\lambda^{'} \times V_\lambda \in \mathcal{W}_{\sigma\vee\alpha}$. Thus $V_\lambda \subset V(\sigma \vee \alpha)$ and so $y \in V(\sigma \vee \alpha)$.

Since Y is a P-space, there is a closed cover $\{K(\sigma)|\sigma \in \Omega^{<\omega}\}$ of Y such that
(6) $K(\sigma) \subset V(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,
(7) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$.

Then we have
(8) $X \times Y = \bigcup\{F(\sigma) \times K(\sigma)|\sigma \in \Omega^{<\omega}\}$.

Proof. Let (x, y) be an arbitrary element of $Y \times Y$ and let us choose $\sigma \in \Omega^\omega$ be $\{F(\sigma \restriction n)|n \in \omega\}$ is a K-net of $C(x)$. Then, by (4) and (7), $y \in K(\sigma \restriction n)$ for some n. Thus $(x, y) \in F(\sigma \restriction n) \times K(\sigma \restriction n)$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{V}_\sigma = \{V_\lambda|\lambda \in \Lambda_\sigma\}$ is a collection of open sets in Y, cover $K(\sigma)$ and $\mathcal{V}_\sigma^{'} = \mathcal{V}_\sigma \cup \{Y \setminus K(\sigma)\}$ is an open cover of Y.

Since Y is strongly $\delta\theta$-refinable, there is a sequence $\langle\mathcal{O}_{\sigma,m}^{'}\rangle_{m\in\omega}$ of open refinements of $\mathcal{V}_\sigma^{'}$ such that for each $y \in Y$, there is an m_y with $\text{ord}(y, \mathcal{O}_{\sigma,m}^{'}) \leq \omega$ for every $m \geq m_y$. Put $\mathcal{O}_{\sigma,m} = \{O \in \mathcal{O}_{\sigma,m}^{'}|O \cap K(\sigma) \neq \emptyset\}$. Then $\mathcal{O}_{\sigma,m}$ are collections of open sets in Y and covers $K(\sigma)$, $\mathcal{O}_{\sigma,m} \prec \mathcal{V}_\sigma$ and

(9) for each $y \in Y$, there is an m_y with $\text{ord}(y, \mathcal{O}_{\sigma,m}) \leq \omega$ for every $m \geq m_y$.

We can represent $\mathcal{O}_{\sigma,m} = \{O_{\sigma,m,\lambda}|\lambda \in \Lambda_\sigma\}$ with $O_{\sigma,m,\lambda} \subset V_\lambda$ for each λ.

Put $\mathcal{L}(\sigma; m) = \{U \times O_{\sigma,m,\lambda}|U \in \mathcal{U}_{\sigma,\lambda}, \lambda \in \Lambda_\sigma\}$ and

$\mathcal{L}_{n,m} = \bigcup_{\sigma\in\Omega^n} \mathcal{L}(\sigma; m) \cup \{(X \times Y \setminus M_n) \cap G_\xi|\xi \in \Xi\}$ where $M_n = \bigcup\{F(\sigma) \times K(\sigma)|\sigma \in \Omega^n\}$. Then
(10) $\mathcal{L}_{n,m}$ is an open cover of $X \times Y$ and a refinement of $\mathcal{G}$.
(11) $\langle\mathcal{L}_{n,m}\rangle_{n,m\in\omega}$ is a $\delta\theta$-sequence.

Proof of (10). Let $(x, y) \in X \times Y$. If $(x, y) \notin M_n$, then $(x, y) \in (X \times Y \setminus M_n) \cap G_\xi$ for some $\xi \in \Xi$. If $(x, y) \in M_n$, then $(x, y) \in F(\sigma) \times K(\sigma)$ for some $\sigma \in \Omega^n$. Since $\bigcup \mathcal{O}_{\sigma,m} \supset K(\sigma)$, $y \in O_{\sigma,m,\lambda}$ for some $\lambda \in \Lambda_\sigma$. Since $F(\sigma) \subset U_\lambda = \bigcup\mathcal{U}_{\sigma,\lambda}$, $x \in U$ for some $U \in \mathcal{U}_{\sigma,\lambda}$. Thus $(x, y) \in U \times O_{\sigma,m,\lambda}$. Hence $\mathcal{L}_{n,m}$ is a cover of $X \times Y$.

By (3), $\mathcal{L}(\sigma; m) \prec \mathcal{G}$ and therefore $\mathcal{L}_{n,m} \prec \mathcal{G}$.

Proof of (11). Let $(x, y) \in X \times Y$. Then there is an n such that $(x, y) \in M_n$. Since $\mathcal{H}_n$ is point finite, there is a finite set $\{\sigma_i|i = 1, 2, ..., k\}$ of Ω^n such that $x \in H(\sigma) \iff \sigma \in \{\sigma_i|i = 1, 2, ..., k\}$. For each i, there is an $m(i) \in \omega$ such that $\text{ord}(y, \mathcal{O}_{\sigma,m}) \leq \omega$ for each $m \geq m(i)$. Put $m^* = \max\{m(i)|i = 1, 2, ..., k\}$. Then $\text{ord}(y, \mathcal{O}_{\sigma,m}) \leq \omega$ for each $m \geq m^*$ and each $i \leq k$. Then $\text{ord}((x, y), \mathcal{L}_{n,m}) \leq \omega$ for each $m \geq m^*$.

To show this, let $m \geq m^*$ and let $(x, y) \in U \times O_{\sigma,m,\lambda} \in \mathcal{L}(\sigma; m)$ with $U \in \mathcal{U}_{\sigma,\lambda}$, $\sigma \in \Omega^n$. Then $U \subset U_\lambda \subset H(\sigma)$. Therefore $x \in H(\sigma)$ and so $\sigma \in \{\sigma_i | i = 1, 2, ..., k\}$. Since $y \in O_{\sigma_i,m,\lambda}$ and $\mathrm{ord}(y, \mathcal{O}_{\sigma_i,m}) \leq \omega$, such λ are at most countably many number. □

3. $\delta\theta$-REFINABILITY-LIKE PROPERTIES

In this section we investigate $\delta\theta$-refinability-like properties.

Definition 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. $\mathcal{V}$ is called a pointwise W-refinement of $\mathcal{U}$ at x if there is a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{V}(x) \prec \mathcal{U}'$. Here $\mathcal{V}(x) = \{V \in \mathcal{V} | x \in V\}$.*

Let $\mathcal{U}$ be a cover of X and $\langle \mathcal{V}_n \rangle_{n\in\omega}$ a sequence of covers of X. A sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ is called a pointwise W-refining sequence for $\mathcal{U}$ if for each x, there exists an n_x such that $\mathcal{V}_{n_x}$ is a pointwise W-refinement of $\mathcal{U}$ at x.

The next characterization of θ-refinable spaces was given by J. M. W. Worrell.

Theorem B ([13], or cf. [14, Theorem 3.4]). A space X is θ-refinable (submetacompact) if and only if every open cover of X has a pointwise W-refining sequence by open covers if and only if every open cover of X has a pointwise W-refining sequence by semi-open covers.

A cover $\mathcal{G}$ of X is said to be a semi-open cover of X if for each $x \in X, x \in \mathrm{Int}(\mathrm{st}(x, \mathcal{G}))$ where $\mathrm{st}(x, \mathcal{L}) = \bigcup\{L | x \in L \in \mathcal{L}\}$ and $\mathrm{Int}(\mathrm{st}(x, \mathcal{G}))$ denotes the interior of $\mathrm{st}(x, \mathcal{G})$.

Definition 3. *([7]). Let $\mathcal{L}$ and $\mathcal{G}$ be covers of X. $\mathcal{L}$ is called "point-star $\dot{F}$-refinement" of $\mathcal{G}$ at $x \in X$ if there is a finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $x \in \bigcap \mathcal{G}'$ and st$(x, \mathcal{L}) \subset \bigcup \mathcal{G}'$.*

A sequence $\langle \mathcal{L}_n \rangle_{n\in\omega}$ of covers of X is called "point-star $\dot{F}$-refining sequence" of $\mathcal{G}$ if for each $x \in X$, there is an $n_x \in \omega$ such that $\mathcal{L}_{n_x}$ is a point-star $\dot{F}$-refinement of $\mathcal{G}$ at x.

H. J. K. Junnila gave the next characterization of submetacompactness.

Theorem C ([7]). A space X is θ-refinable (submetacompact) if and only if every open cover of X has a point star $\dot{F}$-refining sequence by open covers if and only if every open cover of X has a point star $\dot{F}$-refining sequence by semi-open covers.

In [5], w-$\delta\theta$-refinability and ww-$\delta\theta$-refinability were defined. Now we shall define s-w-$\delta\theta$-refinability and s-ww-$\delta\theta$-refinability.

Definition 4. *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X. $\mathcal{V}$ is called a pointwise countable W-refinement of $\mathcal{U}$ at x if there is a countable subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $\mathcal{V}(x) \prec \mathcal{U}'$.*

Let $\mathcal{U}$ be a cover of X and $\langle \mathcal{V}_n \rangle_{n\in\omega}$ a sequence of covers of X. A sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ is called a pointwise countable W-refining sequence for $\mathcal{U}$ if for each x, there exists an n_x such that $\mathcal{V}_{n_x}$ is a pointwise countable W-refinement of $\mathcal{U}$ at x.

We shall say a space X is w-$\delta\theta$-refinable (resp. s-w-$\delta\theta$-refinable) if every open cover of X has a pointwise countable W-refining sequence by open covers (resp. semi-open covers).

Definition 5. *Let $\mathcal{L}$ and $\mathcal{G}$ are covers of X. $\mathcal{L}$ is called "point-star $\dot{C}$-refinement" of $\mathcal{G}$ at $x \in X$ if there is a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $x \in \bigcap \mathcal{G}'$ and st$(x, \mathcal{L}) \subset \bigcup \mathcal{G}'$.*

A sequence $\langle \mathcal{L}_n \rangle_{n\in\omega}$ of covers of X is called "point-star $\dot{C}$-refining sequence" of $\mathcal{G}$ if for each $x \in X$, there is an $n_x \in \omega$ such that $\mathcal{L}_{n_x}$ is a point-star $\dot{C}$-refinement of $\mathcal{G}$ at x.

We shall say a space X is ww-$\delta\theta$-refinable (resp. s-ww-$\delta\theta$-refinable) if every open cover of X has a point star $\dot{C}$-refining sequence by open covers (resp. semi-open covers).

If we define w-θ-refinability, s-w-θ-refinability, ww-θ-refinability and s-ww-θ-refinability similarly. Then Theorems B denotes that a space X is θ-refinable if and only if it is w-θ-refinable if and only if it is s-w-θ-refinable. And Theorem C denotes that a space X is θ-refinable if and only if it is ww-θ-refinable if and only if it is s-ww-θ-refinable.

The following lemma essentially has proved in [5].

Lemma 5. *(1) Let $\mathcal{G}$ be an open cover of X and $\langle \mathcal{V}_n \rangle_{n\in\omega}$ be a pointwise countable W-refining sequence of $\mathcal{G}$. Then there exists a pointwise countable W-refining sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ of $\mathcal{G}$ satisfying the following conditions (a) For each $x \in X$, there exist an $n_x \in \omega$ and a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{H}_n(x) \prec \mathcal{G}'$ for every $n \geq n_x$.*

Here, (b) We can choose $\mathcal{H}_n \prec \mathcal{G}$,
(c) If each $\mathcal{V}_n$ is an open cover, then each $\mathcal{H}_n$ can be an open cover and if each $\mathcal{V}_n$ is a semi-open cover, then each $\mathcal{H}_n$ can be a semi-open cover.
(2) Let $\mathcal{G}$ be an open cover of X and $\langle \mathcal{V}_n \rangle_{n\in\omega}$ be a point-star $\dot{C}$-refining sequence of $\mathcal{G}$. Then there exists a point-star $\dot{C}$-refining sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ of $\mathcal{G}$ satisfying the following conditions (a) For each $x \in X$, there exist an $n_x \in \omega$ and a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ such that $x \in \bigcap \mathcal{G}'$ and $st(x, \mathcal{H}_n) \subset \bigcup \mathcal{G}'$ for every $n \geq n_x$.

Here, (b) We can choose $\mathcal{H}_n \prec \mathcal{G}$,
(c) If each $\mathcal{V}_n$ is an open cover, then each $\mathcal{H}_n$ can be an open cover and if each $\mathcal{V}_n$ is a semi-open cover, then each $\mathcal{H}_n$ can be a semi-open cover.

Proof. Let us put $\mathcal{H}_n = \wedge_{i=0}^{n} \mathcal{V}_i \wedge \mathcal{G} (= \{\cap_{i=0}^{n} V_i \cap G | V_i \in \mathcal{V}_i \text{ for each } i = 0, 1, ..., n, G \in \mathcal{G}\})$. Then $\langle \mathcal{H}_n \rangle_{n\in\omega}$ satisfies the conditions (a), (b) and (c).

We prove only (1).

Proof of (a). Let $x \in X$. Then there are an $n(i) \in \omega$ and a countable subfamily $\mathcal{G}_i$ of $\mathcal{G}$ such that $\mathcal{V}_{n_i}(x) \prec \mathcal{G}_i$.

Define $n_x = \max\{n(i) | i \leq n\}$ and $\mathcal{G}' = \bigcup_{i\leq n} \mathcal{G}_i$. Then it is easy to see that $\mathcal{H}_n(x) \prec \mathcal{G}'$ for every $n \geq n_x$.

(b) is ovbious.

Proof of (c). Suppose each $\mathcal{V}_n$ is a semi-open cover of X. Let $x \in \mathrm{Int}(\mathrm{st}(x, \mathcal{V}_i))$ for each $i \leq n$. Thus there are open sets W_i in X such that $x \in W_i \subset \mathrm{st}(x, \mathcal{V}_i)$ for each $i \leq n$. Let $G \in \mathcal{G}$ so that $x \in G$. Then $W = G \cap \cap_{i\leq n} W_i$ is a neighborhood of x in X such that $W \subset \mathrm{st}(x, \mathcal{H}_n)$. □

Lemma 6. *If X is a metrizable space, then for each $n \in \omega$, there are locally finite open covers $\mathcal{H}_n$ and $\mathcal{B}_n$ of X satisfying the following conditions:*

(1) $\mathcal{H}_n = \{H(\sigma) | \sigma \in \Omega^n\}$, $\mathcal{B}_n = \{B(\sigma) | \sigma \in \Omega^n\}$,

(2) $\overline{B(\sigma)} \subset H(\sigma)$,

(3) $H(\sigma) = \bigcup_{\alpha\in\Omega} H(\sigma \vee \alpha)$, $B(\sigma) = \bigcup_{\alpha\in\Omega} B(\sigma \vee \alpha)$ for each $\sigma \in \Omega^n$,

(4) for each $x \in X$, there is a $\sigma \in \Omega^\omega$ such that $\{H(\sigma \upharpoonright n) | n \in \omega\}$ is a local base of x in X and $\{B(\sigma \upharpoonright n) | n \in \omega\}$ is a local base of x in X.

Proof. For each $n \in \omega$, let $\mathcal{U}_n = \{U(x; 1/2(n+1)) | x \in X\}$. Here $U(x; 1/2(n+1)) = \{y \in X | d(x, y) < 1/2(n+1)\}$ where d is a metric function on X. Then $\mathcal{U}_n$ is an open cover of X. Since X is paracompact, there are a locally finite open cover $\mathcal{U}_n' = \{U(\alpha) | \alpha \in \Omega_n\}$ of X such that $\mathcal{U}_n' \prec \mathcal{U}_n$ and a locally finite open cover $\mathcal{B}_n' = \{B(\alpha)' | \alpha \in \Omega_n\}$ of X such that $\overline{B(\alpha)'} \subset U(\alpha)$ for each $\alpha \in \Omega_n$. Put $\Omega = \bigcup_{n\in\omega} \Omega_n$ and define $U(\alpha) = \emptyset$ for each $\alpha \in \Omega \setminus \Omega_n$. For each $\sigma = (\alpha_0, ..., \alpha_{n-1}) \in \Omega^n$, put $H(\sigma) = \cap_{i=0}^{n-1} U(\alpha_i)$ and $B(\sigma) = \cap_{i=0}^{n-1} B(\alpha_i)'$. Then $\mathcal{H}_n = \{H(\sigma) | \sigma \in \Omega^n\}$ and $\mathcal{B}_n = \{B(\sigma) | \sigma \in \Omega^n\}$ satisfy the conditions.

(1), (2) and (3) are obvious.

To show (4), let $x \in X$. For each $n \in \omega$, let us choose $U(\alpha_n) \in \mathcal{U}_n'$ such that $x \in U(\alpha_n)$. Then $\mathrm{diam}(U(\alpha_n)) \leq 1/n+1$. Therefore $\{U(\alpha_n) | n \in \omega\}$ and $\{B(\alpha_n) | n \in \omega\}$ are local basis of $x \in X$. Thus $\{H(\sigma \upharpoonright n) | n \in \omega\}$ and $\{B(\sigma \upharpoonright n) | n \in \omega\}$ are local basis of $x \in X$ where $\sigma = (\alpha_0, ..., \alpha_{n-1}, ...) \in \Omega^\omega$. □

Theorem 4. *Let X be a metrizable space and Y be a P-space.*
(a) If Y is w-$\delta\theta$-refinable, then so is $X \times Y$.

(b) If Y is s-w-$\delta\theta$-refinable, then so is $X \times Y$.
(c) If Y is ww-$\delta\theta$-refinable, then so is $X \times Y$.
(d) If Y is s-ww-$\delta\theta$-refinable, then so is $X \times Y$.

Proof. Let $\mathcal{H}_n$ and $\mathcal{B}_n$ be open covers of X satisfying the conditions (1) ‘ (4) in Lemma 6. Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an open cover of $X \times Y$. For each $\sigma \in \Omega^{<\omega}$ and each $\xi \in \Xi$, let us define $V_{\sigma,\xi} = \bigcup\{V | V \text{ is an open set in } Y, H(\sigma) \times V \subset G_\xi\}$. Then
(5) $V_{\sigma,\xi}$ is an open set in Y,
(6) $H(\sigma) \times V_{\sigma,\xi} \subset G_\xi$.

For each $\sigma \in \Omega^{<\omega}$, put $V(\sigma) = \bigcup_{\xi\in\Xi} V_{\sigma,\xi}$. Then
(7) Let $\sigma \in \Omega^\omega$. If $\{H(\sigma \restriction n) | n \in \omega\}$ is a local base of a point x, then $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$.
(8) $V(\sigma) \subset V(\sigma \vee \alpha)$ for each $\sigma \in \Omega^{<\omega}$ and each $\alpha \in \Omega$.

Since Y is a P-space, there is a closed cover $\{K(\sigma) | \sigma \in \Omega^{<\omega}\}$ of Y such that
(9) $K(\sigma) \subset V(\sigma)$ for each $\sigma \in \Omega^{<\omega}$,
(10) for each $\sigma \in \Omega^\omega$, if $\bigcup_{n\in\omega} V(\sigma \restriction n) = Y$, then $\bigcup_{n\in\omega} K(\sigma \restriction n) = Y$.

Put $M_n = \bigcup\{\overline{B(\sigma)} \times K(\sigma) | \sigma \in \Omega^n\}$. Then M_n is a closed subset of $X \times Y$ and we have
(11) $X \times Y = \bigcup_{n\in\omega} M_n$.

For each $\sigma \in \Omega^{<\omega}$, $\mathcal{V}_\sigma = \{V_{\sigma,\xi} | \xi \in \Xi\}$ is a collection of open sets in Y, cover $K(\sigma)$ and $\mathcal{V}'_\sigma = \mathcal{V}_\sigma \cup \{Y \setminus K(\sigma)\}$ is an open cover of Y.

Cases (a) and (b). Since Y is w-$\delta\theta$-refinable (resp. s-w-$\delta\theta$-refinable), there is a sequence $\langle \mathcal{O}'_{\sigma,m} \rangle_{m\in\omega}$ of open covers (resp. semi-open covers) of Y such that
(12) for each $y \in Y$, there are an m_y and a countable subfamily $\mathcal{V}'_{(y)}$ of $\mathcal{V}'_\sigma$ such that $(\mathcal{O}'_{\sigma,m})(y) \prec \mathcal{V}'_{(y)}$ for every $m \geq m_y$. Here, if $y \in K(\sigma)$, then we can choose $\mathcal{V}'_{(y)} \subset \mathcal{V}_\sigma$.

Put $\mathcal{O}_{\sigma,m} = \{O \in \mathcal{O}'_{\sigma,m} | O \cap K(\sigma) \neq \emptyset\}$. Then $(\mathcal{O}_{\sigma,m})(y) \prec \mathcal{V}'_{(y)}$ for every $m \geq m_y$. By Lemma 5, we may assume that $\mathcal{O}_{\sigma,m} \prec \mathcal{V}_\sigma$.

Cases (c) and (d). Since Y is ww-$\delta\theta$-refinable (resp. s-ww-$\delta\theta$-refinable), there is a sequence $\langle \mathcal{O}'_{\sigma,m} \rangle_{m\in\omega}$ of open covers (resp. semi-open covers) of Y such that for each $y \in Y$, there are an m_y and a countable subfamily $\mathcal{V}'_{(y)}$ of $\mathcal{V}'_\sigma$ such that $y \in \bigcap \mathcal{V}'_{(y)}$ and $\mathrm{st}(y, \mathcal{O}'_{\sigma,m}) \subset \bigcup \mathcal{V}'_{(y)}$ for every $m \geq m_y$.

Put $\mathcal{O}_{\sigma,m} = \{O \in \mathcal{O}'_{\sigma,m} | O \cap K(\sigma) \neq \emptyset\}$. Then $\mathcal{O}_{\sigma,m}$ are collections of open sets in Y and covers $K(\sigma)$, if $y \in K(\sigma)$, then we can choose $\mathcal{V}'_{(y)} \subset \mathcal{V}_\sigma$. Here we may assume that $\mathcal{O}_{\sigma,m} \prec \mathcal{V}_\sigma$. Put

$\mathcal{L}(\sigma; m) = \{H(\sigma) \times O | O \in \mathcal{O}_{\sigma,m}\}$ and

$\mathcal{L}_{n,m} = \bigcup_{\sigma\in\Omega^n} \mathcal{L}(\sigma; m) \cup \{(X \times Y \setminus M_n) \cap G_\xi | \xi \in \Xi\}$. Then

(13) $\mathcal{L}_{n,m}$ is an open cover (resp. semi-open cover) of $X \times Y$ and a refinement of $\mathcal{G}$.
and
(14) *Case (a).* $\langle \mathcal{L}_{n,m} \rangle_{n,m\in\omega}$ is a w-$\delta\theta$-sequences.
Case (b). $\langle \mathcal{L}_{n,m} \rangle_{n,m\in\omega}$ is a s-w-$\delta\theta$-sequences.
Case (c). $\langle \mathcal{L}_{n,m} \rangle_{n,m\in\omega}$ is a ww-$\delta\theta$-sequences.
Case (d). $\langle \mathcal{L}_{n,m} \rangle_{n,m\in\omega}$ is a s-ww-$\delta\theta$-sequences.

Proof of (13). We prove only case (c). To show that $\mathcal{L}_{n,m}$ is a semi-open cover, let $(x, y) \in X \times Y$. It is ovbious that $(x, y) \in \mathrm{Int}(\mathrm{st}((x, y), \mathcal{L}_{n,m}))$ if $(x, y) \notin M_n$. We consider the case of $(x, y) \in M_n$. Then $(x, y) \in H(\sigma) \times K(\sigma)$ for some $\sigma \in \Omega^n$. Since $\mathcal{O}'_{\sigma,m}$ is a semi-open cover, $y \in \mathrm{Int}(\mathrm{st}(y, \mathcal{O}'_{\sigma,m}))$. Therefore there is an open neighborhood W of y

in Y such that $W \subset \mathrm{st}(y, \mathcal{O}'_{\sigma,m})$. Since $y \in K(\sigma)$, $\mathrm{st}(y, \mathcal{O}'_{\sigma,m}) = \mathrm{st}(y, \mathcal{O}_{\sigma,m})$. Therefore, $H(\sigma) \times W \subset H(\sigma) \times \mathrm{st}(y, \mathcal{O}_{\sigma,m}) \subset \mathrm{st}((x,y), \mathcal{L}_{n,m}))$. Thus $(x,y) \in \mathrm{Int}(\mathrm{st}((x,y)), \mathcal{L}_{n,m}))$. Since $\bigcup \mathcal{L}(\sigma; m) \supset H(\sigma) \times K(\sigma) \supset \overline{B(\sigma)} \times K(\sigma)$, $\mathcal{L}_{n,m}$ is a cover of $X \times Y$. For each $L \in \mathcal{L}_{n,m}$, $L = H(\sigma) \times O$ for some $\sigma \in \Omega^n$ and some $O \in \mathcal{O}_{\sigma,m}$. Since $O \subset V$ for some $V \in \mathcal{V}'_\sigma$ and $O \cap K(\sigma) \neq \emptyset$, $V \neq Y \setminus K(\sigma)$. Thus $V = V_{\sigma,\xi}$ for some $\xi \in \Xi$. Therefore $H(\sigma) \times O \subset H(\sigma) \times V_{\sigma,\xi} \subset G_\xi$ for some $\xi \in \Xi$.

Proof of (14). Let $(x,y) \in X \times Y$. Then there is an n such that $(x,y) \in M_n$. Since $\mathcal{H}_n$ is point finite, there is a finite set $\{\sigma_i | i = 0,1,...,k-1\}$ of Ω^n such that $x \in H(\sigma) \iff \sigma \in \{\sigma_i | i = 0,1,...,k-1\}$.

Cases (a) and (b). For each i, there are an $m(i) \in \omega$ and a countable subfamily $\mathcal{V}'_{(i)}$ of $\mathcal{V}'_{\sigma_i}$ satisfying the condition $(\mathcal{O}'_{\sigma_i,m})(y) \prec \mathcal{V}'_{(i)}$ for each $m \geq m_i$.

Cases (c) and (d). For each i, there are an $m(i) \in \omega$ and a countable subfamily $\mathcal{V}'_i$ of $\mathcal{V}'_{\sigma_i}$ satisfying the conditions: $y \in \bigcap \mathcal{V}'_{(i)}$ and $\mathrm{st}(y, \mathcal{O}'_{\sigma_i,m}) \subset \bigcup \mathcal{V}'_{(i)}$ for every $m \geq m_i$.

Put $\mathcal{V}''_{(i)} = \mathcal{V}'_{(i)} \setminus \{Y \setminus K(\sigma_i)\}$.

Let m be an arbitrary element of ω such that $m \geq \max\{m_i | i = 0,1,..,k-1\}$. Put $\mathcal{L}' = \{H(\sigma_i) \times V | V \in \mathcal{V}''_i, i = 0,1,..,k-1\}$. Then $\mathcal{L}'$ is a countable family and $\mathcal{L}' \prec \mathcal{G}$. For each $L \in \mathcal{L}'$, let us choose $G(L) \in \mathcal{G}$ such that $L \subset G(L)$ and put $\mathcal{G}' = \{G(L) | L \in \mathcal{L}'\}$. Then $\mathcal{G}'$ is a countable subfamily of $\mathcal{G}$ and

Cases (a) and (b). $\mathcal{L}_{n,m}((x,y)) \prec \mathcal{G}'$.

Cases (c) and (d). $(x,y) \in \bigcap \mathcal{G}'$ and $\mathrm{st}((x,y), \mathcal{L}_{n,m}) \subset \bigcup \mathcal{G}'$.

Proof of cases (a) and (b). Let $(x,y) \in L \in \mathcal{L}_{n,m}$. Since $(x,y) \in M_n$, $L = H(\sigma) \times O$ for some $O \in \mathcal{O}_{\sigma,m}$. Since $O \in \mathcal{O}_{\sigma,m}(y)$, $O \subset V$ for some $V \in \mathcal{V}'_i$. Since $O \cap K(\sigma) \neq \emptyset$, $V \in \mathcal{V}''_i$. Since $x \in H(\sigma), \sigma = \sigma_i$ for some i. Hence $L \subset H(\sigma_i) \times V(\mathrm{put} = L') \subset G(L') \in \mathcal{G}'$.

Proof of cases (c) and (d). It is obvious that $(x,y) \in L$ for each $L \in \mathcal{L}'$ and since $L \subset G(L), (x,y) \in G(L)$ for each $G(L) \in \mathcal{G}'$. The proof of $\mathrm{st}((x,y), \mathcal{L}_{n,m}) \subset \bigcup \mathcal{G}'$ is similar to that of $\mathcal{L}_{n,m}((x,y)) \prec \mathcal{G}'$. $\square$

4. Pseudo-open maps and $\delta\theta$-refinability

It is obvious that every $\delta\theta$-refinable space is w-$\delta\theta$-refinable, every w-$\delta\theta$-refinable space is ww-$\delta\theta$-refinable and s-w-$\delta\theta$-refinable, every ww-$\delta\theta$-refinable space is s-ww-$\delta\theta$-refinable and every s-w-$\delta\theta$-refinable space is s-ww-$\delta\theta$-refinable.

However, the converse is not known.

Let $L(X)$ denote the *Lindelöf number* of a space X, i.e., $L(X) = \min\{\kappa \geq \omega \mid$ each open cover $\mathcal{G}$ of X has a subcover $\mathcal{G}'$ with $|\mathcal{G}'| \leq \kappa\}$.

In [4], K. Chiba proved the following.

Theorem D ([4]). Let X be a space with $L(X) \leq \omega_1$. Then the following are equivalent.
(1) X is $\delta\theta$-refinable.
(2) X is w-$\delta\theta$-refinable.
(3) X is ww-$\delta\theta$-refinable.

Now we give the following.

Theorem 5. *Let X be a space with $L(X) \leq \omega_1$. Then the following are equivalent.*
(1) X is $\delta\theta$-refinable.
(2) X is w-$\delta\theta$-refinable.

(3) X is ww-$\delta\theta$-refinable.
(4) X is s-w-$\delta\theta$-refinable.
(5) X is s-ww-$\delta\theta$-refinable.

Proof. This proof is similar to that of Theorem D. It is sufficient to prove that (5) $\Rightarrow$ (1). Let $\mathcal{U}$ be an open cover of X. We may assume that $\mathcal{U} = \{U_\alpha | \alpha < \omega_1\}$. By assumption, there exists a sequence $\langle \mathcal{L}_k \rangle_{k\in\omega}$ of point star $\dot{C}$-refining sequence by semi-open covers of X.

For each $k \in \omega$ and each $\alpha < \omega_1$, define
$V_{k,\alpha} = U_\alpha \cap (\mathrm{Int}(\mathrm{st}(X \setminus \cup_{\beta\neq\alpha} U_\beta, \mathcal{L}_k))$,
$V'_{k,\alpha} = U_\alpha \cap (\cup_{\beta>\alpha} U_\beta) \cap (\mathrm{Int}(\mathrm{st}(X \setminus \cup_{\beta<\alpha} U_\beta, \mathcal{L}_k)))$ and put
$\mathcal{V}_k = \{V_{k,\alpha} | \alpha < \omega_1\} \cup \{V'_{k,\alpha} | \alpha < \omega_1\}$.
Then

(i) $\mathcal{V}_k$ is an open cover of X such that $\mathcal{V}_k \prec \mathcal{U}$.

Proof. It is obvious that each set of $\mathcal{V}_k$ is an open set and $\mathcal{V}_k \prec \mathcal{U}$. To prove that $\mathcal{V}_k$ is a cover of X, let $x \in X$. Put $\alpha = \min\{\beta < \omega_1 | x \in U_\beta\}$. Then $x \in U_\alpha \setminus \cup_{\beta<\alpha} U_\beta$. Since $\mathcal{L}_k$ is semi-open cover of X, $x \in \mathrm{Int}(\mathrm{st}(x, \mathcal{L}_k)) \subset \mathrm{Int}(\mathrm{st}(X \setminus \cup_{\beta<\alpha} U_\beta, \mathcal{L}_k)))$. If $x \notin V'_{k,\alpha}$, then $x \notin \cup_{\beta>\alpha} U_\beta$ and thus $x \in X \setminus \cup_{\beta\neq\alpha} U_\beta$. Since $\mathcal{L}_k$ is semi-open, $x \in \mathrm{Int}(\mathrm{st}(x, \mathcal{L}_k)) \subset \mathrm{Int}(\mathrm{st}(X \setminus \cup_{\beta<\alpha} U_\beta, \mathcal{L}_k))$. Hence $x \in V_{k,\alpha}$.

(ii) $\{\mathcal{V}_k | k \in \omega\}$ is a $\delta\theta$-sequence.

Proof. Let $x \in X$. Then there exist a $k \in \omega$ and a countable subset $\{\alpha_i | i = 1, 2, ...\} \subset \omega_1$ such that $x \in \cap_{i=1}^\infty U_{\alpha_i}$ and $\mathrm{st}(x, \mathcal{L}_k) \subset \cup_{i=1}^\infty U_{\alpha_i}$.

If $x \in V_{k,\alpha}$, then there is an $L \in \mathcal{L}_k$ such that $x \in L$ and $L \cap (X \setminus \cup_{\beta\neq\alpha} U_\beta) \neq \emptyset$. Since $L \subset \cup_{i=1}^\infty U_{\alpha_i}$, $\alpha = \alpha_i$ for some i. Therefore $\{\alpha < \omega_1 | x \in V_{k,\alpha}\} \subset \{\alpha_i | i = 1, 2, ...\}$. Put $\alpha^* = \sup\{\alpha_i | i = 1, 2, ...\}$. Then $\{\alpha < \omega_1 | x \in V'_{k,\alpha}\} \subset \{\alpha | \alpha \leq \alpha^*\}$. To show this, let $\alpha > \alpha^*$. If $x \in L \in \mathcal{L}_k$, then $L \subset \cup_{\beta<\alpha} U_\beta$. Thus $x \notin V'_{k,\alpha}$. Hence $\mathrm{ord}(x, \mathcal{V}_k) \leq \omega$. □

A surjective map $f : X \to Y$ is called *pseudo-open* if for any $y \in Y$ and any open set U in X such that $f^{-1}(y) \subset U$, $y \in \mathrm{Int} f(U)$. Here $\mathrm{Int} f(U)$ denotes the interior of $f(U)$.

Theorem 6. *Let X be a $\delta\theta$-refinable space. If there is a finite to one pseudo-open map f from X onto a space Y, then Y is s-ww- $\delta\theta$-refinable.*

Proof. Let $\mathcal{O}$ be an open cover of Y. Put $\mathcal{U} = \{f^{-1}(O) | O \in \mathcal{O}\}$. Then $\mathcal{U}$ is an open cover of X. Since X is $\delta\theta$-refinable, there is a $\delta\theta$-sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ of open refinements of $\mathcal{U}$. Put $\mathcal{W}_n = \wedge_{k=1}^n \mathcal{V}_k$ and $\mathcal{L}_n = f(\mathcal{W}_n) = \{f(W) | W \in \mathcal{W}_n\}$. Then

(1) $\mathcal{L}_n$ is a semi-open cover of Y and $\mathcal{L}_n \prec \mathcal{O}$.

(2) $\langle \mathcal{L}_n \rangle_n$ is a point $\dot{C}$-refining sequence.

Proof of (1). To show that $\mathcal{L}_n$ is semi-open, let $y \in Y$. Then $f^{-1}(y) \subset \bigcup\{W | W \in \mathcal{W}_n, W \cap f^{-1}(y) \neq \emptyset\}$. Since f is a pseudo-open map, $y \in \mathrm{Int} f(\bigcup\{W | W \in \mathcal{W}_n, W \cap f^{-1}(y) \neq \emptyset\}) = \mathrm{Int} \bigcup\{f(W) | W \in \mathcal{W}_n, y \in f(W)\} = \mathrm{Int}(\mathrm{st}(y, \mathcal{L}_n))$.

Proof of (2). For each $y \in Y$, put $f^{-1}(y) = \{x(i) | i = 1, 2..., j\}$. For each i, there are an $n(i)$ such that $(\mathcal{L}_i)(x(i))$ is countable. Put $\mathcal{V}' = \cup_{i=1}^j (\mathcal{V}_{n(i)})_{x(i)}$. Then $\mathcal{V}'$ is a countable subfamily of $\mathcal{V}$. For each $V \in \mathcal{V}'$, let us choose $O(V) \in \mathcal{O}$ such that $V \subset f^{-1}(O(V))$ and put $\mathcal{O}' = \{O(V) | V \in \mathcal{V}'\}$. Then $\mathcal{O}'$ is a countable subfamily of $\mathcal{O}$ and $y \in \bigcap \mathcal{O}'$. Let us put $m = \max\{n(i) | i = 1, 2, ..., j\}$. Then $\mathrm{st}(y, \mathcal{L}_m) \subset \bigcup \mathcal{O}'$. □

By Theorems 5 and 6, we obtain the following.

Theorem 7. *Let X be a $\delta\theta$-refinable space and Y be a space with $L(Y) \leq \omega_1$. If there is a finite to one pseudo-open map from X onto Y, then Y is $\delta\theta$-refinable.*

Remark. Lemma 1.3 in [10] is not correct. Lemma 1.3 is used only in Lemma 1.4 and Theorem 3.15 in [10]. However, these can be proved without Lemma 1.3.

Lemma 1.3. ([10]). Let $\{\mathcal{F}_i\}$ be a Σ-net of a space X. If for each i, $\mathcal{H}_i$ is a locally finite closed cover of X refining $\mathcal{F}_i$, then $\{\mathcal{H}_i\}$ is a Σ-net.

In fact, the following example exists.

Example. Let $X = (0, \infty)$ with the subspace topology of the Euclidean space $\mathbf{R}$. Let $\mathcal{F}_i = \{(0,1]\} \cup \{(0, \frac{j}{i}], [\frac{j}{i}, \frac{j+1}{i}]; j = 1, 2, ..., i-1\} \cup \{[n, n+\frac{j}{i}]; j = 1, 2, ..., i, n = 1, 2, ...\}$ for each $i \in \mathbf{N}$. Then $\mathcal{F}_i$ is a locally finite closed cover of X.

(1) $\{\mathcal{F}_i\}$ is a Σ-net of X.

Proof. Let $K_1 \supset K_2 \supset ...$ is a sequence of non-empty closed sets of X such that $K_i \subset C(x, \mathcal{F}_i)$ for some point x in X and for each i. Then there are some k and a closed interval $[a, b]$ such that $K_i \subset [a, b]$ for each $i \geq k$. Since $[a, b]$ is compact, $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$.

Let us put $\mathcal{H}_i = \{(0,1]\} \cup \{[n, n+\frac{j}{i}]; j = 1, 2, ..., i, n = 1, 2, ...\}$ for each $i \in \mathbf{N}$. Then $\mathcal{H}_i$ is a locally finite closed cover of X. However

(2) $\{\mathcal{H}_i\}$ is not a Σ-net of X.

Proof. Let $K_i = (0, \frac{1}{i+1}]$ for each $i \in \mathbf{N}$. Then K_i is a closed subset of X, $K_1 \supset K_2 \supset ... K_i \supset ...$ and $K_i \subset (0,1] = C(\frac{1}{2}, \mathcal{F}_i)$ for each i. But $\bigcap_{i=1}^{\infty} K_i = \emptyset$. □

5. Inverse limits

K. Chiba investigated the covering properties of inverse limits and proved the following.

Theorem E ([5]). Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose each projection π_α is a pseudo-open map and X is κ-paracompact where $|\Lambda| = \kappa$. Then
(a) If each X_α is w-$\delta\theta$-refinable, then so is X.
(b) If each X_α is ww-$\delta\theta$-refinable, then so is X.
(cf. [2], [5] or [6] for the definitions of inverse systems and their limits, projections.)

Let κ be an infinite cardinal. A space X is called κ-paracompact if every open cover $\mathcal{G}$ of X with $|\mathcal{G}| \leq \kappa$ has a locally finite open refinement.

In this paper we shall prove the similar result for s-w-$\delta\theta$-refinability and s-ww-$\delta\theta$-refinability.

Theorem 8. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose each projection π_α is a pseudo-open map and X is κ-paracompact where $|\Lambda| = \kappa$. Then*
(c) If each X_α is s-w-$\delta\theta$-refinable, then so is X.
(d) If each X_α is s-ww-$\delta\theta$-refinable, then so is X.

These proofs are quite similar to that of theorem D. But, for completeness, we shall give the proof only for part (c) briefly. (The proofs of (a) and (b) are similar. Therefore in [5] only proof of (b) was given)

Proof of part (c). Let $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of X. For each $\alpha \in \Lambda$ and $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α satisfying $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$ and put $U_\alpha = \cup_{\xi \in \Xi} U_{\alpha,\xi}$. Then $\{\pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\}$ is an open cover of X and there exists an open cover $\{W_\alpha | \alpha \in \Lambda\}$ of X such that
(1) $\overline{W_\alpha} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$, and (2) $W_\alpha \subset W_\beta$ if $\alpha \leq \beta$. (cf. [2]).

Let us put $T_\alpha = X_\alpha \setminus \mathrm{Int}\, \pi_\alpha(X \setminus \overline{W_\alpha})$. Then
(3) T_α is closed in X_α and $T_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

Put $C_\alpha = \text{Int } \pi_\alpha^{-1}(T_\alpha)$ for each $\alpha \in \Lambda$. Then

(4) $\mathcal{C} = \{C_\alpha | \alpha \in \Lambda\}$ is an open cover of X. (cf. [2] or [5]).

Since X is κ-paracompact and $|\mathcal{C}| = \kappa$, there is a locally finite open cover $\mathcal{O} = \{O_\alpha | \alpha \in \Lambda\}$ such that

(5) $O_\alpha \subset C_\alpha$ for each $\alpha \in \Lambda$.

Let us put $\mathcal{U}_\alpha = \{U_{\alpha,\xi} | \xi \in \Xi\}$ and $\mathcal{U}_\alpha^{'} = \mathcal{U}_\alpha \cup \{X_\alpha \setminus T_\alpha\}$. Then $\mathcal{U}_\alpha^{'}$ is an open cover of X_α. Since X_α is s-w-$\delta\theta$-refinable, by Lemma 5, we can choose a sequence $\langle \mathcal{L}_{\alpha,m}^{'} \rangle_{m\in\omega}$ of semi-open covers of X_α satisfying: for each $y \in X_\alpha$, there are an $m(y) \in \omega$ and a countable subfamily $\mathcal{U}_{\alpha,y}^{'}$ of $\mathcal{U}_\alpha^{'}$ such that $\mathcal{L}_{\alpha,m}^{'}(y) \prec \mathcal{U}_\alpha^{'}$ for every $m \geq m(y)$.

Put $\mathcal{L}_{\alpha,m} = \{L \cap U_\alpha | L \in \mathcal{L}_{\alpha,m}^{'}, L \cap T_\alpha \neq \emptyset\}$. Then we have

$(i)_\alpha$. $T_\alpha \subset \bigcup \mathcal{L}_{\alpha,m}$ for each m.

$(ii)_\alpha$. For each $y \in T_\alpha$, there are an $m(y) \in \omega$ and a countable subset $A(y)$ of Ξ such that $y \in \cap_{\xi \in A(y)} U_{\alpha,\xi}$ and $\mathcal{L}_{\alpha,m}(y) \prec \mathcal{U}_\alpha^{'}$ for each $m \geq m(y)$.

Put $\mathcal{L}_m = \{\pi_\alpha^{-1}(L) \cap O_\alpha | L \in \mathcal{L}_{\alpha,m}, \alpha \in \Lambda\}$ for each m. Then

(6) $\mathcal{L}_m$ is a semi-open cover of X for each m.

Proof. Let $x = (x_\alpha)_{\alpha\in\Lambda} \in X$. Then $x \in O_\alpha$ for some α. Since $x_\alpha \in T_\alpha$, by (5), $(i)_\alpha$, $x_\alpha \in L$ for some $L \in \mathcal{L}_{\alpha,m}$. Thus $x \in \pi_\alpha^{-1}(L) \cap O_\alpha \in \mathcal{L}_m$. Thus $\mathcal{L}_m$ is a cover of X.

Since $\mathcal{L}_{\alpha,m}^{'}$ is a semi-open cover of X_α, $x_\alpha \in \text{Int}(\text{st}(x_\alpha, \mathcal{L}_{\alpha,m}^{'}))$. Therefore there is an open set N of X_α such that $x_\alpha \in N \subset \text{Int}(\text{st}(x_\alpha, \mathcal{L}_{\alpha,m}^{'}))$. Then $N \cap U_\alpha$ is a neighborhood of x_α and $N \cap U_\alpha \subset \text{st}(x_\alpha, \mathcal{L}_{\alpha,m}^{'}) \cap U_\alpha = \text{st}(x_\alpha, \mathcal{L}_{\alpha,m})$. Then

$\pi_\alpha^{-1}(N) \cap O_\alpha$ is a neighborhood of x and it is easy to see that $\pi_\alpha^{-1}(N) \cap O_\alpha \subset \text{st}(x_\alpha, \mathcal{L}_m)$.

(7) $\langle \mathcal{L}_m \rangle_{m\in\omega}$ is a pointwise countable W-refining sequence of $\mathcal{G}$.

Proof. Let $x \in X$. Then, since $\mathcal{O}$ is locally finite, there exists a finite set $\{\alpha_i | i = 1, 2, ..., k\}$ of Λ such that $x \in O_\alpha \iff \alpha = \alpha_i$ for some $i = 1, 2, ..., k$. For each $i = 1, 2, ..., k$, since $x_{\alpha_i} \in T_{\alpha_i}$, there exist an $m_i \in \omega$ and a countable subset A_i of Ξ such that $x_{\alpha_i} \in \cap_{\xi \in A_i} U_{\alpha_i,\xi}$ and $\mathcal{L}_{\alpha_i,m}(x_{\alpha_i}) \prec \{U_{\alpha_i,\xi} | \xi \in A_i\}$ for each $m \geq m_i$.

Let us put $m = \max\{m_i | i = 1, 2, ..., k\}$ and $A = \cup_{i=1}^k A_i$. Then $m \in \omega$ and A is a countable subset of Ξ. Then it is easy to see that $\mathcal{L}_m(x) \prec \{G_\xi | \xi \in \Xi\}$. □

Let κ be an infinite cardinal. A space X is called *κ-θ-refinable (κ-submetacompact)* if every open cover $\mathcal{G}$ of X with $|\mathcal{G}| \leq \kappa$ has a θ-sequence $\{\mathcal{H}_n | n \in \omega\}$ of open refinements of $\mathcal{G}$.

If $\mathcal{P}$ is a topological property, X is called *hereditarily* $\mathcal{P}$ if every subspace has $\mathcal{P}$.

It is easy to see that X is hereditarily $\mathcal{P}$ if every open subspace has $\mathcal{P}$ when $\mathcal{P}$ is w-$\delta\theta$-refinability or ww-$\delta\theta$-refinability.

The following theorem is known.

Theorem F ([5]). Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose X is hereditarily κ-submetacompact where $|\Lambda| = \kappa$.

(a) If each X_α is hereditarily w-$\delta\theta$-refinable, then so is X.

(b) If each X_α is hereditarily ww-$\delta\theta$-refinable, then so is X.

Concerning this, we have the following.

Theorem 9. *Let $\{X_\alpha, \pi_\beta^\alpha, \Lambda\}$ be an inverse system and $X = \varprojlim \{X_\alpha, \pi_\beta^\alpha, \Lambda\}$. Suppose X is hereditarily κ-submetacompact where $|\Lambda| = \kappa$.*

(c) If each X_α is hereditarily s-w-$\delta\theta$-refinable, then so is X.

(d) If each X_α is hereditarily s-ww-$\delta\theta$-refinable, then so is X.

We only give the proof of part (c).

Proof of part (c). Let G be an arbitrary open subspace of X and $\mathcal{G} = \{G_\xi | \xi \in \Xi\}$ be an arbitrary open cover of G. For each $\alpha \in \Lambda$ and $\xi \in \Xi$, let $U_{\alpha,\xi}$ be the maximal open set in X_α satisfying $\pi_\alpha^{-1}(U_{\alpha,\xi}) \subset G_\xi$ and put $U_\alpha = \cup_{\xi \in \Xi} U_{\alpha,\xi}$. Then $\mathcal{U} = \{\pi_\alpha^{-1}(U_\alpha) | \alpha \in \Lambda\}$ is an open cover of G with $|\mathcal{U}| = \kappa$. Since G is κ-submetacompact, there is a sequence $\langle \mathcal{O}_n \rangle_{n \in \omega}$ of open refinements of $\mathcal{U}$ satisfying the condition: For each $x \in G$, there exists an $n_x \in \omega$ such that $\mathrm{ord}(x, \mathcal{O}_{n_x}) < \omega$. We can represent $\mathcal{O}_n = \{O_{\alpha,n} | \alpha \in \Lambda\}$ with $O_{\alpha,n} \subset \pi_\alpha^{-1}(U_\alpha)$ for each $\alpha \in \Lambda$. Let us put $\mathcal{U}_\alpha = \{U_{\alpha,\xi} | \xi \in \Xi\}$ for each $\alpha \in \Lambda$. Then $\mathcal{U}_\alpha$ is an open cover of U_α. Since U_α is s-w-$\delta\theta$-refinable, by Lemma 5, there is a sequence $\langle \mathcal{L}_{\alpha,m} \rangle_{m \in \omega}$ of open covers of U_α satisfying:
$y \in U_\alpha$, there are an $m(y) \in \omega$ and a countable subset $A(y)$ of Ξ such that $\mathcal{L}_{\alpha,m}(y) \prec \{U_{\alpha,\xi} | \xi \in A(y)\}$.

For each $n, m \in \omega$, let us put $\mathcal{V}_m^n = \{\pi_\alpha^{-1}(L) \cap O_{\alpha,n} | L \in \mathcal{L}_{\alpha,m}, \alpha \in \Lambda\}$. Then $\langle \mathcal{V}_m^n \rangle_{n,m \in \omega}$ is a pointwise countable W-refining sequence of $\mathcal{G}$ by semi-open covers of X. $\Box$

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