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THE ABYSM OF A HILBERT ALGEBRA

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ABSTRACT. The notion of abysms in Hilbert algebras is introduced, and related properties are investigated.

1. Introduction

Following the introduction of Hilbert algebras by L. Henkin in early 50-ties and A. Diego [6], the algebra and related concepts were developed by D. Busneag (see [1], [2], and [3]). For the general development of Hilbert algebras, the notion of deductive systems plays an important role. For example, it is known that the set of all deductive systems of a Hilbert algebra forms an algebraic lattice which is distributive. (see [4]). Y. B. Jun gave characterizations of deductive systems in Hilbert algebras (see [7] and [8]).

In this paper, we introduced a new notion, called an abysm, in a Hilbert algebras. We give relations among subalgebras, deductive systems, and abysms. Using a deductive system, we make an abysm. Given an element of a Hilbert algebra, we establish the least abysm containing this element.

2. Preliminaries

A Hilbert algebra can be considered as a fragment of propositional logic containing only a logical connective implication "→" and the constant 1 which is interpreted as the value "true".

An algebra $\mathcal{H} := (H; \to, 1)$ of type (2,0) is called a *Hilbert algebra* if it satisfies:

(H1) $(\forall a, b \in H)\ (a \to (b \to a) = 1)$.
(H2) $(\forall a, b, c \in H)\ ((a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1)$.
(H3) $(\forall a, b \in H)\ (a \to b = b \to a = 1 \Rightarrow a = b)$.

If $\mathcal{H} := (H; \to, 1)$ is a Hilbert algebra and we define a binary relation $\leq$ in $\mathcal{H}$ by $a \leq b$ if and only if $a \to b = 1$, then $\leq$ is a partial order in $\mathcal{H} := (H; \to, 1)$. A Hilbert algebra $\mathcal{H} := (H; \to, 1)$ is said to be *commutative* if $(x \to y) \to y = (y \to x) \to x$ for all $x, y \in H$. A nonempty subset S of a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ is called a *subalgebra* of $\mathcal{H}$ if $x * y \in S$ whenever $x, y \in S$. A mapping f from a Hilbert algebra $\mathcal{G} = (G; \to, 1)$ into a Hilbert algebra $\mathcal{H} = (H; \to 1)$ is called a *morphism* if $f(a \to b) = f(a) \to f(b)$ for all $a, b \in G$. Note that if f is a morphism from a Hilbert algebra $\mathcal{G} = (G; \to, 1)$ into a Hilbert algebra $\mathcal{H} = (H; \to 1)$, then $f(1) = 1$.

In a Hilbert algebra $\mathcal{H} := (H; \to, 1)$, we have the following assertions:

(a1) $x \leq y \to x$.
(a2) $x \to 1 = 1,\ 1 \to x = x$.
(a3) $x \to (y \to z) = (x \to y) \to (x \to z)$.

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(a4) $x \leq (x \to y) \to y$.
(a5) $x \to (y \to z) = y \to (x \to z)$.
(a6) $x \to y \leq (y \to z) \to (x \to z)$.
(a7) $x \leq y \Rightarrow z \to x \leq z \to y,\ y \to z \leq x \to z$.

The concept of a deductive system on a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ was also introduced by A. Diego [6] as a subset of H containing 1 and closed under a "deduction", i.e.:

Definition 2.1. A nonempty subset D of a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ is called a *deductive system* of $\mathcal{H}$ if it satisfies:
(Di) $1 \in D$,
(Dii) $(\forall x \in D)\ (\forall y \in H)\ (x \to y \in D \Rightarrow y \in D)$.

Lemma 2.2. [5] *A deductive system D of a Hilbert algebra $\mathcal{H}$ has the following property:*

$$(\forall x \in D)(\forall y \in H)(x \leq y \Rightarrow y \in D).$$

3. Abysms of a Hilbert algebra

For any subsets A and B of a Hilbert algebra $\mathcal{H} := (H; \to, 1)$, we define

$$A \to B := \{x \to y \mid x \in A, y \in B\}.$$

We use the notation $A \to b$ (resp. $a \to B$) instead of $A \to \{b\}$ (resp. $\{a\} \to B$). Note that $A \to B = \bigcup_{a \in A} (a \to B) = \bigcup_{b \in B} (A \to b)$.

Lemma 3.1. *If A is a subset of a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ containing 1, then B is contained in $A \to B$ for every subset B of H.*

Proof. Let $b \in B$. Then $b = 1 \to b \in A \to B$ by (a2), and so B is contained in $A \to B$. □

Lemma 3.2. *Assume that a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ is commutative. For any subset A of H, we have*

$$H \to A = \{x \in H \mid e \leq x \text{ for some } e \in A\}.$$

Proof. Let $\Omega := \{x \in H \mid e \leq x \text{ for some } e \in A\}$. If $a \in H \to A$, then $e \leq b \to e = a$ for some $b \in H$ and $e \in A$. Hence $a \in \Omega$, and so $H \to A \subseteq \Omega$. Conversely, let $a \in \Omega$. Then $e \leq a$ for some $e \in A$. Since H is commutative, it follows from (a2) that

$$a = 1 \to a = (e \to a) \to a = (a \to e) \to e \in H \to A$$

so that $\Omega \subseteq H \to A$. This completes the proof. □

Lemma 3.3. *For any subsets A, B and E of a Hilbert algebra $\mathcal{H}$, we have*
(i) $A \subseteq B \Rightarrow A \to E \subseteq B \to E,\ E \to A \subseteq E \to B$.
(ii) $(A \cap B) \to E \subseteq (A \to E) \cap (B \to E)$.
(iii) $E \to (A \cap B) \subseteq (E \to A) \cap (E \to B)$.
(iv) $(A \cup B) \to E = (A \to E) \cup (B \to E)$.
(v) $E \to (A \cup B) = (E \to A) \cup (E \to B)$.

Proof. (i) Let $x \in A \to E$. Then $x = a \to e$ for some $a \in A$ and $e \in E$. Since $A \subseteq B$, it follows that $x = a \to e$ for some $a \in B$ and $e \in E$ so that $x \in B \to E$. Therefore $A \to E \subseteq B \to E$. Similarly, we get $E \to A \subseteq E \to B$.

(ii) Since $A \cap B \subseteq A, B$, it follows from (i) that $(A \cap B) \to E \subseteq A \to E$ and $(A \cap B) \to E \subseteq B \to E$ so that $(A \cap B) \to E \subseteq (A \to E) \cap (B \to E)$. Similarly, (iii) is valid.

(iv) Since $A, B \subseteq A \cup B$, we have $A \to E \subseteq (A \cup B) \to E$ and $B \to E \subseteq (A \cup B) \to E$ by (i), and so $(A \to E) \cup (B \to E) \subseteq (A \cup B) \to E$. If $x \in (A \cup B) \to E$, then $x = y \to e$

for some $y \in A \cup B$ and $e \in E$. It follows that $x = y \to e$ for some $y \in A$ and $e \in E$; or $x = y \to e$ for some $y \in B$ and $e \in E$ so that $x = y \to e \in A \to E$ or $x = y \to e \in B \to E$. Hence $x \in (A \to E) \cup (B \to E)$, which shows that $(A \cup B) \to E \subseteq (A \to E) \cup (B \to E)$. Therefore (iv) is valid. Similarly we can prove that (v) is valid. □

Definition 3.4. If a nonempty subset A of a Hilbert algebra $\mathcal{H} := (H; \to, 1)$ satisfies the following equality:

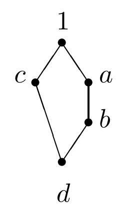
$$H \to A = A,$$

then we say that A is an *abysm* of $\mathcal{H}$.

Note that $\{1\}$ and H itself are abysms of $\mathcal{H}$.

Example 3.5. (1) Let $H = \{a, b, c, d, 1\}$ be a set with the following Cayley table and Hasse diagram:

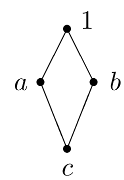
$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	1	1



Then $\mathcal{H} := (H; \to, 1)$ is a Hilbert algebra. The subsets $A = \{1, a\}$, $B = \{1, b\}$, $C = \{1, c\}$, $D = \{1, a, b\}$, $E = \{1, a, c\}$, $F = \{1, a, b, c\}$ are abysms of $\mathcal{H}$.

(2) Let $G = \{1, a, b, c\}$ be a set with the following Cayley table.

$\to$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1



Then $\mathcal{G} := (G; \to, 1)$ is a Hilbert algebra. It is easy to check that $A := \{1, a\}$, $B := \{1, b\}$ and $C := \{1, a, b\}$ are abysms of $\mathcal{G}$, while $D := \{1, c\}$ is not an abysm of $\mathcal{G}$.

Following Example 3.5(2), we know that if e is an element of H such that $H \to e = H$, then any proper subset A of H containing e can not be an abysm of $\mathcal{H}$.

Proposition 3.6. *Every abysm contains the constant* 1.

Proof. Let A be an abysm of $\mathcal{H}$. Then $\emptyset \neq A = H \to A$, and so there exists $a \in A$ and thus $1 = a \to a \in H \to A = A$. This completes the proof. □

Theorem 3.7. *Every abysm is a subalgebra.*

Proof. Let A be an abysm of $\mathcal{H}$ and let $a, b \in A$. Then

$$a \to b \in A \to A \subseteq H \to A = A$$

by Lemma 3.3(i), and so A is a subalgebra of $\mathcal{H}$. □

The converse of Theorem 3.7 is not true. For example, the set $D := \{1, c\}$ in Example 3.5(2) is a subalgebra which is not an abysm of $\mathcal{G}$.

Theorem 3.8. *Every deductive system is an abysm.*

Proof. Let D be a deductive system of $\mathcal{H}$. Then $1 \in D$, and so $D \neq \emptyset$. Since $d \leq b \rightarrow d$ for all $d \in D$ and $b \in H$, we have $b \rightarrow d \in D$. Thus $H \rightarrow D \subseteq D$. Obviously, $D = \{1\} \rightarrow D \subseteq H \rightarrow D$ by Lemma 3.3(i). Therefore $H \rightarrow D = D$, i.e., D is an abysm of $\mathcal{H}$. □

The converse of Theorem 3.8 may not be true. For example, the set $C := \{1, a, b\}$ in Example 3.5(2) is an abysm which is not a deductive system of $\mathcal{G}$ since $a \rightarrow c = b \in C$ and $c \notin C$.

Theorem 3.9. *If D is a deductive system of a Hilbert algebra $\mathcal{H}$, then $A \rightarrow D$ is an abysm of $\mathcal{H}$ for every nonempty subset A of H.*

Proof. Let A be a nonempty subset of H and assume that D is a deductive system of $\mathcal{H}$. Then D is an abysm of $\mathcal{H}$ (see Theorem 3.8). Using (a5), we have

$$H \rightarrow (A \rightarrow D) = A \rightarrow (H \rightarrow D) = A \rightarrow D,$$

and hence $A \rightarrow D$ is an abysm of $\mathcal{H}$. □

Corollary 3.10. *If A is a nonempty proper subset of a Hilbert algebra $\mathcal{H}$, then $A \rightarrow H$ is an abysm of $\mathcal{H}$.*

Theorem 3.11. *Let A and B be abysms of a Hilbert algebra $\mathcal{H}$. Then $A \cap B$ and $A \cup B$ are abysms of $\mathcal{H}$.*

Proof. Let $K = A \cap B$. Then

$$K = 1 \rightarrow K \subseteq H \rightarrow K = H \rightarrow (A \cap B) \subseteq (H \rightarrow A) \cap (H \rightarrow B) = A \cap B = K,$$

and so $H \rightarrow K = K$, that is, $K = A \cap B$ is an abysm of $\mathcal{H}$. Now let $L = A \cup B$. Then

$$L = 1 \rightarrow L \subseteq H \rightarrow L = H \rightarrow (A \cup B) = (H \rightarrow A) \cup (H \rightarrow B) = A \cup B = L,$$

and thus $H \rightarrow L = L$, i.e., $L = A \cup B$ is an abysm of $\mathcal{H}$. □

Generally, we have the following result.

Theorem 3.12. *If $\{A_i \mid i \in \Lambda \subseteq \mathbb{N}\}$ is a family of abysms of a Hilbert algebra $\mathcal{H}$, then $\bigcup_{i \in \Lambda} A_i$ and $\bigcap_{i \in \Lambda} A_i$ are abysms of $\mathcal{H}$.*

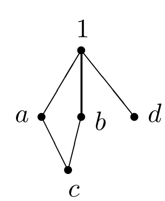
In general, the union of two deductive systems of a Hilbert algebra $\mathcal{H}$ may not be a deductive system of $\mathcal{H}$. For example, in Example 3.5(2), $A = \{1, a\}$ and $B = \{1, b\}$ are deductive systems, but $A \cup B = \{1, a, b\}$ is not a deductive system. But we know that the following result is derived from Theorems 3.8 and 3.11.

Corollary 3.13. *The union of two deductive systems of a Hilbert algebra $\mathcal{H}$ is an abysm of $\mathcal{H}$.*

Let A be an abysm and B a subalgebra of a Hilbert algebra $\mathcal{H}$. Then $A \cup B$ is not an abysm of $\mathcal{H}$ in general as seen in the following example.

Example 3.14. Let $H = \{a, b, c, d, 1\}$ be a set with the following Cayley table and Hasse diagram:

$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	a	b	c	1



Then $\mathcal{H} := (H; \to, 1)$ is a Hilbert algebra. It is easy to check that $A = \{1, a, b\}$ is an abysm of $\mathcal{H}$ and $B = \{1, c\}$ is a subalgebra of $\mathcal{H}$. But $A \cup B = \{1, a, c, d\}$ is not an abysm of $\mathcal{H}$.

Theorem 3.15. *Let $\mathcal{H}$ be a Hilbert algebra. If A is an abysm of $\mathcal{H}$ and B is a subalgebra of $\mathcal{H}$, then $A \cap B$ is an abysm of B.*

Proof. Using Lemma 3.3(iii), we have

$$B \to (A \cap B) \subseteq (B \to A) \cap (B \to B) \subseteq (H \to A) \cap B = A \cap B \subseteq B \to (A \cap B),$$

and so $B \to (A \cap B) = A \cap B$. Therefore $A \cap B$ is an abysm of B. □

Proposition 3.16. *Let A be an abysm of a Hilbert algebra $\mathcal{H}$. If $1 \in B \subseteq H$, then $B \to A = A$.*

Proof. The desired result is by

$$A = 1 \to A \subseteq B \to A \subseteq H \to A = A.$$

□

Theorem 3.17. *Let $f : \mathcal{H} \to \mathcal{G}$ be a morphism of Hilbert algebras.*

(i) *If f is onto and A is an abysm of $\mathcal{H}$, then $f(A)$ is an abysm of $\mathcal{G}$.*
(ii) *If B is an abysm of $\mathcal{G}$, then $f^{-1}(B)$ is an abysm of $\mathcal{H}$.*

Proof. (i) Assume that f is onto and A is an abysm of $\mathcal{H}$. Using (a2) and Lemma 3.3(i), we have $f(A) = 1 \to f(A) \subseteq G \to f(A)$. Let $b \in f(A)$ and $y \in G$. Then $b = f(a)$ and $y = f(x)$ for some $a \in A$ and $x \in H$. Thus

$$y \to b = f(x) \to f(a) = f(x \to a) \in f(H \to A) = f(A),$$

and so $G \to f(A) \subseteq f(A)$. Therefore $f(A)$ is an abysm of $\mathcal{G}$.

(ii) Using Lemma 3.3(i), we have $f^{-1}(B) \subseteq H \to f^{-1}(B)$. Let $a \in f^{-1}(B)$ and $x \in H$. Then $f(a) \in B$ and $f(x) \in G$. It follows that

$$f(x \to a) = f(x) \to f(a) \in G \to B = B$$

so that $x \to a \in f^{-1}(B)$, i.e., $H \to f^{-1}(B) \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is an abysm of $\mathcal{H}$. □

Corollary 3.18. *If $f : \mathcal{H} \to \mathcal{G}$ is a morphism of Hilbert algebras, then $f^{-1}(1)$ is an abysm of $\mathcal{H}$.*

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