*Scientiae Mathematicae Japonicae Online*, **e-2006**, 1125–1141 

# A SEARCH GAME WITH DURABLE SEARCHING RESOURCES

Ryusuke Hohzaki, Naoya Ohsiro, Toru Komiya and Emiko Fukuda

Received February 27, 2006; revised August 7, 2006

Abstract. This paper deals with a two-person zero-sum game called search allocation game (SAG), in which a searcher and a target participate as players. The searcher distributes his searching resources in a search space to detect the target. Searching resources are durable so that their effectiveness lasts for a while. On the other hand, the target moves around in the search space to evade the searcher. There have been so far few researches taking account of the durability of searching resources. We develop two linear programming formulations to solve the SAG with durable resources and we find an optimal strategy of distributing searching resources for the searcher and an optimal moving strategy for the target. We also analyze influences of the durability of resources on optimal strategies by some numerical examples.

## 1 Introduction

In Search Theory, two models have been mainly studied so far for search games with moving targets. One is *search-and-evasion game* and the other is *search allocation game* (SAG) [6]. This paper deals with the SAG. The SAG is a two-person zero-sum game, in which a searcher and a target take part. The searcher distributes his searching resources in a search space to detect the target. On the other hand, the target chooses a path to avoid the searcher.

The problem has many applications such as search-and-rescue activity and military operation in the ocean. At first, the research on search problems started from one-sided problems. Koopman [17] got together the results of the naval Operations Research activities of U.S. Navy in the Second World War. He studied so-called datum search, where a target took a diffusive motion after randomly selecting his course from an exposed point. Meinardi [18] modeled the datum search as a search game. He considered a discrete model, in which the search proceeded in a discrete space during discrete time points. To solve the game, he investigated the target transition so as to make the probability distribution of the target as uniform as possible in the space at all times. That is why his method is difficult to apply to other search problems. His model is one of the search-and-evasion game. The direct application of the datum search model could be military operations such as anti-submarine warfare (ASW). Danskin [3] dealt with a search game of ASW, where a submarine selected a course and a speed at the beginning of the search while an ASW helicopter chose a sequence of points for dipping a sonar, and he found an equilibrium point of the game. The optimal target strategy was a uniform distribution of selecting speed in a representative space of velocity and the optimal searcher's one was a uniform dipping of his sonar in the space. Baston and Bostock [1] and Garnaev [5] discussed games to determine the best points of hiding a submarine and throwing down depth charges by an ASW airplane in a one-dimensional discrete space. Washburn's work [21] was about a multi-stage game with target's and searcher's discrete motions, where both players had no restriction on their motions and the payoff was the total traveling distance until the coincidence of their

---

2000 *Mathematics Subject Classification.* 90B40, 91A05, 90C05.

*Key words and phrases.* search theory, two-person zero-sum game, search allocation game, linear programming.

positions. Nakai [19] dealt with an interesting model in the sense that a safety zone was set up for the target. His model was also a multi-stage game with the payoff of the detection probability of the target. Kikuta [15, 16] studied a game with the payoff of traveling cost. Eagle and Washburn [4] worked on a single-stage game, where the payoff was defined as the cumulative amount of values determined by sequent positions of players.

In those studies, the searcher's strategy was to choose his search paths. But it could be a distribution of searching resources in the search space, especially in the case that the searcher can move faster than the target and can move wherever he likes. Such a game is the search allocation game (SAG). For the SAG, a basic problem is to determine a hiding point for a stationary target and a distribution plan of searching resources of searchers (See Garnaev [6]). Nakai [20], and Iida et al. [13] did research on such stationary target games. Concerning with moving target games, there are Hohzaki's and Iida's papers [14, 8, 10]. Hohzaki and Iida [9] proposed a numerical method to solve more generalized games, where it was just required that the payoff is concave for the searcher's strategy and linear for the target's strategy.

In most of the studies on search games outlined above, authors set comparatively simple constraints on the target motion. That is why the problem preserves a kind of uniformity for optimal solution and then the solution is easy to be estimated. In Washburn and Hohzaki [22, 12], they considered a datum search game with energy constraint in a continuous search space. The energy constraint helps the problem to be more practical but carries off uniformity from optimal solutions. They could not succeed to derive optimal solution because the continuous space is more difficult to deal with than the discrete one for optimization problems, but they proposed an estimation method on lower and upper bounds for the value of the game. In a discrete space, Hohzaki et al. [11] proposed an exact method to solve a SAG with energy constraint. Furthermore, Hohzaki [7] elucidated a relation between the discrete SAG and the continuous SAG.

Reviewing past researches about the SAG, we notice that almost all researches assume comparatively simple types of searching resource such that it is effective just when it is scattered. Dambreville and Le Cadre [2] considered a variety of constraints on the amount of resources. Some linear constraints can give searching resources some characteristics concerning the amount of them, such as renewability. However in almost all past researches, including Dambreville and Le Cadre's work, they handle just constraints on the volume or the amount of resources and we cannot find practical properties on the effectiveness of resources such as durable resources or so. As an example, we can think of sonar buoys or flares, the effectiveness of which lasts for a while. At the same time, we can take some examples for temporary-effective resources such as explosive mines to submarines or human attention to visible objects. In most previous researches, they dealt with only temporary-effective resources but not durable resources. In this paper, we deal with a SAG with durable resources.

In the next section, we describe a SAG model with durable searching resources. In Section 3, we propose a method to solve the SAG by a linear programming problem and give another linear programming formulation to cope with a large size of problems. We take some numerical examples to analyze influences of the durability of the resources on optimal strategies in Section 4. We discuss the extension of our model to other cases in Section 5.

## 2 SAG Model with Durable Searching Resources

Here we define a search allocation game (SAG), where a searcher and a target participate, on a discrete search space. The searcher distributes his searching resources in the search space to detect the target while the target moves around to evade the searcher. The searching resources have an attribute of durability on their effectiveness.

(A1) A search space consists of a discrete geographic cell space $\boldsymbol{K} = \{1, \cdots, K\}$ and a discrete time space $\boldsymbol{T} = \{1, \cdots, T\}$, so that the space is denoted by $\boldsymbol{K} \times \boldsymbol{T}$.

(A2) A target starts from one of cells $S_0 \subseteq \boldsymbol{K}$. From a cell $i$ at time $t$, he can move to a set of cells $N(i,t)$ at next time $t+1$. This is a kind of geographic constraint. It takes some energy $\mu(i,j)$ for the target to move from cell $i$ to $j$. He has initial energy $e_0$ and on the exhaustion of the energy, he is forced to stay his current cell ever since. These are energy constraints. Let us denote a set of all feasible paths satisfying above constraints by $\Omega$, from which the target chooses a path as his pure strategy. A path $\omega$ is assumed to go through cell $\omega(t) \in \boldsymbol{K}$ at time $t$.

(A3) A searcher distributes searching resources to detect the target. His plan is denoted by $\varphi = \{\varphi(i,t),\ i \in \boldsymbol{K},\ t \in \boldsymbol{T}\}$, where $\varphi(i,t)$ is the nonnegative amount of resources to distribute in cell $i$ at time $t$. The effectiveness of the resources lasts for time $t_c$ after its dropping time.

The searcher can begin to distribute his resources from time $\tau$ and then we denote a time period available for searching by $\widehat{\boldsymbol{T}} = \{\tau, \tau+1, \cdots, T\}$. The searcher can use the amount of searching resources $\Phi(t)$ at most at time $t$. $\Phi(t)$ is arbitrarily divisible and then divided pieces are distributed in cells.

(A4) For a target path $\omega$ and a distribution plan of searching resources $\varphi$, the searcher can detect the target with probability $1 - \exp(-g(\varphi,\omega))$, where $g(\varphi,\omega)$ is a weighted amount of effective resources accumulated along path $\omega$. Parameter $\alpha_i$ gives a weight for effective resources in cell $i$. On detection of the target, the searcher gets reward 1 but the target loses the same. We define a payoff of the game by the searcher's reward.

In Assumption (A4), coefficient $\alpha_i$ indicates the efficiency of unit resource accumulated effectively over the target in cell $i$.

¿From Assumption (A2) and (A3), we can represent a set of target paths $\Omega$ and a feasible region of searcher's strategy $\Psi$ by

$$(1)\quad \Omega = \{\omega(t),\ t \in \boldsymbol{T} \mid \omega(1) \in S_0,\ \omega(t+1) \in N(i,t),\ t = 1, \cdots, T-1, \sum_{t=1}^{T-1} \mu(\omega(t), \omega(t+1)) \le e_0\}$$

$$(2)\quad \Psi = \{\varphi \mid \sum_{i \in \boldsymbol{K}} \varphi(i,t) \le \Phi(t),\ t \in \widehat{\boldsymbol{T}},\ \varphi(i,t) \ge 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\}\ .$$

Assume that a target chooses a path $\omega$ and a searcher takes a distribution plan of searching resources $\varphi$. The target is in cell $\omega(t)$ at time $t$ and all searching resources scattered there during a time period $[\max\{\tau, t-t_c\}, t]$ are still effective at the time $t$. Now we have an expression for the weighted amount $g(\varphi,\omega)$, as follows.

$$(3)\qquad g(\varphi,\omega) = \sum_{t \in \widehat{\boldsymbol{T}}} \alpha_{\omega(t)} \sum_{\xi=\max\{\tau, t-t_c\}}^{t} \varphi(\omega(t), \xi) = \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi+t_c, T\}} \alpha_{\omega(t)} \varphi(\omega(t), \xi)\ .$$

Using the expression, the payoff is given by

$$(4)\qquad R(\varphi,\omega) = 1 - \exp(-\sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi+t_c, T\}} \alpha_{\omega(t)} \varphi(\omega(t), \xi))\ .$$

Here let us take a mixed strategy for the target, $\pi = \{\pi(\omega),\ \omega \in \Omega\}$, where $\pi(\omega)$ is the probability for the target to choose path $\omega$. A feasible region of $\pi$ is

$$\Pi = \{\pi(\omega) \mid \sum_{\omega \in \Omega} \pi(\omega) = 1,\ \pi(\omega) \geq 0,\ \omega \in \Omega\}\ . \tag{5}$$

For a pure strategy of the searcher $\varphi$ and a mixed strategy of the target $\pi$, an expected payoff is given by $R(\varphi, \pi) = \sum_{\omega} \pi(\omega) R(\varphi, \omega)$. Because the expected payoff is linear for $\pi$ and strictly concave for $\varphi$, we already know that the game has an equilibrium, that is, a minimax value of the expected payoff coincides with a maximin value (See Hohzaki [9]). From here, we are going to focus on the derivation of an equilibrium point for strategies $\varphi$ and $\pi$.

## 3 Solution Methods for Equilibrium

**3.1 A basic formulation** As we mentioned in the preceding section, there exists an equilibrium point with a pure strategy of the searcher and a mixed strategy of the target, and the value of the game is given by a maximin value or a minimax value of the expected payoff $R(\varphi, \pi)$. In this section, let us start with the following transformation of the maximin problem considering the feasible region $\Pi$.

$$\begin{aligned} \max_{\varphi \in \Psi} \min_{\pi \in \Pi} R(\varphi, \pi) &= \max_{\varphi \in \Psi} \min_{\pi \in \Pi} \sum_{\omega} \pi(\omega) R(\varphi, \omega) = \max_{\varphi \in \Psi} \min_{\omega \in \Omega} R(\varphi, \omega) \\ &= \max_{\varphi \in \Psi, \zeta} \{\zeta \mid 1 - \exp(-g(\varphi, \omega)) \geq \zeta,\ \omega \in \Omega\}\ . \end{aligned}$$

With a replacement $\eta = \ln(1/(1-\zeta))$, we can transform the above expression as follows:

$$= \max_{\varphi \in \Psi, \eta} \{1 - \exp(-\eta) \mid g(\varphi, \omega) \geq \eta,\ \omega \in \Omega\} = 1 - \exp(-\max_{\varphi \in \Psi, \eta} \{\eta \mid g(\varphi, \omega) \geq \eta,\ \omega \in \Omega\})\ .$$

Consequently the maximin problem becomes a linear programming problem $\max_{\varphi, \eta}\ \{\eta \mid g(\varphi, \omega) \geq \eta,\ \omega \in \Omega\}$, which is equivalent to the following problem using Eq. (3).

$$P^S : \quad \max_{\varphi, \eta}\ \eta$$

$$s.t. \quad \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi + t_c, T\}} \alpha_{\omega(t)}\ \varphi(\omega(t), \xi) \geq \eta,\ \omega \in \Omega \tag{6}$$

$$\sum_{i \in \boldsymbol{K}} \varphi(i, t) \leq \Phi(t),\ t \in \widehat{\boldsymbol{T}} \tag{7}$$

$$\varphi(i, t) \geq 0,\ i \in \boldsymbol{K},\ t \in \widehat{\boldsymbol{T}}\ . \tag{8}$$

Using an optimal value of the above problem $\eta^*$, we can calculate a maximin value of the original game or the value of the game by $1 - \exp(-\eta^*)$. At the same time, we can find an optimal strategy $\varphi^*$ for the searcher. Now we see that problem $(P^S)$ is nothing but a maximin problem with a linear expected payoff $R(\varphi, \pi) = \sum_{\omega} \pi(\omega) g(\varphi, \omega)$. From now on, we are going to develop our theory for the game with this reduced expected payoff.

Let us consider a minimax optimization problem next. The expected payoff $R(\varphi, \pi) = \sum_{\omega} \pi(\omega) g(\varphi, \omega)$ can be transformed as follows.

$$R(\varphi, \pi) = \sum_{\omega \in \Omega} \pi(\omega) \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi + t_c, T\}} \alpha_{\omega(t)}\ \varphi(\omega(t), \xi) \tag{9}$$

$$
\begin{aligned}
&= \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{\omega\in\Omega} \pi(\omega)\alpha_{\omega(t)}\ \varphi(\omega(t),\xi) \\
&= \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{i\in\boldsymbol{K}} \sum_{\omega\in\Omega} \delta_{i\omega(t)}\pi(\omega)\ \alpha_i\ \varphi(i,\xi) \\
&= \sum_{\xi=\tau}^{T} \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{i\in\boldsymbol{K}} \left(\sum_{\omega\in\Omega_{it}} \pi(\omega)\right)\ \alpha_i\ \varphi(i,\xi)\ , \\
&= \sum_{\xi=\tau}^{T} \sum_{i\in\boldsymbol{K}} \varphi(i,\xi)\ \alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{\omega\in\Omega_{it}} \pi(\omega)\ .
\end{aligned}
$$

where $\delta_{ij}$ is the Kronecker's delta and $\Omega_{it}$ is a set of paths running through cell $i$ at time $t$, given by $\Omega_{it} \equiv \{\omega\in\Omega \mid \omega(t)=i\}$.

Taking account of Eq. (9) and $\sum_{i\in\boldsymbol{K}} \varphi(i,\xi) \le \Phi(\xi)$, we can transform a maximization problem $\max_\varphi R(\varphi,\pi)$ with respect to $\varphi$ to the following.

$$
\max_{\varphi} R(\varphi,\pi) = \sum_{\xi=\tau}^{T} \Phi(\xi)\ \max_{i\in\boldsymbol{K}}\ \alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{\omega\in\Omega_{it}} \pi(\omega)\ . \tag{10}
$$

Introducing another variable $\nu(\xi)$ which finally gives us the value of $\max_i\ \alpha_i\ \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{\omega\in\Omega_{it}} \pi(\omega)$, we can reach the following linear programming formulation for a minimax optimization $\min_\pi \max_\varphi R(\varphi,\pi)$.

$$
P^T\ :\quad \min_{\pi,\nu} \sum_{\xi\in\widehat{\boldsymbol{T}}} \Phi(\xi)\nu(\xi)
$$

$$
s.t.\qquad \alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \left(\sum_{\omega\in\Omega_{it}} \pi(\omega)\right) \le \nu(\xi),\ i\in\boldsymbol{K},\ \xi\in\widehat{\boldsymbol{T}} \tag{11}
$$

$$
\sum_{\omega\in\Omega} \pi(\omega) = 1 \tag{12}
$$

$$
\pi(\omega) \ge 0,\ \omega\in\Omega. \tag{13}
$$

We can derive an optimal mixed strategy of the target, $\pi^*$, by solving problem $(P^T)$. On the other hand, an optimal searcher's strategy, $\varphi^*$, is given by problem $(P^S)$, as we mentioned before. In practice, we can easily make sure of the duality between these two problems. Then we don't need to solve the problems twice but solve one of them once in order to obtain optimal strategies of both players. Now we state our results.

**Theorem 1** *The value of the game is given as an optimal value of problem $(P^S)$ or $(P^T)$. An optimal strategy of the searcher, $\varphi^*$, is given by an optimal solution of $(P^S)$ or optimal dual variables corresponding to condition (11) in $(P^T)$. An optimal strategy of the target, $\pi^*$, is given by an optimal solution of $(P^T)$ or optimal dual variables corresponding to condition (6) in $(P^S)$.*

**3.2 Another formulation using Markovian motion of target** In the previous section, we obtain two formulations $(P^S)$ and $(P^T)$, in which all target paths $\omega$ is enumerated. However we doubt if we can solve those problems in feasible time in a large size of a search space. For example, we count $|\boldsymbol{K}|^{|\boldsymbol{T}|}$ paths in total in a search space $\boldsymbol{K} \times \boldsymbol{T}$ if there is no limitation on the feasibility of the path. The situation could make the problem infeasible for its practical computation even though it is formulated as a linear problem. Here we are going to discuss alternative formulations without enumerating all target paths in order to cope with a large size of problems.

For simplicity, we assume that energy consumption function $\mu(i,j)$ is integer-valued and initial energy $e_0$ is also integer in Assumption (A2), and we denote a set of energy states of the target by $\boldsymbol{E} = \{0, \cdots, e_0\}$. We can represent a state of the target by a triplet $(i,t,e)$, which means that the target has residual energy $e$ and it is in cell $i$ at time $t$. As a target strategy, we throw away variable $\pi(\omega)$ and newly adopt variable $q(i,t,e)$, which is the existence probability that the target is in state $(i,t,e)$, and variable $v(i,j,t,e)$, which is the transition probability that the target is in $(i,t,e)$ and moves to cell $j$ at next time $t+1$. Taking account of geographic constraint $N(i,t)$ and moving energy constraint, we can denote the cells to which the target can move from state $(i,t,e)$ by $N(i,t,e) = \{j \in N(i,t) | \mu(i,j) \leq e\}$, and the cells from which the target can moves to $(i,t,e)$ by $N^*(i,t,e) = \{j \in \boldsymbol{K} | i \in N(j,t-1,e+\mu(j,i))\}$.

We notice that in Problem $(P^T)$ condition (11) has an expression $\sum_{\omega \in \Omega_{it}} \pi(\omega)$. The expression indicates the existence probability of the target in cell $i$ at time $t$. That is why we can replace the expression with $\sum_{e \in \boldsymbol{E}} q(i,t,e)$ and we can rewrite conditions (11) to $\alpha_i \ \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{e \in \boldsymbol{E}} q(i,t,e) \leq \nu(\xi)$ using new strategy variables $q(\cdot)$. We may replace conditions (12) and (13) with the so-called conservation law of probability. First of all, we require equations $q(i,t,e) = \sum_{j \in N(i,t,e)} v(i,j,t,e)$ and $q(i,t,e) = \sum_{j \in N^*(i,t,e)} v(j,i,t-1,e+\mu(j,i))$, which mean that probability $q(i,t,e)$ is equal to the total out-flow probabilities from state $(i,t,e)$ and the total in-flow probabilities into the state, respectively. At the same time, we need an equation $\sum_{i \in \boldsymbol{K}} \sum_{e \in \boldsymbol{E}} q(i,t,e) = 1$ for the sum of existence probabilities. Initial condition of the target about its initial energy and initial cells is expressed by $\sum_{i \in S_0} q(i,1,e_0) = 1$. Now we've obtained a formulation with existence probability $q(i,t,e)$ and transition probability $v(i,j,t,e)$ as a target strategy, which gives us the value of the game and an optimal target strategy as well as Problem $(P^T)$. In the problem, $q(i,t,e)$, $v(i,j,t,e)$ and $\nu(\xi)$ are variables.

$$\widetilde{P}^T \ : \ \min_{q,v,\nu} \sum_{\xi \in \widehat{\boldsymbol{T}}} \Phi(\xi)\nu(\xi)$$

$$\text{(14)} \quad s.t. \quad \alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{e \in \boldsymbol{E}} q(i,t,e) \leq \nu(\xi), \ i \in \boldsymbol{K}, \ \xi \in \widehat{\boldsymbol{T}}$$

$$\text{(15)} \quad q(i,t,e) = \sum_{j \in N(i,t,e)} v(i,j,t,e), \ i \in \boldsymbol{K}, \ t = 1, \cdots, T-1, \ e \in \boldsymbol{E}$$

$$\text{(16)} \quad q(i,t,e) = \sum_{j \in N^*(i,t,e)} v(j,i,t-1,e+\mu(j,i)), \ i \in \boldsymbol{K}, \ t = 2, \cdots, T, \ e \in \boldsymbol{E}$$

$$\text{(17)} \quad \sum_{i \in S_0} q(i,1,e_0) = 1$$

$$\text{(18)} \quad \sum_{i \in \boldsymbol{K}} \sum_{e \in \boldsymbol{E}} q(i,t,e) = 1, \ t \in \boldsymbol{T}$$

$$v(i,j,t,e) \geq 0, \ i,j \in \boldsymbol{K}, \ t = 1, \cdots, T-1, \ e \in \boldsymbol{E}.$$

In order to obtain an optimal searcher strategy, we are going to develop a dynamic programming formulation. Let $w(i,t,e)$ be a minimal expected payoff given by an optimal movement of the target after time $t$ starting from a state $(i,t,e)$. Because the search doesn't begin during time period $[1,\tau)$ yet and any payoff never occurs then, we have a recursive formulation of the dynamic programming with respect to $w(i,t,e)$ for t ($<\tau$) from its definition.

$$w(i,t,e)=\min_{j\in N(i,t,e)} w(j,t+1,e-\mu(i,j)) \ . \tag{19}$$

After time $\tau$, the search yields payoff $\alpha_i\sum_{\xi=\max\{\tau,t-t_c\}}^{t}\varphi(i,\xi)$ in cell $i$ at time $t$. It brings us to the following dynamic programming formulation for t ($\geq\tau$).

$$w(i,t,e)=\min_{j\in N(i,t,e)}\left\{\alpha_i\sum_{\xi=\max\{\tau,t-t_c\}}^{t}\varphi(i,\xi)+w(j,t+1,e-\mu(i,j))\right\} \ . \tag{20}$$

At the last time $T$, the following payoff occurs without any subsequent payoff.

$$w(i,T,e)=\alpha_i\sum_{\xi=\max\{\tau,T-t_c\}}^{T}\varphi(i,\xi) \ . \tag{21}$$

The minimum payoff during the entire search time is given by $\min_{i\in S_0} w(i,1,e_0)$ because any feasible path must start from one of cells $S_0$ at time $t=1$. The searcher wants to maximize the minimum payoff. It will be done by $\max_{\varphi}\ \min_{i\in S_0} w(i,1,e_0)$, which gives the maximin value for the payoff. Merging conditions (19)–(21) and a feasible region (2) of variable $\varphi(\cdot)$ into a formulation, we can generate an another linear programming formulation $\widetilde{P}^S$.

$$\widetilde{P}^S \ : \ \max_{\varphi,w,\eta}\ \eta$$

$$\text{s.t.}\quad w(i,1,e_0)\geq\eta,\ i\in S_0 \tag{22}$$

$$w(i,t,e)\leq w(j,t+1,e-\mu(i,j)), i\in\boldsymbol{K}, j\in N(i,t,e), t=1,\cdots,\tau-1, e\in\boldsymbol{E} \tag{23}$$

$$w(i,t,e)\leq\alpha_i\sum_{\xi=\max\{\tau,t-t_c\}}^{t}\varphi(i,\xi)+w(j,t+1,e-\mu(i,j)),\quad i\in\boldsymbol{K},\ j\in N(i,t,e),\ t=\tau,\cdots,T-1,\ e\in\boldsymbol{E} \tag{24}$$

$$w(i,T,e)=\alpha_i\sum_{\xi=\max\{\tau,T-t_c\}}^{T}\varphi(i,\xi),\ i\in\boldsymbol{K},\ e\in\boldsymbol{E} \tag{25}$$

$$\sum_{i\in\boldsymbol{K}}\varphi(i,\xi)\leq\Phi(\xi),\ \xi\in\widehat{\boldsymbol{T}} \tag{26}$$

$$\varphi(i,\xi)\geq 0,\ i\in\boldsymbol{K},\ \xi\in\widehat{\boldsymbol{T}}. \tag{27}$$

Conditions (23) and (24) are derivatives from Eqs. (19) and (20), respectively.

We can prove that Problem ($\widetilde{P}^S$) is dual to Problem ($\widetilde{P}^T$), although we leave the proof to Appendix. Because of the duality, both problems give us an identical optimal value, which is the value of the game.

In this section, we develop a linear programming formulation with a Markovian motion strategy for the target. In Reference [11], we already succeeded to propose linear programming formulations for the SAG with non-durable searching resources. We may make use of the formulations by regarding effective resources accumulated on cell $i$ at time $t$, $\widetilde{\varphi}(i,t)\equiv\sum_{\xi=\max\{\tau,t-t_c\}}^{t}\varphi(i,\xi)$, as non-durable resources. However it does not work because the nonnegativity of $\widetilde{\varphi}(i,t)$ cannot guarantee the nonnegativity of $\varphi(i,\xi)$.

## 4 Numerical Examples

Here we elucidate some properties of optimal players' strategies by numerical examples.

### 4.1 Basic properties of optimal strategy

Let us consider a time space $\boldsymbol{T} = \{1, \cdots, 8\}$ and a cell space $\boldsymbol{K} = \{1, \cdots, 8\}$. These cells are aligned in this order so that the difference of two numbers of cells indicates the distance between them. A target starts from cell $S_0 = \{1\}$ at time $t = 1$ and has initial energy $e_0 = 8$. The target spends energy $\mu(i, j) = |i - j|^2$ to move between cells $i$ and $j$. The target can reach cell 8 at the farthest at time 8. He can move from the current cell only within neighborhood cells, that is, $N(i, t) = \{i - 1, i, i + 1\} \cap \boldsymbol{K}$. We set $\alpha_i = 1$ for any $i$ for comprehensibility. A searcher can begin his search from time $\tau = 3$ using available searching resources $\Phi(t) = 1$ every time.

As a basic case with no durability for searching resources, $t_c = 0$, Table 1-a shows an optimal distribution of the target or optimal existence probabilities of the target, $\sum_{e \in \boldsymbol{E}} q(i, t, e)$, time by time. A column indicates a time point and a row a cell. The target certainly exists in cell 1 at initial time $t = 1$ and moves diffusively in the space as time passes. Table 1-b illustrates an optimal distribution of searching resources for the searcher, $\varphi(i, t)$. After the beginning of the search at $\tau = 3$, the target spreads his possible area as widely as possible, and makes his distribution uniform. These two properties are crucial for the target strategy because these properties force the searcher to scatter searching resources over a large area and not to effectively focus searching resources on high-density areas of the target. Corresponding to such a strategy of the target, the searcher distributes his searching resources in a uniform way over possible areas of the target, as seen in Table 1-b. In this case, the value of the game is 1.2.

Table 1-a. Optimal distribution of target ($t_c = 0$)

| Cell\ $t=$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.367 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 2 | 0 | 0.663 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 3 | 0 | 0 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 4 | 0 | 0 | 0 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 5 | 0 | 0 | 0 | 0 | 0.2 | 0.167 | 0.143 | 0.125 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0.167 | 0.143 | 0.125 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.143 | 0.125 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.125 |

Table 1-b. Optimal distribution of searching resources ($t_c = 0$)

| Cell\ $t=$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 2 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 3 | 0.333 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 4 | 0 | 0.25 | 0.2 | 0.167 | 0.143 | 0.125 |
| 5 | 0 | 0 | 0.2 | 0.167 | 0.143 | 0.125 |
| 6 | 0 | 0 | 0 | 0.167 | 0.143 | 0.125 |
| 7 | 0 | 0 | 0 | 0 | 0.143 | 0.125 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.125 |

Table 2-a and 2-b illustrate optimal strategies of the target and the searcher in the case of $t_c = 2$ set for durable searching resources. Table 2-c shows effective searching resources

accumulated on cell $i$ at time $t$, which are calculated by $\alpha_i \sum_{\xi=\max\{\tau,t-t_c\}}^{t} \varphi(i,\xi)$ from Table 2-b. We are going to explain how important the accumulated resources are for the searcher's strategy. Substituting a middle expression of Eq. (3) for $g(\varphi,\omega)$ in the expected payoff $R(\varphi,\pi) = \sum_\omega \pi(\omega) g(\varphi,\omega)$, we obtain the following expression of $R(\varphi,\pi)$ in a similar manner to Eq. (9):

$$R(\varphi,\pi) = \sum_{t=\tau}^{T} \sum_{i\in\boldsymbol{K}} \alpha_i \sum_{\xi=\max\{\tau,t-t_c\}}^{t} \varphi(i,\xi) \sum_{e\in\boldsymbol{E}} q(i,t,e). \tag{28}$$

As well as the existence probability of the target $\sum_{e\in\boldsymbol{E}} q(i,t,e)$, the accumulated effective resources directly affect the expected reward. The searcher cannot help concerning about the uniformity of the accumulated effective resources.

Table 2-a. Optimal distribution of target ($t_c = 2$)

| Cell\ $t=$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.367 | 0.333 | 0.222 | 0.149 | 0.181 | 0.129 | 0.173 |
| 2 | 0 | 0.663 | 0.333 | 0.222 | 0.149 | 0.181 | 0.129 | 0.173 |
| 3 | 0 | 0 | 0.333 | 0.222 | 0.149 | 0.181 | 0.129 | 0.173 |
| 4 | 0 | 0 | 0 | 0.333 | 0.219 | 0 | 0.239 | 0.061 |
| 5 | 0 | 0 | 0 | 0 | 0.333 | 0.125 | 0 | 0.173 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0.333 | 0.125 | 0.023 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0.249 | 0.053 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.173 |

Table 2-b. Optimal distribution of searching resources ($t_c = 2$)

| Cell\ $t=$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | 0.295 | 0.185 | 0.140 | 0.163 | 0.119 | 0.081 |
| 2 | 0.295 | 0.185 | 0.140 | 0.163 | 0.119 | 0.081 |
| 3 | 0.410 | 0.069 | 0.140 | 0.279 | 0.004 | 0.081 |
| 4 | 0 | 0.561 | 0.061 | 0 | 0.364 | 0 |
| 5 | 0 | 0 | 0.489 | 0 | 0 | 0.364 |
| 6 | 0 | 0 | 0.030 | 0.394 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0.394 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.392 |

Table 2-c. Cumulative amount of effective resources ($t_c = 2$)

| Cell\ $t=$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | 0.295 | 0.480 | 0.620 | 0.488 | 0.423 | 0.364 |
| 2 | 0.295 | 0.480 | 0.620 | 0.488 | 0.423 | 0.364 |
| 3 | 0.410 | 0.479 | 0.619 | 0.488 | 0.423 | 0.364 |
| 4 | 0 | 0.561 | 0.622 | 0.622 | 0.424 | 0.364 |
| 5 | 0 | 0 | 0.489 | 0.489 | 0.489 | 0.364 |
| 6 | 0 | 0 | 0.030 | 0.424 | 0.424 | 0.394 |
| 7 | 0 | 0 | 0 | 0 | 0.394 | 0.394 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.392 |

Comparing with Table1-a, there are more probabilities along the boundary of the possible area of the target in Table 2-a, e.g. at $(t,i) = (4,4),(5,5),(6,6)$ and $(7,7)$. We can explain the probability bias as follows. The target moves from cell 1 to outer cells. The searcher begins to pursue the target and drop searching resources in inner cells at earlier time. Taking account of the durability of searching resources, the target had better go to outer cells, where searching resources have not be distributed yet, rather than staying in inner cells. Corresponding to such a movement of the target, the searcher tends to allocate more resources in outer cells, as seen in Table 2-b. As a result, the searcher forms a picture of accumulated effective resources as Table 2-c. In the table, there are some biases and perturbation. For example, there are put more searching resources at $(t,i) = (6,4),(7,5)$, which are just points the target never goes to, as seen in Table 2-a.

Once the searcher distributes a lot of resources in some cell, he doesn't need more there for a while because of the durability of the resources. In Table 2-b, there are some zeros in cells $4 \sim 7$ during times $6 \sim 8$. We might call the property the periodicity of distributing resources. We also have to pay attention to the allocation of resources 0.03 in cell 6 at time 5, where the target never exists. This comes from the durability of the resources, too. The searcher expects cumulative effects of resources afterwards. Let us call the property the pre-allocation policy of searching resources. Even though Table 2-b shows several kinds of disturbing characteristics of distributing resources such as the periodicity and the pre-allocation, we can see comparative uniformity in Table 2-c, as explained above. In this case, the value of the game is 2.7, which is larger than the no-durability case, of course.

**4.2 Effect of durability period** Here we vary durability period $t_c$ and investigate its effects on optimal strategies of players. Let us set parameters like $T = 10,\ K = 10,\ \tau = 3,\ S_0 = \{1\},\ \mu(i,j) = (i-j)^2,\ e_0 = 9,\ \Phi(t) = 1,\ \alpha_i = 1,\ N(i,t) = \{i-3,\cdots,i,\cdots,i+3\} \cap \boldsymbol{K}$. For $t_c = 0, 2$ and 4, we illustrate optimal strategies of players in Table 3, 4 and 5, respectively.

Expansion of the possible area of the target, and uniformity of the target probability and the allocated searching resources features Table 3-a and 3-b. But these features are not perfect, as seen at $(t,i) = (4,6),(7,8),(9,9),(10,9),(10,10)$ in Table 3-a, because initial energy is not enough for the target to make his distribution totally uniform. In this case, we have $G = 1.18$ as the value of the game.

Table 4-a shows an optimal strategy of the target in the case of $t_c = 2$. Table 4-b and 4-c are results for an optimal distribution of searching resources and the accumulated amount of effective resources, respectively. We can observe biased target probabilities on a line $(t,i) = (2,2),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)$ in Table 4-a, as we already explained the reason of this characteristic about Table 2-a. We can also see the focus of searching resources at $(t,i) = (4,5),(5,6),(6,7),(8,8)$ in Table 4-b. The searcher seems to chase a target with high-existence probabilities by the pre-allocation policy. We can say that this aim succeeds to some extent judging from the peaks of the accumulated effective resources at $(t,i) = (4,4),(5,5),(6,6),(8,8)$ in Table 4-c. There appears another feature called the periodic allocation of searching resources in Table 4-b. The value of the game is $G = 2.92$ in this case.

Table 5-a, 5-b and 5-c show the results for $t_c = 4$. As well as Table 4, the basic properties for players' strategies are still kept in this case. On the whole, the properties become a little intensive for this longer durability period. In this case, the value of the game is $G = 3.97$.

For $t_c = 0$, we can estimate the whole of available cumulative resources by $E = \sum_{t=\tau}^{T} \Phi(t)$. We can do that by $E = \sum_{t=\tau}^{T} \sum_{\xi=t}^{\min\{t+t_c,T\}} \Phi(\xi)$ for durable resources and obtain values $E = 8,\ 21$ and 30 for $t_c = 0,\ 2$ and 4, respectively. We already have the values of the games $G = 1.18,\ 2.92,\ 3.97$ for $t_c = 0,\ 2,\ 4$. We can easily reason the increasingness of the values for longer periods of the durability. We get the value of the game $G = 4.57$ and the total amount of cumulative resources $E = 35$ for $t_c = 6$. Now we calculate $G/E$, which indicates an increasing rate of the resultant game value per unit cumulative resource, to obtain 0.148, 0.139, 0.132 and 0.130 for $t_c = 0,\ 2,\ 4$ and 6, respectively. The decreasingness of these values implies that the effectiveness of unit cumulative resource is diminishing for longer period of the durability. Generally the searcher cannot manipulate the durability of searching resources so well to generate his desired form of effective searching resources.

Table 3-a. Optimal distribution of target ($t_c = 0$)

| Cell\ $t =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.260 | 0.200 | 0.183 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 2 | 0 | 0.394 | 0.200 | 0.183 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 3 | 0 | 0.345 | 0.200 | 0.183 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 4 | 0 | 0.001 | 0.200 | 0.183 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 5 | 0 | 0 | 0.200 | 0.183 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 6 | 0 | 0 | 0 | 0.085 | 0.167 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0.143 | 0.133 | 0.125 | 0.118 | 0.112 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0.068 | 0.125 | 0.118 | 0.112 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.057 | 0.073 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.033 |

Table 3-b. Optimal distribution of searching resources ($t_c = 0$)

| Cell\ $t =$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.200 | 0.200 | 0.167 | 0.134 | 0.123 | 0.108 | 0.123 | 0.125 |
| 2 | 0.200 | 0.200 | 0.167 | 0.134 | 0.123 | 0.108 | 0.123 | 0.125 |
| 3 | 0.200 | 0.200 | 0.167 | 0.134 | 0.123 | 0.108 | 0.123 | 0.125 |
| 4 | 0.200 | 0.200 | 0.167 | 0.134 | 0.123 | 0.108 | 0.123 | 0.125 |
| 5 | 0.200 | 0.200 | 0.167 | 0.134 | 0.123 | 0.108 | 0.123 | 0.125 |
| 6 | 0 | 0 | 0.167 | 0.221 | 0.236 | 0.108 | 0.123 | 0.125 |
| 7 | 0 | 0 | 0 | 0.109 | 0.148 | 0.108 | 0.123 | 0.125 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.244 | 0.136 | 0.125 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4-a. Optimal distribution of target ($t_c = 2$)

| Cell\ $t =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.252 | 0.210 | 0.179 | 0.168 | 0.140 | 0.118 | 0.136 | 0.110 | 0.110 |
| 2 | 0 | 0.439 | 0.210 | 0.179 | 0.168 | 0.140 | 0.118 | 0.136 | 0.110 | 0.110 |
| 3 | 0 | 0.309 | 0.210 | 0.179 | 0.168 | 0.140 | 0.118 | 0.136 | 0.110 | 0.110 |
| 4 | 0 | 0 | 0.210 | 0.238 | 0.108 | 0.140 | 0.177 | 0.077 | 0.110 | 0.110 |
| 5 | 0 | 0 | 0.159 | 0.179 | 0.219 | 0.088 | 0.118 | 0.137 | 0.109 | 0.110 |
| 6 | 0 | 0 | 0 | 0.046 | 0.168 | 0.211 | 0.046 | 0.136 | 0.110 | 0.110 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0.140 | 0.241 | 0.012 | 0.110 | 0.110 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0.064 | 0.229 | 0.064 | 0.064 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.165 | 0.055 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.110 |

Table 4-b. Optimal distribution of searching resources ($t_c = 2$)

| Cell\ $t =$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.227 | 0.127 | 0.138 | 0.159 | 0.100 | 0.092 | 0.159 | 0.068 |
| 2 | 0.227 | 0.127 | 0.138 | 0.159 | 0.100 | 0.092 | 0.159 | 0.068 |
| 3 | 0.227 | 0.127 | 0.138 | 0.159 | 0.100 | 0.092 | 0.159 | 0.068 |
| 4 | 0.227 | 0.241 | 0.024 | 0.159 | 0.214 | 0 | 0.137 | 0.182 |
| 5 | 0.091 | 0.378 | 0.046 | 0 | 0.351 | 0 | 0 | 0.319 |
| 6 | 0 | 0 | 0.515 | 0.110 | 0 | 0.241 | 0.110 | 0.008 |
| 7 | 0 | 0 | 0 | 0.254 | 0.137 | 0 | 0.173 | 0.185 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.484 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0.103 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.103 |

Table 4-c. Cumulative amount of effective resources ($t_c = 2$)

| Cell\ $t =$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.227 | 0.354 | 0.493 | 0.424 | 0.397 | 0.351 | 0.351 | 0.319 |
| 2 | 0.227 | 0.354 | 0.493 | 0.424 | 0.397 | 0.351 | 0.351 | 0.319 |
| 3 | 0.227 | 0.354 | 0.493 | 0.424 | 0.397 | 0.351 | 0.351 | 0.319 |
| 4 | 0.227 | 0.469 | 0.493 | 0.424 | 0.397 | 0.373 | 0.351 | 0.319 |
| 5 | 0.091 | 0.469 | 0.515 | 0.424 | 0.397 | 0.351 | 0.351 | 0.319 |
| 6 | 0 | 0 | 0.515 | 0.625 | 0.625 | 0.351 | 0.351 | 0.359 |
| 7 | 0 | 0 | 0 | 0.254 | 0.391 | 0.391 | 0.310 | 0.359 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.484 | 0.484 | 0.484 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0.103 | 0.103 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.103 |

Table 5-a. Optimal distribution of target ($t_c = 4$)

| Cell\ $t =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.272 | 0.215 | 0.183 | 0.161 | 0.133 | 0.098 | 0.135 | 0.116 | 0.124 |
| 2 | 0 | 0.448 | 0.215 | 0.180 | 0.164 | 0.133 | 0.098 | 0.135 | 0.112 | 0.127 |
| 3 | 0 | 0.279 | 0.215 | 0.179 | 0.165 | 0.133 | 0.098 | 0.135 | 0.112 | 0.128 |
| 4 | 0 | 0 | 0.215 | 0.217 | 0.127 | 0.133 | 0.098 | 0.135 | 0.150 | 0.090 |
| 5 | 0 | 0 | 0.140 | 0.177 | 0.215 | 0.085 | 0.173 | 0.060 | 0.110 | 0.130 |
| 6 | 0 | 0 | 0 | 0.064 | 0.167 | 0.249 | 0.099 | 0.064 | 0.064 | 0.130 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0.133 | 0.273 | 0.088 | 0.088 | 0.022 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0.063 | 0.248 | 0.063 | 0.063 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.185 | 0.055 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.130 |

Table 5-b. Optimal distribution of searching resources ($t_c = 4$)

| Cell\ $t =$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.225 | 0.166 | 0.050 | 0.122 | 0.198 | 0.041 | 0.118 | 0 |
| 2 | 0.225 | 0.166 | 0.050 | 0.122 | 0.198 | 0.041 | 0.118 | 0 |
| 3 | 0.246 | 0.146 | 0.050 | 0.122 | 0.198 | 0.061 | 0.097 | 0 |
| 4 | 0.225 | 0.187 | 0.050 | 0.155 | 0.144 | 0.094 | 0.085 | 0 |
| 5 | 0.078 | 0.334 | 0.184 | 0 | 0.111 | 0 | 0.232 | 0.135 |
| 6 | 0 | 0 | 0.617 | 0.058 | 0 | 0 | 0 | 0.515 |
| 7 | 0 | 0 | 0 | 0.421 | 0.151 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.763 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0.350 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.350 |

Table 5-c. Cumulative amount of effective resources ($t_c = 4$)

| Cell\ $t =$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.225 | 0.392 | 0.442 | 0.564 | 0.761 | 0.576 | 0.528 | 0.478 |
| 2 | 0.225 | 0.392 | 0.442 | 0.564 | 0.761 | 0.576 | 0.528 | 0.478 |
| 3 | 0.246 | 0.392 | 0.442 | 0.564 | 0.761 | 0.576 | 0.528 | 0.478 |
| 4 | 0.225 | 0.412 | 0.462 | 0.617 | 0.761 | 0.630 | 0.528 | 0.478 |
| 5 | 0.078 | 0.412 | 0.597 | 0.597 | 0.708 | 0.630 | 0.528 | 0.478 |
| 6 | 0 | 0 | 0.617 | 0.675 | 0.675 | 0.675 | 0.675 | 0.573 |
| 7 | 0 | 0 | 0 | 0.421 | 0.573 | 0.573 | 0.573 | 0.573 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0.763 | 0.763 | 0.763 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0.350 | 0.350 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.350 |

**5 Extension to Other Models** We can apply our basic formulations $\widetilde{P}^T$ and $\widetilde{P}^S$ to special cases by modifying them a little. The first case is a model where there are safe shelters for the target to hide. If the target is never detected in cells $K_0 \subseteq \boldsymbol{K}$, $K_0$ provides the target the shelters. In order to embed the effect of the shelters in our formulation, we only have to replace condition (14) to

$$\alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{e\in\boldsymbol{E}} q(i,t,e) \le \nu(\xi),\ i \in \boldsymbol{K}\backslash K_0,\ \xi \in \widehat{\boldsymbol{T}} \tag{29}$$

or to set parameter $\alpha_i = 0,\ i \in K_0$.

If the target has his final destinations or goals $K_0 \subseteq \boldsymbol{K}$ where he must go at final time $T$, we need an additional condition $\sum_{i\in K_0}\sum_{e\in\boldsymbol{E}} q(i,T,e) = 1$ besides condition (29).

The second case is a model with supply depots. In the model, the target can visit depots $K_1 \subseteq \boldsymbol{K}$ to supply his energy and increase it by $e_f$. There is an upper limit $e_0 + e_f$ on energy, though. To suit our formulation to the model, we first need to change a set of energy states $\boldsymbol{E} = \{0, \cdots, e_0\}$ to $\boldsymbol{E} = \{0, \cdots, e_0 + e_f\}$. When the target with residual energy $e$ stops by the supply depots, his energy increases to $\min\{e+e_f, e_0+e_f\}$. Let $Q(i,t,e)$ be the probability that the target is at supply depot $i \in K_1$ at time $t$ and it refuels his energy up to $e$. The in-flow probability $q(i,t,e)$ coming into depot $i$ changes to $Q(i,t,\min\{e+e_f, e_0+e_f\})$ as the out-flow probability. For the supply-depot model, we can propose a revised version of formulation $(\widetilde{P}^T)$ as follows.

$$\widetilde{P}^T_D \;:\; \min_{q,v,\nu} \sum_{\xi\in\widehat{\boldsymbol{T}}} \Phi(\xi)\nu(\xi)$$

$$s.t. \quad \alpha_i \sum_{t=\xi}^{\min\{\xi+t_c,T\}} \sum_{e\in\boldsymbol{E}} q(i,t,e) \le \nu(\xi),\; i\in\boldsymbol{K},\; \xi\in\widehat{\boldsymbol{T}}$$

(30) $$q(i,t,e) = \sum_{j\in N(i,t,e)} v(i,j,t,e),\; i\in\boldsymbol{K}\backslash K_1,\; t=1,\cdots,T-1,\; e\in\boldsymbol{E}$$

(31) $$Q(i,t,e+e_f) = q(i,t,e),\; i\in K_1,\; t=1,\cdots,T-1,\; e\in\{0,\cdots,e_0-1\}$$

(32) $$Q(i,t,e_0+e_f) = \sum_{e=e_0}^{e_0+e_f} q(i,t,e),\; i\in K_1,\; t=1,\cdots,T-1$$

(33) $$Q(i,t,e) = \sum_{j\in N(i,t,e)} v(i,j,t,e),$$
$$i\in K_1,\; t=1,\cdots,T-1,\; e\in\{e_f,e_f+1,\cdots,e_0+e_f\}$$

$$q(i,t,e) = \sum_{j\in N^*(i,t,e)} v(j,i,t-1,e+\mu(j,i)),\; i\in\boldsymbol{K},\; t=2,\cdots,T,\; e\in\boldsymbol{E}$$

$$\sum_{i\in S_0} q(i,1,e_0) = 1$$

$$\sum_{i\in\boldsymbol{K}}\sum_{e\in\boldsymbol{E}} q(i,t,e) = 1,\; t\in\boldsymbol{T}$$

$$v(i,j,t,e)\ge 0,\; i,j\in\boldsymbol{K},\; t=1,\cdots,T-1,\; e\in\boldsymbol{E}.$$

For the out-flow probabilities in cells other than supply depots, the conservation law of the probability is given by Eq. (30) as same as the original formulation. However it should be Eq. (33) at depots $K_1$. Equation (31) or (32) represents a relation between $q(i,t,e)$ and $Q(i,t,e)$ at depot $i\in K_1$. Using this formulation, we can analyze optimal target strategies. They tell the target where and when he must refuel in an optimal way.

**6 Conclusion** In this paper, we deal with a two-person zero-sum game called search allocation game (SAG), where searching resources have the property of temporal durability, and we propose two methods to solve the SAG. By the first method, we give an optimal searcher's plan of distributing resources and an optimal target plan of selecting paths in a search space. To cope with the case of the huge number of target paths, we can use the second method, where a target strategy is given by a Markovian motion.

There have been several researches taking account of practical feasibility conditions on the target motion, such as moving energy constraint. However there seems to be no research handling practical properties on the effectiveness of searching resources, such as durability. Thinking of practical search sensors, we notice that a wide variety of searching resources possess this attribute. We expect our proposed methods to be applied to practical search problem or more complicated search situations.

## Appendix: Proof of the duality between Problem $(\widetilde{P}^T)$ and $(\widetilde{P}^S)$.

Here let us prove the duality between problems $(\widetilde{P}^T)$ and $(\widetilde{P}^S)$. We set dual variables $\phi(j,\xi)$, $y(i,t,e)$, $z(i,t,e)$, $\eta$ and $\zeta(t)$ corresponding to conditions (14), (15), (16), (17) and

(18), respectively, to obtain a dual problem to $\widetilde{P}^T$.

$$(D_1) \quad \max\ \eta + \sum_{t \in \boldsymbol{T}} \zeta(t)$$

$$s.t. \quad \sum_{j \in \boldsymbol{K}} \phi(j,\xi) \le \Phi(\xi),\ \xi \in \widehat{\boldsymbol{T}}$$

$$\text{(A1)} \qquad -\alpha_i \sum_{\xi=\max\{\tau, T-t_c\}}^{T} \phi(i,\xi) + z(i,T,e) + \zeta(T) = 0,\ i \in \boldsymbol{K},\ e \in \boldsymbol{E}$$

$$\text{(A2)} \qquad -\alpha_i \sum_{\xi=\max\{\tau, t-t_c\}}^{t} \phi(i,\xi) - y(i,t,e) + z(i,t,e) + \zeta(t) = 0,$$

$$i \in \boldsymbol{K},\ t = \tau, \cdots, T-1,\ e \in \boldsymbol{E}$$

$$\text{(A3)} \qquad -y(i,t,e) + z(i,t,e) + \zeta(t) = 0,\ i \in \boldsymbol{K},\ t = 2, \cdots, \tau-1,\ e \in \boldsymbol{E}$$

$$\text{(A4)} \qquad -y(i,1,e_0) + \eta + \zeta(1) = 0,\ i \in S_0$$

$$\text{(A5)} \qquad -y(i,1,e_0) + \zeta(1) = 0,\ i \in \boldsymbol{K} \backslash S_0$$

$$\text{(A6)} \qquad -y(i,1,e) + \zeta(1) = 0,\ i \in \boldsymbol{K},\ e \in \boldsymbol{E} \backslash \{e_0\}$$

$$\text{(A7)} \qquad y(i,t,e) - z(j,t+1,e-\mu(i,j)) \le 0,\ i \in \boldsymbol{K}, j \in N(i,t,e), t = 1, \cdots, T-1, e \in \boldsymbol{E}$$

$$\phi(i,\xi) \ge 0,\ i \in \boldsymbol{K},\ \xi \in \widehat{\boldsymbol{T}}.$$

Variable $z(i,t,e)$ is not defined yet for index $t = 1$. With the adoption of a new definition

$$\text{(A8)} \qquad z(i,1,e) = \begin{cases} \eta, & i \in S_0 \textit{ and } e = e_0 \\ 0, & \textit{otherwise}, \end{cases}$$

we can express conditions (A4), (A5) and (A6) by an equality $y(i,1,e) = z(i,1,e) + \zeta(1)$, which makes condition (A3) valid for $t = 1$. By using Eqs. (A2) and (A3), we replace variable $y(i,t,e)$ with $z(i,t,e)$ to obtain another problem.

$$(D_2) \quad \max\ \eta + \sum_{t \in \boldsymbol{T}} \zeta(t)$$

$$s.t. \quad z(i,1,e_0) = \eta,\ i \in S_0$$

$$z(i,1,e_0) = 0,\ i \in \boldsymbol{K} \backslash S_0$$

$$z(i,1,e) = 0,\ i \in \boldsymbol{K},\ e \in \boldsymbol{E} \backslash \{e_0\}$$

$$z(i,t,e) + \zeta(t) \le z(j,t+1,e-\mu(i,j)),$$

$$i \in \boldsymbol{K},\ j \in N(i,t,e),\ t = 1, \cdots, \tau-1,\ e \in \boldsymbol{E}$$

$$z(i,t,e) + \zeta(t) \le \alpha_i \sum_{\xi=\max\{\tau, t-t_c\}}^{t} \phi(i,\xi) + z(j,t+1,e-\mu(i,j)),$$

$$i \in \boldsymbol{K},\ j \in N(i,t,e),\ t = \tau, \cdots, T-1,\ e \in \boldsymbol{E}$$

$$z(i,T,e) + \zeta(T) = \alpha_i \sum_{\xi=\max\{\tau, T-t_c\}}^{T} \phi(i,\xi),\ i \in \boldsymbol{K},\ e \in \boldsymbol{E}$$

$$\sum_{i \in \boldsymbol{K}} \phi(i,\xi) \le \Phi(\xi),\ \xi \in \widehat{\boldsymbol{T}}$$

$$\phi(i,\xi) \ge 0,\ i \in \boldsymbol{K},\ \xi \in \widehat{\boldsymbol{T}}.$$

By introducing $w(i,t,e) \equiv z(i,t,e) + \sum_{\xi=t}^{T} \zeta(\xi)$, we have the following problem $(D_3)$.

$$(D_3) \quad \max \ \eta + \sum_{t \in \boldsymbol{T}} \zeta(t)$$

$$\text{(A9) } s.t. \quad w(i,1,e_0) = \eta + \sum_{t \in \boldsymbol{T}} \zeta(t), \ i \in S_0$$

$$\text{(A10)} \quad w(i,1,e_0) = \sum_{t \in \boldsymbol{T}} \zeta(t), \ i \in \boldsymbol{K} \backslash S_0$$

$$\text{(A11)} \quad w(i,1,e) = \sum_{t \in \boldsymbol{T}} \zeta(t), \ i \in \boldsymbol{K}, \ e \in \boldsymbol{E} \backslash \{e_0\}$$

$$\text{(A12)} \quad w(i,t,e) \leq w(j,t+1,e-\mu(i,j)), i \in \boldsymbol{K}, j \in N(i,t,e), t = 1, \cdots, \tau - 1, e \in \boldsymbol{E}$$

$$\text{(A13)} \quad w(i,t,e) \leq \alpha_i \sum_{\xi=\max\{\tau, t-t_c\}}^{t} \phi(i,\xi) + w(j,t+1,e-\mu(i,j)),$$
$$i \in \boldsymbol{K}, \ j \in N(i,t,e), \ t = \tau, \cdots, T-1, \ e \in \boldsymbol{E}$$

$$\text{(A14)} \quad w(i,T,e) = \alpha_i \sum_{\xi=\max\{\tau, T-t_c\}}^{T} \phi(i,\xi), \ i \in \boldsymbol{K}, \ e \in \boldsymbol{E}$$

$$\sum_{i \in \boldsymbol{K}} \phi(i,\xi) \leq \Phi(\xi), \ \xi \in \widehat{\boldsymbol{T}}$$

$$\phi(i,\xi) \geq 0, \ i \in \boldsymbol{K}, \ \xi \in \widehat{\boldsymbol{T}}.$$

In the above problem, the optimization proceeds as follows. First of all, a nonnegative value $w(i,t,e)$ is given by Eq. (A14) for the last time $t = T$. Inequalities (A13) and (A12) are difference equations between values of $w(i,t,e)$ at neighbored times $t+1$ and $t$. While satisfying these inequalities, the problem is to maximize a value $w(i,1,e_0), \ i \in S_0$ at initial time $t = 1$. The objective function is the same as $w(i,1,e_0), \ i \in S_0$ but never depends on $w(i,1,e)$ for $i \notin S_0$ nor $e \neq e_0$ directly and indirectly. That is why we can set $\zeta(t) = 0, \ t \in \boldsymbol{T}$, to make the right-hand sides of Eqs. (A10) and (A11) zeros or we can erase these equations. Furthermore, we can change Eq. (A9) to $w(i,1,e_0) \geq \eta, \ i \in S_0$ keeping an optimal value unchanged. Finally we have reached Problem $\widetilde{P}^S$ as a dual problem to Problem $\widetilde{P}^T$.

## Acknowledgments

The authors thank the editor and an anonymous referee for their valuable comments and helpful suggestions, which help me to improve this paper.

## References

[1] V.J. Baston and F.A. Bostock, A One-Dimensional Helicopter-Submarine Game, *Naval Research Logistics*, **36**, pp.479–490, 1989.

[2] F. Dambreville and J.-P. Le Cadre, Search Game for a Moving Target with Dynamically Generated Informations, Proceedings of the 5th International Conference on Information Fusion (FUSION'2002), pp.243–250, 2002.

[3] J.M. Danskin, A Helicopter versus Submarine Search Game, *Operations Research*, **16**, pp.509–517, 1968.

[4] J.N. Eagle and A.R. Washburn, Cumulative Search-Evasion Games, *Naval Research Logistics*, **38**, pp.495–510, 1991.

[5] A.Y. Garnaev, A Remark on a Helicopter-Submarine Game, *Naval Research Logistics*, **40**, pp.745–753, 1993.

[6] A.Y. Garnaev, *Search Games and Other Applications of Game Theory*, Springer-Verlag, Tokyo, 2000.

[7] R. Hohzaki, Search Allocation Game, *European Journal of Operational Research*, **172**, pp.101–119, 2006.

[8] R. Hohzaki and K. Iida, A Search Game with Reward Criterion, *Journal of the Operations Research Society of Japan*, **41**, pp.629–642, 1998.

[9] R. Hohzaki and K. Iida, A Solution for a Two-Person Zero-Sum Game with a Concave Payoff Function, *Nonlinear Analysis and Convex Analysis*, World Science Publishing Co., London, pp.157–166, 1999.

[10] R. Hohzaki and K. Iida, A Search Game When a Search Path Is Given, *European Journal of Operational Research*, **124**, pp.114–124, 2000.

[11] R. Hohzaki, K. Iida and T. Komiya, Discrete search allocation game with energy constraints, *Journal of the Operations Research Society of Japan*, **45**, pp.93–108, 2002.

[12] R. Hohzaki, and A. Washburn, An Approximation for a Continuous Datum Search Game with Energy Constraint, *Journal of the Operations Research Society of Japan*, **46**, pp.306–318, 2003.

[13] K. Iida, R. Hohzaki and K. Sato, Hide-and-Search Game with the Risk Criterion, *Journal of the Operations Research Society of Japan*, **37**, pp.287–296, 1994.

[14] K. Iida, R. Hohzaki and S. Furui, A Search Game for a Mobile Target with the Conditionally Deterministic Motion Defined by Paths, *Journal of the Operations Research Society of Japan*, **39**(4), pp.501–511, 1996.

[15] K. Kikuta, A Hide and Seek Game with Traveling Cost, *Journal of the Operations Research Society of Japan*, **33**, pp.168–187, 1990.

[16] K. Kikuta, A Search Game with Traveling Cost, *Journal of the Operations Research Society of Japan*, **34**, pp.365–382, 1991.

[17] B.O. Koopman, *Search and Screening*, Pergamon, pp.221–227, 1980.

[18] J.J. Meinardi, A Sequentially Compounded Search Game, *Theory of Games: Techniquea and Applications*, The English Universities Press, London, pp.285–299, 1964.

[19] T. Nakai, A sequential evasion-search game with a goal. *Journal of the Operations Research Society of Japan*, **29**, pp.113–122, 1986.

[20] T. Nakai, Search Models wiht Continuous Effort under Various Criteria, *Journal of the Operations Research Society of Japan*, **31**, pp.335–351, 1988.

[21] A.R. Washburn, Search-Evasion Game in a Fixed Region, *Operations Research*, **28**, pp.1290–1298, 1980.

[22] A.R. Washburn and R. Hohzaki, The Diesel Submarine Flaming Datum Problem, *Military Operations Research*, **4**, pp.19–30, 2001.

Ryusuke Hohzaki
Department of Computer Science,
National Defense Academy,
1-10-20 Hashirimizu, Yokosuka,
239-8686, Japan
Phone: 046-841-3810 Ext. 3621
Fax: 046-844-5911
E-mail: `hozaki@cc.nda.ac.jp`