SOME DECOMPOSITIONS OF IDEALS IN BF-ALGEBRAS

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Abstract. In this paper we study some properties of (normal, closed) ideals in BF-algebras, especially we show that any ideal of BF-algebra can be decomposed into the union of some sets, and obtain the greatest closed ideal \( I^0 \) of an ideal \( I \) of a BF-algebra \( X \) contained in \( I \).

1. Introduction

The concept of \( B \)-algebras was introduced by J. Neggers and H. S. Kim ([1, 4, 5, 6]). They defined a \( B \)-algebra as an algebra \( (X; *, 0) \) of type \((2, 0)\) (i.e., a non-empty set \( X \) with a binary operation \(*\) and a constant \( 0 \)) satisfying \((B1), (B2)\) and \((BH) \ x * y = 0 = y * x \) implies \( x = y \). Recently, C. B. Kim and H. S. Kim ([3]) defined a \( BG \)-algebra as an algebra \( (X; *, 0) \) of type \((2, 0)\) satisfying \((B1), (B2)\) and \((BG) \ x * y = (x * y) * (0 * y) \), for any \( x, y \in Z \). A. Walendziak ([9]) introduced the notion of BF-algebras, which is a generalization of \( B \)-algebras, and investigated some properties of (normal) ideals in BF-algebras. For another generalization of \( B \)-algebras we refer to [7, 8]. S. W. Wei and Y. B. Jun ([10]) studied ideals in \( BCI \)-algebras and decomposed some ideals into the union of some sets. We apply this concept to BF-algebras. In this paper we study some properties of (normal, closed) ideals in BF-algebras, especially we show that any ideal of BF-algebra can be decomposed into the union of some sets, and obtain the greatest closed ideal \( I^0 \) of an ideal \( I \) of a BF-algebra \( X \) contained in \( I \).

2. Decompositions of ideals in BF-algebras

Let us review some definitions and results. By a BF-algebra ([9]) we mean a non-empty set \( X \) with a binary operation “\(*\)” and a constant \( 0 \) satisfying the following conditions:

\[(B1) \ x * x = 0,\]
\[(B2) \ x * 0 = x,\]
\[(BF) \ 0 * (x * y) = y * x\]
for any \( x, y, z \in X \).

A non-empty subset \( I \) of a BF-algebra \( X \) is said to be a subalgebra if \( x \in I \) and \( y \in I \) imply \( x * y \in I \).

An ideal of a BF-algebra \( X \) is a subset \( I \) containing 0 such that if \( x * y \in I \) and \( y \in I \) then \( x \in I \).

An ideal \( I \) of a BF-algebra \( X \) is said to be normal if for any \( x, y, z \in X \), \( x \in I \) implies \( (z * x) * (z * y) \in I \).

**Lemma 2.1.** ([9]) If \( I \) is a normal ideal of a BF-algebra \( X \), then

(a) \( x \in I \Rightarrow 0 * x \in I \),

(b) \( x * y \in I \Rightarrow y * x \in I \),

for any \( x, y \in X \).

An ideal \( I \) of \( X \) is said to be closed if \( x \in I \) then \( 0 * x \in I \). By Lemma 2.1-(a), it is known that every normal ideal of a BF-algebra \( X \) is a closed ideal of \( X \). Note that a closed ideal need not be a subalgebra. See the following example.

**Example 2.2.** Let \( X := \{0, 1, 2, 3\} \) be a set with the following table:

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 3 & 2 & 1 \\
1 & 1 & 0 & 2 & 2 \\
2 & 2 & 2 & 0 & 2 \\
3 & 3 & 2 & 2 & 0 \\
\end{array}
\]

Then \( (X; *, 0) \) is a BF-algebra, and \( I := \{0, 1, 3\} \) is a closed ideal of \( X \), but not a subalgebra of \( X \), since \( 1 * 3 = 2 \notin I \). Moreover, \( J := \{0, 1\} \) is an ideal of \( X \), but not closed, since \( 0 * 1 = 3 \notin J \). The set \( K := \{0, 2\} \) is a subalgebra of \( X \), but not an ideal of \( X \), since \( 3 * 2 = 2 \in K, 2 \in K, 3 \notin K \).

For any BF-algebra \( X \) and \( x, y \in X \), we denote

\[
A(x, y) = \{z \in X|(z * x) * y = 0\}.
\]

**Theorem 2.3.** If \( I \) is an ideal of a BF-algebra \( X \), then

\[
I = \bigcup_{x, y \in I} A(x, y).
\]

**Proof.** Let \( I \) be an ideal of a BF-algebra \( X \). If \( z \in I \), then \( (z * 0) * z = z * z = 0 \). Hence \( z \in A(0, z) \). It follows that

\[
I \subseteq \bigcup_{z \in I} A(0, z) \subseteq \bigcup_{x, y \in I} A(x, y).
\]
Let $z \in \bigcup_{x,y \in I} A(x,y)$. Then there exist $a, b \in I$ such that $z \in A(a,b)$, so that $(z*a)*b = 0$. Since $I$ is an ideal, it follows that $z \in I$. Thus $\bigcup_{x,y \in I} A(x,y) \subseteq I$, and consequently, $I = \bigcup_{x,y \in I} A(x,y)$.

**Corollary 2.4.** If $I$ is an ideal of a $BF$-algebra $X$, then

$$I = \bigcup_{x \in I} A(0,x).$$

**Proof.** By Theorem 2.3, we have that $\bigcup_{x \in I} A(0,x) \subseteq \bigcup_{x,y \in I} A(x,y) = I$. If $x \in I$, then $x \in \bigcup_{x \in I} A(0,x)$, since $(x*0)*x = 0$. Hence $I \subseteq \bigcup_{x \in I} A(0,x)$. This completes the proof. 

We give an example satisfying Theorem 2.3 and Corollary 2.4. See the following example.

**Example 2.5.** Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

<table>
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<tr>
<th>*</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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Then $(X; *, 0)$ is a $BF$-algebra and $I := \{0, 2, 3\}$ is an ideal of $X$. Moreover, it is easy to check that $I = A(2,0) \cup A(3,2)$ and $I = A(0,0) \cup A(0,2)$.

**Theorem 2.6.** Let $I$ be a subset of a $BF$-algebra $X$ such that $0 \in I$ and

$$I = \bigcup_{x,y \in I} A(x,y).$$

Then $I$ is an ideal of $X$.

**Proof.** Let $x * y, y \in I = \bigcup_{x,y \in I} A(x,y)$. Since $(x * y) * (x * y) = 0$, it follows that $x \in A(y, x * y) \subseteq I$. Hence $I$ is an ideal of $X$. 

Combining Theorems 2.3 and 2.6, we have the following corollary.

**Corollary 2.7.** Let $X$ be a $BF$-algebra and $I$ be a subset of $X$ containing 0. Then $I$ is an ideal of $X$ if and only if

$$I = \bigcup_{x,y \in I} A(x,y).$$

Now, we give a characterization of normal and closed ideal in $BF$-algebras.

**Proposition 2.8.** Let $I$ be a normal ideal of a $BF$-algebra $X$. If $x * z \in I$, $y * z \in I$ and $z \in I$, then $x * y \in I$. 


Proof. Let $I$ be a normal ideal of $X$. Assume that $x \star z \in I$, $y \star z \in I$ and $z \in I$. Since $I$ is an ideal of $X$, we obtain $x, y \in I$. By Lemma 2.1-(a), $0 \star y \in I$ and by definition of normal, $(x \star 0) \star (x \star y) \in I$, i.e., $x \star (x \star y) \in I$. Also, by Lemma 2.1-(b), we have $(x \star y) \star x \in I$. Since $I$ is an ideal of $X$ and $x \in I$, we obtain $x \star y \in I$. 

The converse of Proposition 2.8 need not be true in general. See the following example.

Example 2.9. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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Then $(X; \ast, 0)$ is a BF-algebra and $I := \{0\}$ is an ideal of $X$. Although $I$ satisfies the condition: $x \star z \in I$, $y \star z \in I$ and $z \in I$ imply $x \star y \in I$, $I$ is not a normal ideal of $X$, since $1 \ast 3 = 0 \in I$, $(2 \ast 1) \ast (2 \ast 3) = 2 \notin I$. 

Corollary 2.10. If $I$ is a subset of a BF-algebra $X$ with satisfying the conditions:

(1) $0 \in I$,

(2) $x \star z \in I$, $y \star z \in I$ and $z \in I$ imply $x \star y \in I$

for any $x, y, z \in X$, then $I$ is a subalgebra of $X$.

Proof. Given $x, y \in I$, by (B2), we have $x = x \ast 0, y = y \ast 0$. It follows from (2) that $x \ast y \in I$. 

Proposition 2.11. Let $I$ be a subset of a BF-algebra $X$ with the following conditions:

(1) $0 \in I$,

(2) $x \star z \in I$, $y \star z \in I$ and $z \in I$ imply $x \star y \in I$

Then $I$ is a closed ideal of $X$.

Proof. Assume that $I$ satisfies (1) and (2). We claim that $I$ is a closed ideal of $X$. Let $x \ast y, y \ast x \in I$. Since $0 \ast 0, y \ast 0, 0 \in I$, by (2), we have $0 \ast y \in I$, which proves that $I$ is closed. Since $x \ast y, 0 \ast y, y \ast x \in I$, by applying (2) again, we obtain that $x = x \ast 0 \in I$, so that $I$ is an ideal of $X$. 

Lemma 2.12. ([9]) If $(X; \ast, 0)$ is a BF-algebra, then $0 \ast (0 \ast x) = x$ for any $x \in X$.

Theorem 2.13. Let $I$ be an ideal of a BF-algebra $X$. Then the set

$I^0 := \{ x \in I | 0 \ast x \in I \}$
is the greatest closed ideal of $X$ which is contained in $I$.

Proof. First, we show that $I^0$ is an ideal of $X$. Clearly, $0 \in I^0$. If $x \ast y, y \in I^0$, then $x \ast y, y \in I$, since $I^0 \subseteq I$. Since $I$ is an ideal of $X$, $x \in I$. By applying Lemma 2.12, we have $0 \ast (0 \ast x) = x \in I$. This means that $0 \ast x \in I^0$. Since $I^0 \subseteq I$, $0 \ast x \in I$ and hence $x \in I^0$. Hence $I^0$ is an ideal of $X$.

If $x \in I^0$, by definition of $I^0$, we have $0 \ast x \in I$ and $x \in I$. Since $I$ is an ideal of $X$, $x \in I$. By applying Lemma 2.12, we have $0 \ast (0 \ast x) = x \in I$, which proves that $I^0$ is closed.

Now, assume that $A$ is a closed ideal of $X$ which is contained in $I$. If $x \in A$, then $0 \ast x \in A$. Since $A$ is contained in $I$, we have $x, 0 \ast x \in I$, and so $x \in I^0$. Thus $A \subseteq I^0$. Therefore, $I^0$ is the greatest closed ideal of $X$ which is contained in $I$.

Example 2.14. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<tbody>
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<td>3</td>
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</table>

Then $(X; \ast, 0)$ is a $BF$-algebra and $I := \{0, 1, 3\}$ is an ideal of $X$. Let $I_1 := \{0, 1\}$, $I_2 := \{0, 3\}$ and $I_3 := \{0, 1, 3\}$ be subsets of $I$. We can see that $I_1$ is not an ideal, since $3 \ast 1 = 1 \in I_1$, but $3 \notin I_1$. $I_2$ is a closed ideal, but $I_3$ is not closed, since $0 \ast 1 = 2 \notin I_3$. Hence $I_2$ is the greatest closed ideal of $X$ which is contained in $I$, i.e., $I^0 = I_2$.

References


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