THE ESSENCE OF SUBTRACTION ALGEBRAS

YOUNG BAE JUN, HEE SIK KIM* AND KYOUNG JA LEE

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Abstract. The notion of essences in subtraction algebras is introduced, and many properties are investigated. Relations among subalgebras, ideals and essences are given. Homomorphic image and inverse image are considered.

1. Introduction

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim [5] showed that the subtraction algebra is equivalent to the implicative BCK-algebra, and the subtraction semigroup is a special case of the BCI-semigroup. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [2], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [4] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In this paper, we introduce the notion of essences in subtraction algebras, and investigate several properties. We discuss relations among subalgebras, ideals and essences. We consider the homomorphic image and inverse image of an essence.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \le b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \le)$ is

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* Corresponding author.

a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3, 4]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1. [3] A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

- $0 \in I$
- $(\forall x \in X)(\forall y \in I)(x - y \in I \Rightarrow x \in I)$.

Lemma 2.2. [4] *An ideal I of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in I)(x \leq y \Rightarrow x \in I).$$

3. ESSENCES

Let X be a subtraction algebra. For any subsets G and H of X, we define

$$G - H := \{x - y \mid x \in G, y \in H\}.$$

Lemma 3.1. *Let X be a subtraction algebra. If $0 \in H \subseteq X$, then*

$$(\forall G \subseteq X)(G \subseteq G - H).$$

Proof. Let $x \in G$. Then $x = x - 0 \in G - H$ by (a2), and so $G \subseteq G - H$. □

Lemma 3.2. *Let X be a subtraction algebra. For any $G \subseteq X$, we have*

$$G - X = \{x \in X \mid x \leq e \text{ for some } e \in G\}.$$

Proof. Let $A(G) := \{x \in X \mid x \leq e \text{ for some } e \in G\}$. If $x \in G - X$, then $x = e - y \leq e$ for some $e \in G$ and $y \in X$. Therefore $x \in A(G)$, and so $G - X \subseteq A(G)$. Now let $x \in A(G)$. Then $x \leq e$ for some $e \in G$, and thus $x - e = 0$. Hence

$$x = x - 0 = x - (x - e) = e - (e - x) \in G - X,$$

and therefore $A(G) \subseteq G - X$. This completes the proof. □

Lemma 3.3. *For any subsets A, B and E of a subtraction algebra X, we have*

(i) $A \subseteq B \Rightarrow A - E \subseteq B - E, E - A \subseteq E - B$.
(ii) $(A \cap B) - E \subseteq (A - E) \cap (B - E)$.
(iii) $E - (A \cap B) \subseteq (E - A) \cap (E - B)$.

(iv) $(A \cup B) - E = (A - E) \cup (B - E)$.
(v) $E - (A \cup B) = (E - A) \cup (E - B)$.

Proof. (i) Let $x \in A - E$. Then $x = a - e$ for some $a \in A$ and $e \in E$. Since $A \subseteq B$, it follows that $x = a - e$ for some $a \in B$ and $e \in E$ so that $x \in B - E$. Therefore $A - E \subseteq B - E$. Similarly, we get $E - A \subseteq E - B$.

(ii) Since $A \cap B \subseteq A, B$, it follows from (i) that $(A \cap B) - E \subseteq A - E$ and $(A \cap B) - E \subseteq B - E$ so that $(A \cap B) - E \subseteq (A - E) \cap (B - E)$. Similarly, (iii) is valid.

(iv) Since $A, B \subseteq A \cup B$, we have $A - E \subseteq (A \cup B) - E$ and $B - E \subseteq (A \cup B) - E$ by (i), and so $(A - E) \cup (B - E) \subseteq (A \cup B) - E$. If $x \in (A \cup B) - E$, then $x = y - e$ for some $y \in A \cup B$ and $e \in E$. It follows that $x = y - e$ for some $y \in A$ and $e \in E$; or $x = y - e$ for some $y \in B$ and $e \in E$ so that $x = y - e \in A - E$ or $x = y - e \in B - E$. Hence $x \in (A - E) \cup (B - E)$, which shows that $(A \cup B) - E \subseteq (A - E) \cup (B - E)$. Therefore (iv) is valid. Similarly we can prove that (v) is valid. □

Definition 3.4. If a nonempty subset G of a subtraction algebra X satisfies $G - X = G$, then we say that G is an *essence* of X.

Obviously, $\{0\}$ is an essence of a subtraction algebra X which is called a *trivial* essence of X, and X itself is an essence of a subtraction algebra X. Note that if a is an element of a subtraction algebra X such that $\{a\} - X = X$, then any proper subset G of X containing a can not be an essence of X.

Example 3.5. Let $X_1 := \{0, a, b, c\}$ be a set with the following Cayley table.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then X_1 is a subtraction algebra. It is easy to check that $G_1 := \{0, a\}$, $G_2 := \{0, b\}$ and $G_3 := \{0, a, b\}$ are essences of X_1. But $H := \{0, c\}$ is not an essence of X_1.

Theorem 3.6. *Let X be a subtraction algebra. Then*
(i) *Every essence of X contains the zero element* 0.
(ii) *Every essence of X is a subalgebra of X.*
(iii) *Every ideal of X is an essence of X.*

Proof. (i) Let G be an essence of X. Then $\emptyset \neq G = G - X$, and so there exists $x \in G$ and thus $0 = x - x \in G - X = G$.

(ii) Let $x, y \in G$. Then $x - y \in G - G \subseteq G - X = G$ by Lemma 3.3(i), and thus G is a subalgebra of X.

(iii) Let I be an ideal of X. Then $0 \in I$, and so $I \neq \emptyset$. By Lemma 2.2, for any $x \in X$ and $y \in I$, we have $y - x \in I$ since $y - x \leq y$. Thus $I - X \subseteq I$. Obviously, $I \subseteq I - X$ by Lemma 3.1. Therefore $I - X = I$, i.e., I is an essence of X. □

The following example shows that the converse of (ii) and (iii) in Theorem 3.6 may not be true.

Example 3.7. In Example 3.5, $G_3 := \{0, a, b\}$ is an essence which is not an ideal, and $H := \{0, c\}$ is a subalgebra which is not an essence.

Theorem 3.8. *Let G and H be essences of a subtraction algebra X. Then $G \cap H$ and $G \cup H$ are essences of X.*

Proof. By Lemma 3.1 and Lemma 3.3 (ii),

$$G \cap H \subseteq (G \cap H) - X \subseteq (G - X) \cap (H - X) = G \cap H,$$

and so $(G \cap H) - X = G \cap H$, i.e., $G \cap H$ is an essence of X. Now by Lemma 3.1 and Lemma 3.3 (iv), we get

$$G \cup H \subseteq (G \cup H) - X = (G - X) \cup (H - X) = G \cup H,$$

and thus $(G \cup H) - X = G \cup H$, i.e., $G \cup H$ is an essence of X. □

Generally, we have the following observation.

Theorem 3.9. *If $\{G_i \mid i \in \Lambda \subseteq \mathbb{N}\}$ is a family of essences of a subtraction algebra X, then $\bigcap_{i \in \Lambda} G_i$ and $\bigcup_{i \in \Lambda} G_i$ are essences of X.*

In general, the union of two ideals of a subtraction algebra X may not be an ideal of X. For example, in Example 3.5, $G_1 := \{0, a\}$ and $G_2 := \{0, b\}$ are ideals of X_1, but $G_1 \cup G_2 = \{0, a, b\}$ is not an ideal of X_1. But we know that the above Theorems 3.6 and 3.8 lead to the following result.

Corollary 3.10. *The intersection and union of two ideals of a subtraction algebra X are essences of X.*

Theorem 3.11. *Let X and Y be subtraction algebras. If G and H are essences of X and Y, respectively, then $G \times H$ is an essence of $X \times Y$.*

Proof. Since $(G \times H) - (X \times Y) = (G - X) \times (H - X) = G \times H$, we know that $G \times H$ is an essence of $X \times Y$. □

Let G be an essence and H be a subalgebra of a subtraction algebra X. Then $G \cup H$ is not an essence of X in general as seen in the following example.

Example 3.12. (i) In Example 3.5, $G_1 := \{0, a\}$ is an essence and $H := \{0, c\}$ is a subalgebra of X_1, but $G_1 \cup H = \{0, a, c\}$ is not an essence of X_1.

(ii) Consider a subtraction algebra $X_2 := \{0, a, b, c, d\}$ with the following Cayley table.

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Then $G := \{0, a, d\}$ is an essence of X_2 and $H := \{0, c\}$ is a subalgebra of X_2, but $G \cup H = \{0, a, c, d\}$ is not an essence of X_2.

Theorem 3.13. *Let X be a subtraction algebra. If G is an essence of X and H is a subalgebra of X, then $G \cap H$ is an essence of H.*

Proof. Using Lemma 3.1 and Lemma 3.3 (i), (ii), we have

$$(G \cap H) - H \subseteq (G - H) \cap (H - H) \subseteq (G - X) \cap H = G \cap H \subseteq (G \cap H) - H,$$

and so $(G \cap H) - H = G \cap H$. Therefore $G \cap H$ is an essence of H. □

Lemma 3.14. *Let G be an essence of a subtraction algebra X. Then*

$$(\forall x \in X)(\forall a \in G)(x \leq a \Rightarrow x \in G).$$

Proof. Let $x \in X$ and $a \in G$. If $x \leq a$, then $x - a = 0 \in G = G - X$ and so $x = x - 0 = x - (x - a) = a - (a - x) \in G - X = G$, proving the lemma. □

Theorem 3.15. *Let X be a subtraction algebra. For any $a \in X$, the set*

$$(a] := \{x \in X \mid x \leq a\}$$

is the least essence of X containing a.

Proof. Obviously $a \in (a]$. Note that $(a] \subseteq (a] - X$ by Lemma 3.1. For any $y \in X$ and any $x \in X$ with $x \leq a$, we have $x - y \leq a - y \leq a$, and so $x - y \in (a]$. Hence $(a] - X = (a]$, that is, $(a]$ is an essence of X. Now let G be an essence of X containing a, and let $x \in (a]$. Then $x \leq a$, and so $x \in G$ by Lemma 3.14, i.e., $(a] \subseteq G$. This completes the proof. □

Theorem 3.16. *Let G be an essence of a subtraction algebra X. For any nonempty subset H of X, we have $G - H$ is an essence of X and $G - H \subseteq G$. In particular, if $0 \in H$, then $G - H = G$.*

Proof. Obviously, $0 \in G - H$. By using (S3), we have

$$(G - H) - X = (G - X) - H = G - H,$$

and so $G - H$ is an essence of X. Let $a \in X$ and $y \in H$. Then $a - y \leq a$, and hence $a - y \in G$ by Lemma 3.14. It follows that $G - H \subseteq G$. Now, if $0 \in H$, then $a = a - 0 \in G - H$ for any $a \in G$. Therefore $G \subseteq G - H$. This completes the proof. □

Corollary 3.17. *Let H be any nonempty subset of a subtraction algebra X.*

(i) *If I is an ideal of X, then $I - H$ is an essence of X and $I - H \subseteq I$.*
(ii) *$X - H$ is an essence of X.*

Theorem 3.18. *If G is an essence of a subtraction algebra X and $a \notin G$, then the set*

$$G_a := \{x \in X \mid x - a \in G\}$$

is an essence of X containing G and a.

Proof. Obviously $G \subseteq G_a$ and $a \in G_a$. Let $x \in G_a$ and $y \in X$. Then

$$(x - y) - a = (x - a) - y \leq x - a \in G.$$

Since G is an essence, it follows from Lemma 3.14 that $(x - y) - a \in G$ so that $x - y \in G_a$. This shows that $G_a - X \subseteq G_a$. Now, we have $G_a \subseteq G_a - X$ by Lemma 3.1. Hence $G_a - X = G_a$, i.e., G_a is an essence of X. □

Let X and Y be subtraction algebras. A mapping $f : X \to Y$ is called a *subtraction homomorphism* if $f(x - y) = f(x) - f(y)$ for all $x, y \in X$. Note that $f(0) = 0$. If we define a mapping f from the subtraction algebra X_2 in Example 3.12 to the subtraction algebra X_1 in Example 3.5 by $f(0) = f(d) = 0, f(a) = b, f(b) = a$, and $f(c) = c$, then it is a subtraction homomorphism. We can easily see that $f(G) = \{0, b\}$ is an essence of X_1, where $G = \{0, a, d\}$ is an essence of X_2, and $f^{-1}(G_1) = \{0, b, d\}$ is also an essence of X_2, where $G_1 = \{0, a\}$ is an essence of X_1. We generalize:

Theorem 3.19. *Let $f : X \to Y$ be a subtraction homomorphism.*

(i) *If f is onto and G is an essence of X, then $f(G)$ is an essence of Y.*
(ii) *If H is an essence of Y, then $f^{-1}(H)$ is a essence of X.*

Proof. (i) Assume that f is onto and G is an essence of X. By Lemma 3.1, we have $f(G) \subseteq f(G) - Y$. Let $b \in f(G)$ and $y \in Y$. Then $b = f(a)$ and $y = f(x)$ for some $a \in G$ and $x \in X$. Thus

$$b - y = f(a) - f(x) = f(a - x) \in f(G - X) = f(G),$$

and so $f(G) - Y \subseteq f(G)$. Therefore $f(G) - Y = f(G)$, that is, $f(G)$ is an essence of Y.

(ii) By Lemma 3.1, we have $f^{-1}(H) \subseteq f^{-1}(H) - X$. Let $a \in f^{-1}(H)$ and $x \in X$. Then $f(a) \in H$ and $f(x) \in Y$. It follows that $f(a - x) = f(a) - f(x) \in H - Y = H$ so that $a - x \in f^{-1}(H)$, i.e., $f^{-1}(H) - X \subseteq f^{-1}(H)$. Hence $f^{-1}(H) - X = f^{-1}(H)$, that is, $f^{-1}(H)$ is an essence of X. □

Corollary 3.20. *If $f : X \to Y$ is a subtraction homomorphism, then the kernel $Ker(f) := \{x \in X \mid f(x) = 0\}$ of f is an essence of X.*

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Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
ybjun@gnu.ac.kr or jamjana@korea.com

Hee Sik Kim
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
heekim@hanyang.ac.kr

Kyoung Ja Lee
Department of Mathematics
Kookmin University
Seoul 136-702, Korea
lsj1109@kookmin.ac.kr