GA-OPTIMAL PARTIALLY BALANCED FRACTIONAL $2^{m_1+m_2}$ FACTORIAL DESIGNS OF RESOLUTION R($\{00, 10, 01, 11\}|\Omega$) WITH $2 \le m_1 \le m_2 \le 4$

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Received November 8, 2005; revised August 29, 2006

Abstract. Consider a partially balanced fractional $2^{m_1+m_2}$ factorial design derived from a simple partially balanced array such that the general mean, all the $m_1 + m_2$ main effects, some linear combinations of the $\binom{m_1}{2}$ two-factor interactions and of the $\binom{m_2}{2}$ two-factor ones and all the m_1m_2 two-factor ones are estimable, where the three-factor and higher-order interactions are assumed to be negligible, and $2 \le m_k$ ($k = 1, 2$). Furthermore we consider the situation in which the number of assemblies is less than the number of non-negligible factorial effects. Under these situations, this paper presents optimal designs with respect to the generalized A-optimality criterion, where $2 \le m_1 \le m_2 \le 4$.

1 Introduction The characteristic roots of the information matrix of a balanced fractional 2^m factorial (2^m-BFF) design of resolution V were obtained by Srivastava and Chopra [12]. Using the algebraic structure of the triangular multidimensional partially balanced (TMDPB) association scheme, Yamamoto et al. [15] extended their results to a 2^m-BFF design of resolution $2\ell + 1$, where $2\ell \le m$. A balanced array of two symbols and m constraints, which is a generalization of an orthogonal array, turns out to be a 2^m-BFF design under certain conditions (see [11, 14]).

As a special case of an asymmetrical balanced array introduced by Nishii [9], a partially balanced array (PBA) of two symbols was presented by Kuwada [3]. A PBA of strength m_1+m_2 is said to be simple, and such an array is briefly denoted by SPBA(m_1+m_2; $\{\lambda_{i_1,i_2}\}$), where λ_{i_1,i_2} are the indices of an SPBA (e.g., [7, 8]). A fractional factorial design derived from an array of two symbols and m_1+m_2 constraints is called a partially balanced fractional $2^{m_1+m_2}$ factorial ($2^{m_1+m_2}$-PBFF) design if the variance-covariance matrix of the estimators of the factorial effects to be of interest is invariant under any permutation within m_k factors for $k = 1, 2$ each. Under certain conditions, a PBA of two symbols and $m_1 + m_2$ constraints becomes a $2^{m_1+m_2}$-PBFF design (e.g., [3, 4]).

Kuwada et al. [7, 8] have obtained optimal $2^{m_1+m_2}$-PBFF designs with respect to the generalized A-optimality (GA-optimality) criterion such that the general mean and all the $m_1 + m_2$ main effects are estimable, and furthermore (A) (a) all the $\binom{m_1}{2}$ two-factor interactions, all the $\binom{m_2}{2}$ two-factor ones and some linear combinations of the m_1m_2 two-factor ones are estimable, and they are called resolution R($\{00, 10, 01, 20, 02\}|\Omega$) designs,

2000 *Mathematics Subject Classification.* 62K05, 05B30.
Key words and phrases. Estimable parametric functions, ETMDPB association algebra, GA-optimality criterion, PBFF designs, Resolution.

where $\Omega = \{00, 10, 01, 20, 02, 11\}$ and $2 \le m_1 \le m_2 \le 4$, (b) all the $\binom{m_1}{2}$ two-factor ones, some linear combinations of the $\binom{m_2}{2}$ two-factor ones and all the $m_1 m_2$ two-factor ones are estimable, and they are called resolution $\mathrm{R}(\{00, 10, 01, 20, 11\}|\Omega)$ designs, where $2 \le m_1, m_2 \le 4$, and (B) all the $\binom{m_1}{2}$ two-factor ones, and some linear combinations of the $\binom{m_2}{2}$ two-factor ones and of the $m_1 m_2$ two-factor ones are estimable, and they are called resolution $\mathrm{R}(\{00, 10, 01, 20\}|\Omega)$ designs, where $2 \le m_1, m_2 \le 4$.

In this paper, we consider a $2^{m_1+m_2}$-PBFF design such that the general mean, all the $m_1 + m_2$ main effects, some linear combinations of the $\binom{m_1}{2}$ two-factor interactions and of the $\binom{m_2}{2}$ two-factor ones and all the $m_1 m_2$ two-factor ones are estimable, where the three-factor and higher-order interactions are assumed to be negligible and $2 \le m_1 \le m_2 \le 4$, and it is called a resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ design. Furthermore, we present GA-optimal $2^{m_1+m_2}$-PBFF designs of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ when the number of assemblies (or treatment combinations) is less than the number of factorial effects up to the two-factor interactions.

2 Preliminaries We consider a fractional $2^{m_1+m_2}$ factorial design T with N assemblies, where the three-factor and higher-order interactions are assumed to be negligible and $2 \le m_k$ $(k = 1, 2)$. Then the $1 \times \nu(m_1, m_2)$ vector of the non-negligible factorial effects is given by $(\boldsymbol{\theta}'_{00}; \boldsymbol{\theta}'_{10}; \boldsymbol{\theta}'_{01}; \boldsymbol{\theta}'_{20}; \boldsymbol{\theta}'_{02}; \boldsymbol{\theta}'_{11})$ $(= \boldsymbol{\Theta}'$, say), where $\nu(m_1, m_2) = 1 + (m_1 + m_2) + \binom{m_1+m_2}{2}$, and $\boldsymbol{\theta}'_{00} = \{\theta(\phi; \phi)\}$, $\boldsymbol{\theta}'_{10} = \{\theta(u; \phi)|1 \le u \le m_1\}$, $\boldsymbol{\theta}'_{01} = \{\theta(\phi; v)|1 \le v \le m_2\}$, $\boldsymbol{\theta}'_{20} = \{\theta(u_1 u_2; \phi)|1 \le u_1 < u_2 \le m_1\}$, $\boldsymbol{\theta}'_{02} = \{\theta(\phi; v_1 v_2)|1 \le v_1 < v_2 \le m_2\}$ and $\boldsymbol{\theta}'_{11} = \{\theta(u; v)|1 \le u \le m_1, 1 \le v \le m_2\}$. Here $\theta(\phi; \phi)$ is the general mean, $\theta(u; \phi)$ and $\theta(\phi; v)$ are the main effects of the u-th factor in the m_1 factors and of the v-th factor in the m_2 factors, respectively, and $\theta(u_1 u_2; \phi)$, $\theta(\phi; v_1 v_2)$ and $\theta(u; v)$ are the two-factor interactions of the u_1-th and u_2-th factors in the m_1 factors, of the v_1-th and v_2-th factors in the m_2 factors, and of the u-th factor in the m_1 factors and the v-th factor in the m_2 factors, respectively. Thus the ordinary linear model based on T is given by

$$\boldsymbol{y}(T) = E_T \boldsymbol{\Theta} + \boldsymbol{e}_T,$$

where $\boldsymbol{y}(T)$, E_T and $\boldsymbol{e}_T$ are an observation vector of size $N \times 1$, the $N \times \nu(m_1, m_2)$ design matrix whose elements are either 1 or -1, and an $N \times 1$ error vector with mean $\boldsymbol{0}_N$ and variance-covariance matrix $\sigma^2 I_N$, respectively. Here $\boldsymbol{0}_p$ and I_p denote the $p \times 1$ null vector and the identity matrix of order p, respectively. The normal equations for estimating $\boldsymbol{\Theta}$ are given by

$$M_T \hat{\boldsymbol{\Theta}} = E'_T \boldsymbol{y}(T), \tag{2.1}$$

where M_T $(= E'_T E_T)$ is the information matrix of order $\nu(m_1, m_2)$.

Let $A^{(a_1 a_2, b_1 b_2)}_{\alpha_1 \alpha_2}$ and $D^{(a_1 a_2, b_1 b_2)}_{\alpha_1 \alpha_2}$ for $a_1 a_2, b_1 b_2 \in S_{\alpha_1 \alpha_2}$ $(\alpha_1 \alpha_2 \in S^*)$ be the local association matrices of size $n_{a_1 a_2} \times n_{b_1 b_2}$ and the ordered association matrices of order $\nu(m_1, m_2)$ of the extended TMDPB (ETMDPB) association scheme, respectively, where

$S_{00} = \{00, 10, 01, 20, 02, 11\}$, $S_{10} = \{10, 20 \text{ (if } m_1 \geq 3), 11\}$, $S_{01} = \{01, 02 \text{ (if } m_2 \geq 3), 11\}$, $S_{20} = \{20\}$ (if $m_1 \geq 4$), $S_{02} = \{02\}$ (if $m_2 \geq 4$), $S_{11} = \{11\}$, $S^* = \{00, 10, 01, 20 \text{ (if } m_1 \geq 4), 02 \text{ (if } m_2 \geq 4), 11\}$, and $n_{p_1p_2} = \binom{m_1}{p_1}\binom{m_2}{p_2}$ (see [3]). Note that $A^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$ and $D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$ are the (0,1) matrices. Further let $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ and $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ for $a_1a_2, b_1b_2 \in S_{\beta_1\beta_2}$ ($\beta_1\beta_2 \in S^*$) be the matrices of size $n_{a_1a_2} \times n_{b_1b_2}$ and of order $\nu(m_1, m_2)$, respectively. Then the relationships between $A^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$ and $A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$, and $D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$ and $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ are given by

$$\begin{aligned} &A^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2} \ (= A^{(b_1b_2,a_1a_2)'}_{\alpha_1\alpha_2}) = \textstyle\sum_{\beta_1\beta_2} z^{(a_1a_2,b_1b_2)}_{\beta_1\beta_2\alpha_1\alpha_2} A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}, \\ &D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2} \ (= D^{(b_1b_2,a_1a_2)'}_{\alpha_1\alpha_2}) = \textstyle\sum_{\beta_1\beta_2} z^{(a_1a_2,b_1b_2)}_{\beta_1\beta_2\alpha_1\alpha_2} D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \\ &\qquad\qquad\qquad\qquad \text{for } \alpha_k \leq a_k, b_k \leq 2 \text{ and } 0 \leq \alpha_k \leq 2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \ (= A^{\#(b_1b_2,a_1a_2)'}_{\beta_1\beta_2}) = \textstyle\sum_{\alpha_1\alpha_2} z^{\beta_1\beta_2\alpha_1\alpha_2}_{(a_1a_2,b_1b_2)} A^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}, \\ &D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} \ (= D^{\#(b_1b_2,a_1a_2)'}_{\beta_1\beta_2}) = \textstyle\sum_{\alpha_1\alpha_2} z^{\beta_1\beta_2\alpha_1\alpha_2}_{(a_1a_2,b_1b_2)} D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2} \\ &\qquad\qquad\qquad\qquad \text{for } \beta_k \leq a_k, b_k \leq 2 \text{ and } 0 \leq \beta_k \leq 2, \end{aligned} \tag{2.3}$$

where $z^{(a_1a_2,b_1b_2)}_{\beta_1\beta_2\alpha_1\alpha_2} = z^{(a_1,b_1)}_{\beta_1\alpha_1} z^{(a_2,b_2)}_{\beta_2\alpha_2}$ and $z^{\beta_1\beta_2\alpha_1\alpha_2}_{(a_1a_2,b_1b_2)} = z^{\beta_1\alpha_1}_{(a_1,b_1)} z^{\beta_2\alpha_2}_{(a_2,b_2)}$. Here

$$\begin{aligned} z^{(a_k,b_k)}_{\beta_k\alpha_k} \ (= z^{(b_k,a_k)}_{\beta_k\alpha_k}) &= \textstyle\sum_{p=0}^{\alpha_k} (-1)^{\alpha_k - p} \binom{a_k-\beta_k}{p} \binom{a_k-p}{a_k-\alpha_k} \binom{m_k-a_k-\beta_k+p}{p} \\ &\quad \times \sqrt{\binom{m_k-a_k-\beta_k}{b_k-a_k}\binom{b_k-\beta_k}{b_k-a_k}} \Big/ \binom{b_k-a_k+p}{p} \qquad \text{for } a_k \leq b_k, \\ z^{\beta_k\alpha_k}_{(a_k,b_k)} \ (= z^{\beta_k\alpha_k}_{(b_k,a_k)}) &= \phi_{\beta_k} z^{(a_k,b_k)}_{\beta_k\alpha_k} \Big/ \Big\{\binom{m_k}{a_k}\binom{a_k}{\alpha_k}\binom{m_k-a_k}{b_k-a_k+\alpha_k}\Big\} \qquad \text{for } a_k \leq b_k, \\ \phi_\beta &= \binom{m}{\beta} - \binom{m}{\beta-1} \end{aligned}$$

(see [10, 15]). The properties of these matrices are cited in the following:

$$\begin{aligned} &\textstyle\sum_{\beta_1\beta_2} A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} = I_{n_{a_1a_2}}, \ A^{\#(a_1a_2,c_1c_2)}_{\beta_1\beta_2} A^{\#(c_1c_2,b_1b_2)}_{\gamma_1\gamma_2} = \delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2} A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}, \\ &\text{rank}\{A^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}\} = \phi_{\beta_1\beta_2}, \\ &\textstyle\sum_{a_1a_2}\sum_{\beta_1\beta_2} D^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2} = I_{\nu(m_1,m_2)}, \\ &D^{\#(a_1a_2,c_1c_2)}_{\beta_1\beta_2} D^{\#(d_1d_2,b_1b_2)}_{\gamma_1\gamma_2} = \delta_{c_1d_1}\delta_{c_2d_2}\delta_{\beta_1\gamma_1}\delta_{\beta_2\gamma_2} D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}, \\ &\text{rank}\{D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}\} = \phi_{\beta_1\beta_2} \end{aligned} \tag{2.4}$$

(see [3]), where $\phi_{\beta_1\beta_2} = \phi_{\beta_1}\phi_{\beta_2}$ and δ_{pq} is the Kronecker delta.

Let $\mathcal{A}$ be the algebra generated by the linear closure of the ordered association matrices $D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2}$, and it is denoted by $[D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2} | a_1a_2, b_1b_2 \in S_{\alpha_1\alpha_2} \ (\alpha_1\alpha_2 \in S^*)]$. Then from (2.2) and (2.3), we obtain $\mathcal{A} = [D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2} | a_1a_2, b_1b_2 \in S_{\beta_1\beta_2} \ (\beta_1\beta_2 \in S^*)]$. Note that $\mathcal{A}$ is called the ETMDPB association algebra (see [3]). Using the properties of $\mathcal{A}$, the information matrix M_T is given by

$$\begin{aligned} M_T &= \textstyle\sum_{a_1a_2}\sum_{b_1b_2}\sum_{\alpha_1\alpha_2} \gamma_{|a_1-b_1|+2\alpha_1,|a_2-b_2|+2\alpha_2} D^{(a_1a_2,b_1b_2)}_{\alpha_1\alpha_2} \\ &= \textstyle\sum_{a_1a_2}\sum_{b_1b_2}\sum_{\beta_1\beta_2} \kappa^{a_1a_2,b_1b_2}_{\beta_1\beta_2} D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}, \end{aligned} \tag{2.5}$$

where T is a $2^{m_1+m_2}$-PBFF design derived from an SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$, and the relationships between γ_{i_1,i_2} and λ_{j_1,j_2}, and $\kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2}$ and γ_{i_1,i_2} are given by

$$\begin{aligned}\gamma_{i_1,i_2} &= \textstyle\sum_{j_1,j_2}[\sum_{p_1,p_2}\{\prod_{k=1}^{2}(-1)^{p_k}\binom{i_k}{p_k}\binom{m_k-i_k}{j_k-i_k+p_k}\}]\lambda_{j_1,j_2},\\ \kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2} &= \textstyle\sum_{\alpha_1\alpha_2} z_{\beta_1\beta_2\alpha_1\alpha_2}^{(a_1a_2,b_1b_2)}\gamma_{|a_1-b_1|+2\alpha_1,|a_2-b_2|+2\alpha_2},\end{aligned} \tag{2.6}$$

respectively. Thus from the properties of the algebra $\mathcal{A}$, the information matrix M_T is isomorphic to $||\kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2}||$ $(=K_{\beta_1\beta_2},$ say) of order 6 for $\beta_1\beta_2=00$, of order 3 (if $m_1\geq 3$) (or 2 (if $m_1=2$)) for $\beta_1\beta_2=10$, of order 3 (if $m_2\geq 3$) (or 2 (if $m_2=2$)) for $\beta_1\beta_2=01$, of order 1 (if $m_1\geq 4$) for $\beta_1\beta_2=20$, of order 1 (if $m_2\geq 4$) for $\beta_1\beta_2=02$ and of order 1 for $\beta_1\beta_2=11$ with multiplicities $\phi_{00},\ \phi_{10},\ \phi_{01},\ \phi_{20},\ \phi_{02}$ and ϕ_{11}, respectively (see [3]). Note that $K_{\beta_1\beta_2}$ are called the irreducible representations of M_T with respect to the ideals $[D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)}|a_1a_2,b_1b_2\in S_{\beta_1\beta_2}]$ $(=\mathcal{A}_{\beta_1\beta_2},$ say) of $\mathcal{A}$ for $\beta_1\beta_2\in S^*$. From (2.6), we have

$$K_{\beta_1\beta_2}=(D_{\beta_1\beta_2}F_{\beta_1\beta_2}\Lambda_{\beta_1\beta_2})(D_{\beta_1\beta_2}F_{\beta_1\beta_2}\Lambda_{\beta_1\beta_2})' \tag{2.7}$$

(see [7]), where

$$D_{00}=\mathrm{diag}[1;-1/\sqrt{m_1};-1/\sqrt{m_2};1/\sqrt{2m_1(m_1-1)};1/\sqrt{2m_2(m_2-1)};1/\sqrt{m_1m_2}\,],$$

$$D_{10}=\begin{cases}\mathrm{diag}[2;-2/\sqrt{m_2}\,] & \text{if } m_1=2,\\ \mathrm{diag}[2;-2/\sqrt{m_1-2};-2/\sqrt{m_2}\,] & \text{if } m_1\geq 3,\end{cases}$$

$$D_{01}=\begin{cases}\mathrm{diag}[2;-2/\sqrt{m_1}\,] & \text{if } m_2=2,\\ \mathrm{diag}[2;-2/\sqrt{m_2-2};-2/\sqrt{m_1}\,] & \text{if } m_2\geq 3,\end{cases}$$

$$D_{20}=\begin{cases}\text{vanishes} & \text{if } m_1=2,3,\\ 2^2 & \text{if } m_1\geq 4,\end{cases}\quad D_{02}=\begin{cases}\text{vanishes} & \text{if } m_2=2,3,\\ 2^2 & \text{if } m_2\geq 4,\end{cases}\quad D_{11}=2^2,$$

the column vectors of F_{00} corresponding to $\lambda_{a,x}$ $(0\leq a\leq m_1;\ 0\leq x\leq m_2)$, of F_{10} corresponding to $\lambda_{b,y}$ $(1\leq b\leq m_1-1;\ 0\leq y\leq m_2)$, of F_{01} corresponding to $\lambda_{c,z}$ $(0\leq c\leq m_1;\ 1\leq z\leq m_2-1)$, of F_{20} (if $m_1\geq 4$) corresponding to $\lambda_{d,u}$ $(2\leq d\leq m_1-2;\ 0\leq u\leq m_2)$, of F_{02} (if $m_2\geq 4$) corresponding to $\lambda_{e,v}$ $(0\leq e\leq m_1;\ 2\leq v\leq m_2-2)$ and of F_{11} corresponding to $\lambda_{f,w}$ $(1\leq f\leq m_1-1;\ 1\leq w\leq m_2-1)$ are given by $\sqrt{\lambda_{a,x}}(1,m_1-2a,m_2-2x,(m_1-2a)^2-m_1,(m_2-2x)^2-m_2,(m_1-2a)(m_2-2x))'$, $\sqrt{\lambda_{b,y}}(1,m_1-2b,m_2-2y)'$ (if $m_1\geq 3$) (or $\sqrt{\lambda_{1,y}}(1,m_2-2y)'$ (if $m_1=2$)), $\sqrt{\lambda_{c,z}}(1,m_2-2z,m_1-2c)'$ (if $m_2\geq 3$) (or $\sqrt{\lambda_{c,1}}(1,m_1-2c)'$ (if $m_2=2$)), $\sqrt{\lambda_{d,u}}$, $\sqrt{\lambda_{e,v}}$ and $\sqrt{\lambda_{f,w}}$, respectively, and the diagonal elements of $\Lambda_{\beta_1\beta_2}$ $(\beta_1\beta_2\in S^*)$ corresponding to $\lambda_{g,s}$ $(\beta_1\leq g\leq m_1-\beta_1;\ \beta_2\leq s\leq m_2-\beta_2)$ are given by $\sqrt{\binom{m_1-2\beta_1}{g-\beta_1}\binom{m_2-2\beta_2}{s-\beta_2}}$ ((i) if $\beta_1=\beta_2=0$, then $g=a$ and $s=x$, (ii) if $\beta_1=1$ and $\beta_2=0$, then $g=b$ and $s=y$, (iii) if $\beta_1=0$ and $\beta_2=1$, then $g=c$ and $s=z$, (iv) if $m_1\geq 4$, $\beta_1=2$ and $\beta_2=0$, then $g=d$ and $s=u$, (v) if $m_2\geq 4$, $\beta_1=0$ and $\beta_2=2$, then $g=e$ and $s=v$, and (vi) if $\beta_1=\beta_2=1$, then $g=f$ and $s=w$) and the off-diagonal elements of them are all zero. Note that F_{00} is of size $6\times\{(m_1+1)(m_2+1)\}$, F_{10} is of size $3\times\{(m_1-1)(m_2+1)\}$ (if $m_1\geq 3$) (or $2\times(m_2+1)$ (if $m_1=2$)), F_{01} is of size $3\times\{(m_1+1)(m_2-1)\}$ (if $m_2\geq 3$) (or $2\times(m_1+1)$ (if $m_2=2$)), F_{20} (if $m_1\geq 4$) is of size $1\times\{(m_1-3)(m_2+1)\}$, F_{02} (if $m_2\geq 4$) is of size $1\times\{(m_1+1)(m_2-3)\}$

and F_{11} is of size $1\times\{(m_1-1)(m_2-1)\}$, and $\Lambda_{\beta_1\beta_2}$ are of order $(m_1+1-2\beta_1)(m_2+1-2\beta_2)$.

Remark 2.1. From (2.5), the a_1a_2-th row block and the b_1b_2-th column block of $D^{\#(a_1a_2,b_1b_2)}_{\beta_1\beta_2}$ are concerned with $\boldsymbol{\theta}_{a_1a_2}$ and $\boldsymbol{\theta}_{b_1b_2}$, respectively. Thus from (2.7), the first, second, third, fourth, fifth and last rows of F_{00} correspond to $\boldsymbol{\theta}_{00},\ \boldsymbol{\theta}_{10},\ \boldsymbol{\theta}_{01},\ \boldsymbol{\theta}_{20},\ \boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$, respectively, the first, second (if $m_1\geq 3$) and last rows of F_{10} correspond to $\boldsymbol{\theta}_{10},\ \boldsymbol{\theta}_{20}$ and $\boldsymbol{\theta}_{11}$, respectively, the first, second (if $m_2\geq 3$) and last rows of F_{01} correspond to $\boldsymbol{\theta}_{01},\ \boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$, respectively, and the rows of F_{20} (if $m_1\geq 4$), F_{02} (if $m_2\geq 4$) and F_{11} corresponds to $\boldsymbol{\theta}_{20},\ \boldsymbol{\theta}_{02}$ and $\boldsymbol{\theta}_{11}$, respectively.

It follows from the definitions of $D_{\beta_1\beta_2}$, $F_{\beta_1\beta_2}$ and $\Lambda_{\beta_1\beta_2}$ that $\text{rank}\{K_{\beta_1\beta_2}\}$ $=\text{r-rank}\{F_{\beta_1\beta_2}\}$, where r-rank$\{A\}$ denotes the row rank of a matrix A.

Definition 2.1. Let $(T^{(1)};T^{(2)})$ $(=T,$ say) be an SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$, where $T^{(k)}$ are of size $N\times m_k$ $(k=1,2)$, and further let $\tilde{T}=(\bar{T}^{(1)};T^{(2)})$, $\breve{T}=(T^{(1)};\bar{T}^{(2)})$ and $\bar{T}=(\bar{T}^{(1)};\bar{T}^{(2)})$, where $\bar{T}^{(k)}$ denotes the complement of $T^{(k)}$. Then $\tilde{T}$, $\breve{T}$ and $\bar{T}$ are called the former complementary array (FCA) of T, the latter complementary array (LCA) of T and the completely complementary array (CCA) of T, respectively.

Note that if T is an SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$, then $\tilde{T}$, $\breve{T}$ and $\bar{T}$ are the SPBA$(m_1+m_2;\{\lambda_{m_1-i_1,i_2}\})$, SPBA$(m_1+m_2;\{\lambda_{i_1,m_2-i_2}\})$ and SPBA$(m_1+m_2;\{\lambda_{m_1-i_1,m_2-i_2}\})$, respectively. Let $M_{\tilde{T}}$, $M_{\breve{T}}$ and $M_{\bar{T}}$ be the information matrices associated with $\tilde{T}$, $\breve{T}$ and $\bar{T}$, respectively, where T is an SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$. Further let $\tilde{K}_{\beta_1\beta_2}$, $\breve{K}_{\beta_1\beta_2}$ and $\bar{K}_{\beta_1\beta_2}$, respectively, denote the irreducible representations of $M_{\tilde{T}}, M_{\breve{T}}$ and $M_{\bar{T}}$ with respect to the ideals $\mathcal{A}_{\beta_1\beta_2}$ of the algebra $\mathcal{A}$. Then from (2.7), we can get

$$\begin{aligned}(2.8)\qquad \tilde{K}_{\beta_1\beta_2}=\tilde{\Delta}_{\beta_1\beta_2}K_{\beta_1\beta_2}\tilde{\Delta}_{\beta_1\beta_2},\ \breve{K}_{\beta_1\beta_2}&=\breve{\Delta}_{\beta_1\beta_2}K_{\beta_1\beta_2}\breve{\Delta}_{\beta_1\beta_2},\\ \bar{K}_{\beta_1\beta_2}&=\bar{\Delta}_{\beta_1\beta_2}K_{\beta_1\beta_2}\bar{\Delta}_{\beta_1\beta_2}\quad \text{for } \beta_1\beta_2\in S^*\end{aligned}$$

(see [7]), where $\tilde{\Delta}_{00}=\text{diag}[1;-1;1;1;1;-1]$, $\breve{\Delta}_{00}=\text{diag}[1;1;-1;1;1;-1]$, $\bar{\Delta}_{00}=\text{diag}[1;-1;-1;1;1;1]$, $\tilde{\Delta}_{10}=\text{diag}[1;-1;1]$ (if $m_1\geq 3$) (or diag$[1;1]$ (if $m_1=2$)), $\breve{\Delta}_{10}=\text{diag}[1;1;-1]$ (if $m_1\geq 3$) (or diag$[1;-1]$ (if $m_1=2$)), $\bar{\Delta}_{10}=\text{diag}[1;-1;-1]$ (if $m_1\geq 3$) (or diag$[1;-1]$ (if $m_1=2$)), $\tilde{\Delta}_{01}=\text{diag}[1;1;-1]$ (if $m_2\geq 3$) (or diag$[1;-1]$ (if $m_2=2$)), $\breve{\Delta}_{01}=\text{diag}[1;-1;1]$ (if $m_2\geq 3$) (or diag$[1;1]$ (if $m_2=2$)), $\bar{\Delta}_{01}=\text{diag}[1;-1;-1]$ (if $m_2\geq 3$) (or diag$[1;-1]$ (if $m_2=2$)), $\tilde{\Delta}_{20}=\breve{\Delta}_{20}=\bar{\Delta}_{20}=1$ (if $m_1\geq 4$) (or vanishes (if $m_1=2,3$)), $\tilde{\Delta}_{02}=\breve{\Delta}_{02}=\bar{\Delta}_{02}=1$ (if $m_2\geq 4$) (or vanishes (if $m_2=2,3$)) and $\tilde{\Delta}_{11}=\breve{\Delta}_{11}=\bar{\Delta}_{11}=1$.

3 Estimable parametric functions Linear parametric functions $C\boldsymbol{\Theta}$ of $\boldsymbol{\Theta}$ are estimable for some matrix C of order $\nu(m_1,m_2)$ if and only if there exists a matrix X of order $\nu(m_1,m_2)$ such that $XM_T=C$ (e.g., [13]). In this section, we consider a $2^{m_1+m_2}$-PBFF design of resolution R$(\{00,10,01,11\}|\Omega)$ derived from an SPBA$(m_1+m_2;\{\lambda_{i_1,i_2}\})$.

Since M_T belongs to the ETMDPB association algebra $\mathcal{A}$, we impose some restrictions on C and X such that

$$
\begin{aligned}
(3.1)\quad C = {} & D_{00}^{\#(00,00)} + \{D_{00}^{\#(10,10)} + D_{10}^{\#(10,10)}\} + \{D_{00}^{\#(01,01)} + D_{01}^{\#(01,01)}\} \\
& +\{g_{00}^{20,20} D_{00}^{\#(20,20)} + g_{10}^{20,20} D_{10}^{\#(20,20)} \text{ (if } m_1 \geq 3) + g_{20}^{20,20} D_{20}^{\#(20,20)} \text{ (if } m_1 \geq 4)\} \\
& +\{g_{00}^{20,02} D_{00}^{\#(20,02)} + g_{00}^{02,20} D_{00}^{\#(02,20)}\} \\
& +\{g_{00}^{02,02} D_{00}^{\#(02,02)} + g_{01}^{02,02} D_{01}^{\#(02,02)} \text{ (if } m_2 \geq 3) + g_{02}^{02,02} D_{02}^{\#(02,02)} \text{ (if } m_2 \geq 4)\} \\
& +\{D_{00}^{\#(11,11)} + D_{10}^{\#(11,11)} + D_{01}^{\#(11,11)} + D_{11}^{\#(11,11)}\}, \\
X = {} & \textstyle\sum_{a_1a_2} \sum_{b_1b_2} \sum_{\beta_1\beta_2} \chi_{\beta_1\beta_2}^{a_1a_2,b_1b_2} D_{\beta_1\beta_2}^{\#(a_1a_2,b_1b_2)},
\end{aligned}
$$

respectively, where $g_{\gamma_1\gamma_2}^{a_1a_2,b_1b_2}$ are some constants, and $\chi_{\beta_1\beta_2}^{a_1a_2,b_1b_2}$ are also some constants which depend on $\kappa_{\beta_1\beta_2}^{a_1a_2,b_1b_2}$ and $g_{\beta_1\beta_2}^{a_1a_2,b_1b_2}$. It then follows from the properties of $\mathcal{A}$ that the matrices X and C are isomorphic to $X_{\beta_1\beta_2}$ and $\Gamma_{\beta_1\beta_2}$ for $\beta_1\beta_2 \in S^*$, respectively. Here

$$
\begin{aligned}
& X_{\beta_1\beta_2} = ||\chi_{\beta_1\beta_2}^{a_1a_2,b_1b_2}||, \\
& \Gamma_{00} = \mathrm{diag}[I_3; \begin{pmatrix} g_{00}^{20,20} & g_{00}^{20,02} \\ g_{00}^{02,20} & g_{00}^{02,02} \end{pmatrix}; 1], \\
& \Gamma_{10} = \begin{cases} I_2 & \text{if } m_1 = 2, \\ \mathrm{diag}[1; g_{10}^{20,20}; 1] & \text{if } m_1 \geq 3, \end{cases} \qquad \Gamma_{01} = \begin{cases} I_2 & \text{if } m_2 = 2, \\ \mathrm{diag}[1; g_{01}^{02,02}; 1] & \text{if } m_2 \geq 3, \end{cases} \\
& \Gamma_{20} = \begin{cases} \text{vanishes} & \text{if } m_1 = 2, 3, \\ g_{20}^{20,20} & \text{if } m_1 \geq 4, \end{cases} \qquad \Gamma_{02} = \begin{cases} \text{vanishes} & \text{if } m_2 = 2, 3, \\ g_{02}^{02,02} & \text{if } m_2 \geq 4, \end{cases} \qquad \Gamma_{11} = 1.
\end{aligned}
$$

Let $M_T^* = P'M_TP$, $X^* = P'XP$, $C^* = P'CP$, and $\boldsymbol{\Theta}^* = P'\boldsymbol{\Theta}$, where $P = \mathrm{diag}[I_{1+m_1+m_2}; \begin{pmatrix} 0 & I_{\binom{m_1}{2}} & 0 \\ 0 & 0 & I_{\binom{m_2}{2}} \\ I_{m_1m_2} & 0 & 0 \end{pmatrix}]$. If there exists X such that $XM_T = C$, then there also exists X^* such that $X^*M_T^* = C^*$, and vice versa. Thus the estimability of $C\boldsymbol{\Theta}$ is equivalent to that of $C^*\boldsymbol{\Theta}^*$. The matrices M_T^*, X^* and C^* are isomorphic to $K_{\beta_1\beta_2}^*$, $X_{\beta_1\beta_2}^*$ and $\Gamma_{\beta_1\beta_2}^*$ for $\beta_1\beta_2 \in S^*$, respectively. Here $K_{00}^* = P_{00}'K_{00}P_{00}$, $K_{\gamma_1\gamma_2}^* = P_{\gamma_1\gamma_2}'K_{\gamma_1\gamma_2}P_{\gamma_1\gamma_2}$, $K_{\omega_1\omega_2}^* = K_{\omega_1\omega_2}$, $X_{00}^* = P_{00}'X_{00}P_{00}$, $X_{\gamma_1\gamma_2}^* = P_{\gamma_1\gamma_2}'X_{\gamma_1\gamma_2}P_{\gamma_1\gamma_2}$, $X_{\omega_1\omega_2}^* = X_{\omega_1\omega_2}$, $\Gamma_{00}^* = P_{00}'\Gamma_{00}P_{00}$, $\Gamma_{\gamma_1\gamma_2}^* = P_{\gamma_1\gamma_2}'\Gamma_{\gamma_1\gamma_2}P_{\gamma_1\gamma_2}$ and $\Gamma_{\omega_1\omega_2}^* = \Gamma_{\omega_1\omega_2}$ for $\gamma_1\gamma_2 = 10$ (if $m_1 \geq 3$), 01 (if $m_2 \geq 3$), and $\omega_1\omega_2 = 10$ (if $m_1 = 2$), 01 (if $m_2 = 2$), 20 (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$), 11, where $P_{00} = \mathrm{diag}[I_3; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}]$ and $P_{\gamma_1\gamma_2} = \mathrm{diag}[1; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]$. Then $X^*M_T^* = C^*$ is isomorphic to $X_{\beta_1\beta_2}^*K_{\beta_1\beta_2}^* = \Gamma_{\beta_1\beta_2}^*$ for $\beta_1\beta_2 \in S^*$. Note that if $C\boldsymbol{\Theta}$ is estimable (and hence $C^*\boldsymbol{\Theta}^*$ is also estimable), where C is given by (3.1), then a design is of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$.

If $N \geq \nu(m_1, m_2)$, then there exists a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\Omega|\Omega)$, i.e., of resolution V (e.g., [3]). Thus in this paper, we would like to focus the attention on obtaining a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$. Since $N < \nu(m_1, m_2)$, the information matrix M_T is singular, and hence at least one of $K^*_{\beta_1\beta_2}$ ($\beta_1\beta_2 \in S^*$) is singular, which yields that at least one of $F_{\beta_1\beta_2}$ is not of full row rank. If $F_{\gamma_1\gamma_2}$ ($\gamma_1\gamma_2 = 00, 10, 01, 20$ (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$)) is of full row rank (and hence $K^*_{\gamma_1\gamma_2}$ is of full rank), then in the matrix equation $X^*_{\gamma_1\gamma_2}K^*_{\gamma_1\gamma_2} = \mathit{\Gamma}^*_{\gamma_1\gamma_2}$, there always exists $X^*_{\gamma_1\gamma_2}$ such that $X^*_{\gamma_1\gamma_2} = (K^*_{\gamma_1\gamma_2})^{-1}$. Hence $\mathit{\Gamma}^*_{\gamma_1\gamma_2}$ is the identity matrix. Thus if $F_{\gamma_1\gamma_2}$ is of full row rank, then without loss of generality, we can put $g^{a_1a_2,b_1b_2}_{\gamma_1\gamma_2} = 1$ ($\gamma_1\gamma_2 = 00, 10$ (if $m_1 \geq 3$), 01 (if $m_2 \geq 3$), 20 (if $m_1 \geq 4$), 02 (if $m_2 \geq 4$)) if $a_1a_2 = b_1b_2$, and $g^{a_1a_2,b_1b_2}_{00} = 0$ if $a_1a_2 \neq b_1b_2$.

Theorem 3.1. *Let T be a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an $\mathrm{SPBA}(m_1+m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$. Then we have that $\text{r-rank}\{F_{11}\} = 1$, and hence $A^{\#(11,11)}_{11}\boldsymbol{\theta}_{11}$ is estimable, and furthermore that the following holds*:

(I) *If the matrix $F_{\beta_1\beta_2}$ is of full row rank, then $A^{\#(a_1a_2,a_1a_2)}_{\beta_1\beta_2}\boldsymbol{\theta}_{a_1a_2}$ are estimable for $a_1a_2 \in S_{\beta_1\beta_2}$ and $\beta_1\beta_2 \in S^*$,*

(II) (i) (A) *if $\text{r-rank}\{F_{00}\} = 4$ and the fourth and fifth rows of F_{00} are zero, then $A^{\#(a_1a_2,a_1a_2)}_{00} \times \boldsymbol{\theta}_{a_1a_2}$ ($a_1a_2 = 00, 10, 01, 11$) are estimable, and*

(B) *if $\text{r-rank}\{F_{00}\} = 5$ and the first three and last rows of F_{00} are linearly independent, then $A^{\#(a_1a_2,a_1a_2)}_{00}\boldsymbol{\theta}_{a_1a_2}$ ($a_1a_2 = 00, 10, 01, 11$) are estimable, and moreover*

(a) *if the fifth row of F_{00} is zero, then $g^{20,20}_{00}A^{\#(20,20)}_{00}\boldsymbol{\theta}_{20}$ and $g^{02,20}_{00}A^{\#(02,20)}_{00}\boldsymbol{\theta}_{20} = g^{02,20}_{00}A^{\#(02,20)}_{00}(A^{\#(20,20)}_{00}\boldsymbol{\theta}_{20})$ are estimable,*

(b) *if the fourth row of F_{00} is zero, then $g^{02,02}_{00}A^{\#(02,02)}_{00}\boldsymbol{\theta}_{02}$ and $g^{20,02}_{00}A^{\#(20,02)}_{00}\boldsymbol{\theta}_{02} = g^{20,02}_{00}A^{\#(20,02)}_{00}(A^{\#(02,02)}_{00}\boldsymbol{\theta}_{02})$ are estimable, and*

(c) *if the fifth row of F_{00} equals w_{00} ($\neq 0$) times the fourth, then*
$g^{20,20}_{00}A^{\#(20,20)}_{00}\boldsymbol{\theta}_{20} + g^{20,02}_{00}A^{\#(20,02)}_{00}\boldsymbol{\theta}_{02} = g^{20,20}_{00}(A^{\#(20,20)}_{00}\boldsymbol{\theta}_{20} + w^*_{00}A^{\#(20,02)}_{00}\boldsymbol{\theta}_{02})$,
$g^{02,20}_{00}A^{\#(02,20)}_{00}\boldsymbol{\theta}_{20} + g^{02,02}_{00}A^{\#(02,02)}_{00}\boldsymbol{\theta}_{02}$
$= g^{02,20}_{00}A^{\#(02,20)}_{00}(A^{\#(20,20)}_{00}\boldsymbol{\theta}_{20} + w^*_{00}A^{\#(20,02)}_{00}\boldsymbol{\theta}_{02})$ *are estimable, where*
$g^{a_1a_2,02}_{00} = w^*_{00}g^{a_1a_2,20}_{00}$ *($a_1a_2 = 20, 02$) and* $w^*_{00} = \sqrt{m_1(m_1-1)/\{m_2(m_2-1)\}}w_{00}$,

(ii) *if $m_1 \geq 3$, $\text{r-rank}\{F_{10}\} = 2$ and the second row of F_{10} is zero, then $A^{\#(b_1b_2,b_1b_2)}_{10} \times \boldsymbol{\theta}_{b_1b_2}$ ($b_1b_2 = 10, 11$) are estimable,*

(iii) *if $m_2 \geq 3$, $\text{r-rank}\{F_{01}\} = 2$ and the second row of F_{01} is zero, then $A^{\#(c_1c_2,c_1c_2)}_{01} \times \boldsymbol{\theta}_{c_1c_2}$ ($c_1c_2 = 01, 11$) are estimable.*

Proof. From (2.4), (3.1), Remark 2.1 and Lemma A.1, the results can be easily proved.

Remark 3.1. It follows from Lemma A.1 that in Theorem 3.1(II)(i)(B), since $g^{a_1a_2,b_1b_2}_{00}$ ($a_1a_2, b_1b_2 = 20, 02$) are arbitrary, without loss of generality, we can put $g^{20,20}_{00} = 1$ and $g^{02,20}_{00} \neq 0$ for (a), and $g^{02,02}_{00} = 1$ and $g^{20,02}_{00} \neq 0$ for (b). Furthermore we define $g^{20,20}_{00}$ ($= g^{20,20}_{00}(\alpha)$, say) $= 1$ if $\alpha = 0$, $1/(1+|w^*_{00}|)$ if $\alpha = 1$ and $1/\sqrt{1+(w^*_{00})^2}$ if $\alpha = 2$, and $g^{02,20}_{00} \neq 0$ for (c).

From the relations among the rows of $F_{\beta_1\beta_2}$, and applying Lemma A.1 to the matrix equations $X^*_{\beta_1\beta_2}K^*_{\beta_1\beta_2} = \mathit{\Gamma}^*_{\beta_1\beta_2}$, we have the following:

Lemma 3.1. *A necessary condition for T to be a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an $\mathrm{SPBA}(m_1+m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$ is that $\text{r-rank}\{F_{11}\} = 1$, and in addition*

(a) r-rank$\{F_{00}\} = 4$ *and the fourth and fifth rows of* F_{00} *are zero,*

(b) r-rank$\{F_{00}\} = 5$ *and the fourth row of* F_{00} *is zero, and furthermore*

(i) $m_2 \geq 3$, r-rank$\{F_{01}\} = 2$ *and the second row of* F_{01} *is zero, or*

(ii) $m_2 \geq 4$ *and* r-rank$\{F_{02}\} = 0$,

(c) r-rank$\{F_{00}\} = 5$ *and the fifth row of* F_{00} *equals* w_{00} $(\neq 0)$ *times the fourth,*

(d) $m_1 \geq 3$, r-rank$\{F_{10}\} = 2$ *and the second row of* F_{10} *is zero, and furthermore*

(i) r-rank$\{F_{00}\} = 5$ *and the fifth row of* F_{00} *is zero,*

(ii) $m_2 \geq 3$, r-rank$\{F_{01}\} = 2$ *and the second row of* F_{01} *is zero, or*

(iii) $m_2 \geq 4$ *and* r-rank$\{F_{02}\} = 0$, *or*

(e) $m_1 \geq 4$ *and* r-rank$\{F_{20}\} = 0$, *and furthermore*

(i) r-rank$\{F_{00}\} = 5$ *and the fifth row of* F_{00} *is zero,*

(ii) $m_2 \geq 3$, r-rank$\{F_{01}\} = 2$ *and the second row of* F_{01} *is zero, or*

(iii) $m_2 \geq 4$ *and* r-rank$\{F_{02}\} = 0$.

In Lemma 3.1, it can be easily shown that there does not exist a $2^{m_1+m_2}$-PBFF design of resolution R$(\{00, 10, 01, 11\}|\Omega)$ derived from an SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$ and $2 \leq m_1, m_2 \leq 4$ satisfying the conditions (a)-(d) and (e)(i),(ii).

If $(T^{(1)}; T^{(2)})$ is an SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$, then $(T^{(2)}; T^{(1)})$ is also the SPBA$(m_2 + m_1; \{\lambda^*_{i_1,i_2}\})$, where $\lambda^*_{i_1,i_2} = \lambda_{i_2,i_1}$. Thus if $(T^{(1)}; T^{(2)})$ derived from an SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ is of resolution R$(\{00, 10, 01, 11\}|\Omega)$, then $(T^{(2)}; T^{(1)})$ is also of resolution R$(\{00, 10, 01, 11\}|\Omega)$, and hence we only consider the case $2 \leq m_1 \leq m_2$.

Theorem 3.2. *Let* T *be an* SPBA$(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ *with* $N < \nu(m_1, m_2)$, *where* $2 \leq m_1 \leq m_2 \leq 4$. *Then* T *is a* $2^{m_1+m_2}$-PBFF *design of resolution* R$(\{00, 10, 01, 11\}|\Omega)$ *if and only if one of the following holds, or one of its* FCA, LCA *and* CCA *holds*:

(I) *When* $m_1 = 2, 3$ *and* $m_1 \leq m_2 \leq 4$, *there does not exist a design of resolution* R$(\{00, 10, 01, 11\}|\Omega)$,

(II) *when* $m_1 = m_2 = 4$ $(\nu(4, 4) = 37)$, $\lambda_{1,1} = 1$ *and* $\lambda_{a,2} = \lambda_{1,3} = \lambda_{2,x} = \lambda_{3,1} = \lambda_{3,3} = 0$ $(0 \leq a \leq 4;\ x = 0, 1, 3, 4)$, *and furthermore*

(i) *exactly three of* $\{\lambda_{0,1}, \lambda_{0,3}, \lambda_{4,1}, \lambda_{4,3}\}$ *are* 1, *exactly two of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{1,0}, \lambda_{1,4}\}$ *are* 1 *and* $\lambda_{0,0} = \lambda_{0,4} = \lambda_{4,0} = \lambda_{4,4} = 0$, *or*

(ii) (a) *exactly two of* $\{\lambda_{0,1}, \lambda_{0,3}, \lambda_{4,1}, \lambda_{4,3}\}$ *except for* $\{\lambda_{0,1}, \lambda_{4,1}\}$, $\{\lambda_{0,3}, \lambda_{4,1}\}$ *and* $\{\lambda_{0,3}, \lambda_{4,3}\}$ *are* 1, *and moreover*

(1) *exactly three of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *are* 1 *and* $\lambda_{0,0} = \lambda_{0,4} = \lambda_{4,0} = \lambda_{4,4} = 0$, *or*

(2) *exactly two of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{1,0}, \lambda_{1,4}\}$ *are* 1 *and* $1 \leq \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \leq 4$,

(b) $\lambda_{0,3} = \lambda_{4,1} = 1$ *and* $\lambda_{0,1} = \lambda_{4,3} = 0$, *and moreover*

(1) *exactly three of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *are* 1 *and* $\lambda_{0,0} = \lambda_{0,4} = \lambda_{4,0} = \lambda_{4,4} = 0$,

(2) *exactly two of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{1,0}, \lambda_{1,4}\}$ *and* $\{\lambda_{3,0}, \lambda_{3,4}\}$ *are* 1 *and* $1 \leq \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \leq 4$, *or*

(3) $\lambda_{3,0} = \lambda_{3,4} = 1,\ 1 \le \lambda_{0,0} + \lambda_{4,0} + \lambda_{4,4},\ \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 4$ *and* $\lambda_{1,0} = \lambda_{1,4} = 0$, *or*

(c) $\lambda_{0,3} = \lambda_{4,3} = 1$ *and* $\lambda_{0,1} = \lambda_{4,1} = 0$, *and moreover*

(1) *exactly three of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *are* 1 *and* $\lambda_{0,0} = \lambda_{0,4} = \lambda_{4,0} = \lambda_{4,4} = 0$,

(2) *exactly two of* $\{\lambda_{1,0}, \lambda_{1,4}, \lambda_{3,0}, \lambda_{3,4}\}$ *except for* $\{\lambda_{1,0}, \lambda_{1,4}\}$ *and* $\{\lambda_{1,4}, \lambda_{3,0}\}$ *are* 1 *and* $1 \le \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 4$, *or*

(3) $\lambda_{1,4} = \lambda_{3,0} = 1,\ 1 \le \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,4},\ \lambda_{0,0} + \lambda_{0,4} + \lambda_{4,0} + \lambda_{4,4} \le 4$ *and* $\lambda_{1,0} = \lambda_{3,4} = 0$.

Proof. Checking the sufficiency of Lemma 3.1 for given indices λ_{i_1,i_2}, the results can be easily obtained.

Remark 3.2. In Theorem 3.2, the matrices $F_{\gamma_1\gamma_2}$ ($\gamma_1\gamma_2 = 00, 10, 01, 11$) are of full row rank and r-rank$\{F_{20}\}$ =r-rank$\{F_{02}\} = 0$. Furthermore we have $N = 36$ for (II)(i), (ii)(a)(1), (b)(1), (c)(1), and $33 \le N \le 36$ for (II)(ii)(a)(2), (b)(2), (3), (c)(2), (3).

Example 3.1. Let T be the SPBA($m_1 + m_2 = 4 + 4; \{\lambda_{0,0} = \lambda_{0,3} = \lambda_{1,1} = \lambda_{3,0} = \lambda_{3,4} = \lambda_{4,3} = 1,\ \lambda_{j_1,j_2} = 0\ (j_1 j_2 \ne 00, 03, 11, 30, 34, 43)\}$), which is GA-optimal as in Table 4.1 of Section 4. Then T is given by

$$T' = \begin{pmatrix} 0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&1&1&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&0&1&1&1&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&1&1&1&0&1&1&1&1&1&1&1 \\ 0&1&1&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0 \\ 0&1&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&1&1&1&1&1&0&1 \\ 0&1&0&1&1&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&1&1&1&1&1&0&1&1 \\ 0&0&1&1&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&1&1&1&1&0&1&1&1 \end{pmatrix}$$

which yields that

$$F_{00} = \begin{pmatrix} 1&0&0&1&0&0&1&0&&0&1&0&&0&1&0&&0&1&0 \\ 4&0&0&4&0&0&2&0&&0&-2&0&&0&-2&0&&0&-4&0 \\ 4&0&0&-2&0&0&2&0&\cdots\cdots&0&4&0&\cdots&0&-4&0&\cdots&0&-2&0 \\ 12&0&0&12&0&0&0&0&&0&0&0&&0&0&0&&0&12&0 \\ 12&0&0&0&0&0&0&0&&0&12&0&&0&12&0&&0&0&0 \\ 16&0&0&-8&0&0&4&0&&0&-8&0&&0&8&0&&0&8&0 \end{pmatrix} \quad (6 \times 25),$$

$$F_{10} = \begin{pmatrix} 0&1&0&&0&1&0&&0&1 \\ 0&2&0&\cdots\cdots&0&-2&0&\cdots&0&-2 \\ 0&2&0&&0&4&0&&0&-4 \end{pmatrix} \quad (3 \times 15),$$

$$F_{01} = \begin{pmatrix} 0&0&1&1&0&&0&1 \\ 0&0&-2&2&0&\cdots\cdots&0&-2 \\ 0&0&4&4&0&&0&-4 \end{pmatrix} \quad (3 \times 15),$$

$$F_{20} = \boldsymbol{0}\ (1 \times 5), \quad F_{02} = \boldsymbol{0}\ (1 \times 5), \quad F_{11} = \begin{pmatrix} 1 & 0 \cdots\cdots 0 \end{pmatrix}\ (1 \times 9).$$

Thus the matrices $F_{\beta_1\beta_2}$ $(\beta_1\beta_2 = 00, 10, 01, 11)$ are of full row rank, and r-rank$\{F_{20}\}$ =r-rank$\{F_{02}\} = 0$. Hence from (2.4), $A_{00}^{\#(00,00)}\boldsymbol{\theta}_{00}$, i.e., $\boldsymbol{\theta}_{00}$, $A_{00}^{\#(10,10)}\boldsymbol{\theta}_{00}$ and $A_{10}^{\#(10,10)}\boldsymbol{\theta}_{10}$, i.e., $\boldsymbol{\theta}_{10}$, $A_{00}^{\#(01,01)}\boldsymbol{\theta}_{01}$ and $A_{01}^{\#(01,01)}\boldsymbol{\theta}_{01}$, i.e., $\boldsymbol{\theta}_{01}$, $A_{00}^{\#(20,20)}\boldsymbol{\theta}_{20}$ and $A_{10}^{\#(20,20)}\boldsymbol{\theta}_{20}$, $A_{00}^{\#(02,02)}\boldsymbol{\theta}_{02}$ and $A_{01}^{\#(02,02)}\boldsymbol{\theta}_{02}$, $A_{00}^{\#(11,11)}\boldsymbol{\theta}_{11}$, $A_{10}^{\#(11,11)}\boldsymbol{\theta}_{11}$, $A_{01}^{\#(11,11)}\boldsymbol{\theta}_{11}$ and $A_{11}^{\#(11,11)}\boldsymbol{\theta}_{11}$, i.e., $\boldsymbol{\theta}_{11}$ are estimable, but $A_{20}^{\#(20,20)}\boldsymbol{\theta}_{20}$ and $A_{02}^{\#(02,02)}\boldsymbol{\theta}_{02}$ are not estimable. Therefore T is of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$.

4 GA-optimal designs In this section, we present GA-optimal $2^{m_1+m_2}$-PBFF designs of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from SPBAs$(m_1{+}m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$, where $2 \le m_1 \le m_2 \le 4$. Since $\boldsymbol{\Theta}^* = P'\boldsymbol{\Theta}$ and $C^* = P'CP$, where P is the permutation matrix given in the previous section, $C^*\boldsymbol{\Theta}^*$ is estimable if and only if $C\boldsymbol{\Theta}$ is estimable. Thus if $C\boldsymbol{\Theta}$ is estimable (and hence there exists a matrix X such that $XM_T = C$), then its unbiased estimator is given by $C\hat{\boldsymbol{\Theta}}$, where $\hat{\boldsymbol{\Theta}}$ is a solution of the equations (2.1), and furthermore $\mathrm{Var}[C\hat{\boldsymbol{\Theta}}] = \sigma^2 XM_TX'$. Here $\mathrm{Var}[\boldsymbol{y}]$ denotes the variance-covariance matrix of a random vector $\boldsymbol{y}$. By use of the algebraic structure of the ETMDPB association scheme, XM_TX' is isomorphic to $X_{\beta_1\beta_2}K_{\beta_1\beta_2}X'_{\beta_1\beta_2}$ for $\beta_1\beta_2 \in S^*$.

Let $\sigma^2 V_T$ be the variance-covariance matrix of the linearly independent estimators in $C\hat{\boldsymbol{\Theta}}$. Then from Lemma A.2, we have the following:

Lemma 4.1. *Let T be a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an $\mathrm{SPBA}(m_1{+}m_2; \{\lambda_{i_1,i_2}\})$ with $N < \nu(m_1, m_2)$. Then the matrix V_T $(= V_T(\alpha)$, say$)$ is isomorphic to $V_{\beta_1\beta_2}(\alpha)$ $(\beta_1\beta_2 \in S^*)$ for $0 \le \alpha \le 2$, where*

$$V_{\beta_1\beta_2}(\alpha) = (K_{\beta_1\beta_2})^{-1} \quad \text{if } F_{\beta_1\beta_2} \text{ is of full row rank},$$

$$V_{00}(\alpha) = \begin{cases} (K^a_{00})^{-1} & \text{if r-rank}\{F_{00}\} = 4 \text{ and the fourth and fifth rows of } F_{00} \text{ are zero}, \\ (K^b_{00})^{-1} & \text{if r-rank}\{F_{00}\} = 5 \text{ and the fifth row of } F_{00} \text{ is zero}, \\ (K^c_{00})^{-1} & \text{if r-rank}\{F_{00}\} = 5 \text{ and the fourth row of } F_{00} \text{ is zero}, \\ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} (K^b_{00})^{-1} \begin{pmatrix} I_3 & 0 & 0 \\ 0 & g_{00}^{20,20}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} & \\ \quad \text{if r-rank}\{F_{00}\} = 5 \text{ and the fifth row of } F_{00} \text{ equals } w_{00}\ (\neq 0) \text{ times the fourth}, & \end{cases}$$

$$V_{10}(\alpha) = (K^a_{10})^{-1} \quad \text{if } m_1 \ge 3,\ \text{r-rank}\{F_{10}\} = 2 \text{ and the second row of } F_{10} \text{ is zero},$$

$$V_{01}(\alpha) = (K^a_{01})^{-1} \quad \text{if } m_2 \ge 3,\ \text{r-rank}\{F_{01}\} = 2 \text{ and the second row of } F_{01} \text{ is zero},$$

$$V_{20}(\alpha) = \begin{cases} 0 & \textit{if } m_1 \geq 4 \textit{ and } \text{r-rank}\{F_{20}\} = 0, \\ \textit{vanishes} & \textit{if } m_1 = 2, 3, \end{cases}$$

$$V_{02}(\alpha) = \begin{cases} 0 & \textit{if } m_2 \geq 4 \textit{ and } \text{r-rank}\{F_{02}\} = 0, \\ \textit{vanishes} & \textit{if } m_2 = 2, 3. \end{cases}$$

Here K_{00}^a, K_{00}^b and K_{00}^c *are the* 4×4, 5×5 *and* 5×5 *submatrices of* K_{00} *corresponding to the first three, and furthermore the last, the fourth and last, and the fifth and last rows and columns, respectively, and both* K_{10}^a *and* K_{01}^a *are, respectively, the* 2×2 *submatrices of* K_{10} *and of* K_{01} *corresponding to the first and last rows and columns, and* $g_{00}^{20,20}(\alpha)$ *for* $0 \leq \alpha \leq 2$ *are given in Remark* 3.1.

From Lemma 4.1, the following holds:

Theorem 4.1. *Let* T *be a* $2^{m_1+m_2}$-PBFF *design of resolution* $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ *derived from an* $\mathrm{SPBA}(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ *with* N *assemblies, where* $N < \nu(m_1, m_2)$. *Then we get*

$$\begin{aligned} \mathrm{tr}\{V_T(\alpha)\} = \phi_{00}\mathrm{tr}\{V_{00}(\alpha)\} + \phi_{10}\mathrm{tr}\{V_{10}(\alpha)\} + \phi_{01}\mathrm{tr}\{V_{01}(\alpha)\} \\ +\phi_{20}\mathrm{tr}\{V_{20}(\alpha)\}\,(\textit{if } m_1 \geq 4) + \phi_{02}\mathrm{tr}\{V_{02}(\alpha)\}\,(\textit{if } m_2 \geq 4) + \phi_{11}\mathrm{tr}\{V_{11}(\alpha)\} \\ \textit{for } 0 \leq \alpha \leq 2. \end{aligned}$$

Remark 4.1. As shown in Section 3, if $(T^{(1)}; T^{(2)})$ $(= T,$ say) is a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$, then $(T^{(2)}; T^{(1)})$ $(= T^*,$ say) is also the $2^{m_2+m_1}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$. Thus we have $\mathrm{tr}\{V_T(\alpha)\} = \mathrm{tr}\{V_{T^*}(\alpha)\}$ for $0 \leq \alpha \leq 2$.

As a generalization of the A-optimality criterion, Kuwada et al. [6] introduced the GA-optimality criterion for selecting a design. For resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ designs, we recall the definition of GA_α-optimality criteria:

Definition 4.1. Let T be a $2^{m_1+m_2}$-PBFF design of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from an $\mathrm{SPBA}(m_1 + m_2; \{\lambda_{i_1,i_2}\})$ with N assemblies, where $N < \nu(m_1, m_2)$. If $\mathrm{tr}\{V_T(\alpha)\} \leq \mathrm{tr}\{V_{T^\star}(\alpha)\}$ for any $T^\star$, which is a resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ design derived from an $\mathrm{SPBA}(m_1 + m_2; \{\lambda^\star_{i_1,i_2}\})$ with the same number of assemblies, then T is said to be GA_α-optimal for $0 \leq \alpha \leq 2$.

The GA_1- and GA_2-optimality criteria are suitable for comparison of designs in the sense that they reflect the confounding (or aliasing) structure of the parametric vectors (see [8]). Using Theorems 3.2 and 4.1, we can obtain GA_α-optimal 2^{4+4}-PBFF designs of resolution $\mathrm{R}(\{00, 10, 01, 11\}|\Omega)$ derived from $\mathrm{SPBAs}(4+4; \{\lambda_{i_1,i_2}\})$ with $N < \nu(4, 4)$ $(= 37)$, which are given by Table 4.1. In this table, from Remark 3.2 and Lemma 4.1, we have $33 \leq N \leq 36$, and $V_T(0) = V_T(1) = V_T(2)$. Furthermore GA_α-optimal designs for each N except for $N = 36$ are derived from the same SPBAs for $0 \leq \alpha \leq 2$. Note that in Table 4.1,

$\boldsymbol{\lambda}' = (\lambda_{0,0}, \lambda_{0,1}, \ldots, \lambda_{0,4}, \lambda_{1,0}, \ldots, \lambda_{1,4}, \ldots, \lambda_{4,0}, \ldots, \lambda_{4,4})$, and the number (II)(ii)(c)(2) of the last column corresponds to Theorem 3.2 (II)(ii)(c)(2). Moreover from (2.8), if a resolution R($\{00, 10, 01, 11\}|\Omega$) design derived from an SPBA($m_1 + m_2; \{\lambda_{i_1,i_2}\}$) is GA$_\alpha$-optimal for $0 \le \alpha \le 2$, then the designs derived from its FCA, LCA and/or CCA are also GA$_\alpha$-optimal.

Note that GA-optimal $2^{m_1+m_2}$-PBFF designs with (A) $\det(K_{\gamma_1\gamma_2}) \neq 0$ ($\gamma_1\gamma_2 = 00, 10, 01, 20$ (if $m_1 \ge 4$), 02 (if $m_2 \ge 4$)) and $K_{11} = 0$ for $4 \le m_1 + m_2 \le 6$, and (B) $\det(K_{\gamma_1\gamma_2}) \neq 0$ ($\gamma_1\gamma_2 = 00, 10, 01$), and furthermore (a) $K_{20} \neq 0$ (if $m_1 \ge 4$) or vanishes ($m_1 = 2, 3$) and $K_{02} = K_{11} = 0$ for $2 \le m_1 \le 4$ and $m_2 = 4$, and (b) $K_{20} = K_{02} = K_{11} = 0$ for $m_1 = m_2 = 4$ were obtained by Kuwada [2] and Kuwada and Matsuura [5], respectively, where $\det(A)$ denotes the determinant of a matrix A. Moreover GA$_\alpha$-optimal $2^{m_1+m_2}$-PBFF designs of resolutions R($\{00, 10, 01, 20, 02\}|\Omega$) and R($\{00, 10, 01, 20, 11\}|\Omega$), and of resolution R($\{00, 10, 01, 20\}|\Omega$) with $N < \nu(m_1, m_2)$ and $2 \le m_1, m_2 \le 4$ have been obtained by Kuwada et al. [7, 8], respectively.

Table 4.1. GA$_\alpha$-optimal 2^{4+4}-PBFF designs.

N	$\boldsymbol{\lambda}'$	tr$\{V_T(0)\}$	tr$\{V_T(1)\}$	tr$\{V_T(2)\}$	Theorem
33	10010 01000 00000 10001 00010	1.48337	1.48337	1.48337	(II)(ii)(c)(2)
34	20010 01000 00000 10001 00010	1.43209	1.43209	1.43209	(II)(ii)(c)(2)
35	30010 01000 00000 10001 00010	1.41500	1.41500	1.41500	(II)(ii)(c)(2)
36	30011 01000 00000 10001 00010	1.40511	1.40511	1.40511	(II)(ii)(c)(2)
	30010 01000 00000 10001 10010	1.40511	1.40511	1.40511	(II)(ii)(c)(2)

Appendix Matrix equation Consider a matrix equation $ZL = H$ with a variable matrix Z of order n, where $L = ||L_{ij}||$ and $H = ||H_{ij}||$ $(1 \le i, j \le 3)$ are the positive semidefinite matrix of order n with $\text{rank}\{L\} = \text{rank}\{\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}\} = n_1 + n_2$ (≥ 1) and a matrix of order n with $H_{11} = I_{n_1}$, $H_{12} = H'_{21} = O_{n_1 \times n_2}$ and $H_{13} = H'_{31} = O_{n_1 \times n_3}$, respectively. Here L_{ij} and H_{ij} are of size $n_i \times n_j$, $n_1 + n_2 + n_3 = n$, and $O_{p\times q}$ is the null matrix of size $p \times q$. The matrix equation $ZL = H$ has a solution if and only if $\text{rank}\{L'\} = \text{rank}\{(L'; H')\}$. Thus we have the following (see [1]):

Lemma A.1. *A matrix equation $ZL = H$ has a solution, where Z is a variable matrix of order n, if and only if*

(I) $n_3 = 0$, *where H_{22} (if $n_2 \ge 1$) is arbitrary, or*

(II) $n_3 \ge 1$ *and in addition*

(i) *when $n_2 = 0$, $L_{33} = O_{n_3 \times n_3}$, and furthermore $H_{33} = O_{n_3\times n_3}$, or*

(ii) *when $n_2 \ge 1$, there exists a matrix W of size $n_3 \times n_2$ such that $(L_{31}; L_{32}; L_{33}) = W(L_{21}; L_{22}; L_{23})$, and furthermore $H'_{23} = WH'_{22}$ and $H'_{33} = WH'_{32}$, where H_{22} and H_{32} are arbitrary.*

In Lemma A.1, the matrix equation $ZL = H$ has a solution Z such that $Z = HL^{-1}$ for the case (I), $Z = \begin{pmatrix} L_{11}^{-1} & Z_{13} \\ 0 & Z_{33} \end{pmatrix}$ for the case (II)(i), where Z_{i3} $(i = 1, 3)$ are arbitrary, and $Z = \left(\begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} - \begin{pmatrix} 0 & Z_{13}W' \\ 0 & Z_{23}W' \\ 0 & Z_{33}W' \end{pmatrix} ; \begin{pmatrix} Z_{13} \\ Z_{23} \\ Z_{33} \end{pmatrix} \right)$ for the case (II)(ii), where Z_{i3} $(i = 1, 2, 3)$ are arbitrary. Thus we obtain the following (see [7]):

$$ZLZ' = \begin{cases} L_{11}^{-1} & \text{if } n_2 = n_3 = 0, \\ \begin{pmatrix} I_{n_1} \\ 0 \end{pmatrix} L_{11}^{-1} \begin{pmatrix} I_{n_1} ; & 0 \end{pmatrix} & \text{if } n_2 = 0 \text{ and } n_3 \geq 1, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 \\ 0 & H'_{22} \end{pmatrix} & \text{if } n_2 \geq 1 \text{ and } n_3 = 0, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & H'_{22} & H'_{32} \end{pmatrix} & \text{if } n_2 \geq 1 \text{ and } n_3 \geq 1, \end{cases}$$

where H_{22} and H_{32} are arbitrary. Let Z^* be an $(n_1 + n_2) \times n$ submatrix of a solution Z whose rows are linearly independent. Then from ZLZ' given above, we have the following lemma:

Lemma A.2.

$$Z^*LZ^{*\prime} = \begin{cases} L_{11}^{-1} & \textit{if } n_2 = 0, \\ \begin{pmatrix} I_{n_1} & 0 \\ 0 & H_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_{n_1} & 0 \\ 0 & H'_{22} \end{pmatrix} & \textit{if } n_2 \geq 1. \end{cases}$$

Acknowledgments

The authors would like to express their hearty thanks to the referee for his/her valuable comments and suggestions which have improved the early draft of this paper. The last author's work was partially supported by Grant-in-Aid for Scientific Research (C) of the JSPS under Contract Number 14580348.

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