NOTE ON ASYMPTOTICS OF WHITTLE ESTIMATORS FOR SQUARE TRANSFORMED ARCH(∞) MODELS

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ABSTRACT. Whittle estimators are important and fundamental in time series estimation. We apply Whittle estimation to the square transformed ARCH(∞) models, which can be expressed as linear processes. Whittle estimators for linear processes are known to be asymptotically normal with asymptotic variance $V_W = V_2 + V_4$, where $V_2$ is written in terms of the second-order spectra only, and $V_4$ includes the fourth-order cumulant spectra. This note gives a useful and explicit expression of $V_4$, and shows that there exists a case of $V_4 < 0$. Since $V_2$ can be regarded as the inverse of Fisher information $F^{-1}$ in terms of the second-order spectra, the result implies that there is a case when $V_W < F^{-1}$. For ARCH models with various innovation distributions, we evaluate $V_W$, $V_2$ and $V_4$ numerically. The numerical studies elucidate some interesting features of the Whittle estimators.

1. Introduction

ARCH model arises frequently in economic time series, which was introduced by Engle (1982). This model assumes the dependence of the one period forecast variance on a finite number of passed variables. Robinson (1991) extended this model to ARCH(∞) model, the one period forecast variance depends on an infinite number of passed variables. Giraitis et al. (2000) have derived sufficient conditions for the existence of a stationary solution of ARCH(∞) model. The square transformed ARCH(∞) models have representations as linear processes.

To estimate a parameter $\theta$ of linear process, Whittle estimation is widely used. Recently, Whittle estimators for a class of parametric ARCH(∞) models are shown to be asymptotically normal in Giraitis and Robinson (2001). For a general class of linear processes, Hosoya and Taniguchi (1982) introduced a Whittle estimator, which is obtained by minimizing $\int_{-\pi}^{\pi} \left\{ \log f_\theta(\lambda) + \frac{I_x(\lambda)}{f_\theta(\lambda)} \right\} d\lambda$, where $I_x(\lambda)$ is the periodogram and $f_\theta(\lambda)$ is the spectral density of the process concerned, and derived the asymptotic variance, $V_W = V_2 + V_4$, where $V_2$ is written in terms of the second-order spectra only, and $V_4$ includes the fourth-order cumulant spectra. $V_2$ is known to be the inverse of time series Fisher information $F^{-1}$. In this note we apply the Whittle estimators to the squared ARCH(∞) models, and investigate behavior of $V_W$. Then it is shown that there is a case when $V_W < F^{-1}$. Numerical evaluation for $V_W$, $V_2$ and $V_4$ are also provided.

This note is organized as follows. Section 2 describes Whittle estimators for the square transformed ARCH(∞) models and gives results of these asymptotics and provides a useful and explicit representation of $V_4$. We also give two examples satisfying $V_W = F^{-1}$ and $V_W < F^{-1}$. Section 3 provides numerical studies of $V_W$, $V_2$ and $V_4$. The results elucidate some interesting features of the asymptotics of the Whittle estimator for the parameter of ARCH(∞) model. Proof is relegated to Section 4.

Key words and phrases. ARCH(∞) model, Whittle estimator, Asymptotic efficiency, Spectral density.
2. Estimation and Asymptotics

Throughout this paper we deal with the following ARCH(∞) model.

\[ X_t = u_t \sqrt{a_0 + \sum_{j=1}^{\infty} a_j X_{t-j}^2} \]

where \(a_0 > 0, a_j \geq 0, j = 1, \ldots, \) and \(\{u_t\}\) is a sequence of i.i.d random variables with mean 0, variance 1. Let \(Y_t = X_t^2, \xi_t = u_t^2, \sigma_t^2 = a_0 + a_1 Y_{t-1} + \cdots, \) then we may write \(Y_t = \sigma_t^2 \xi_t\) and \(E[\xi_t] = 1.\) If we define \(\epsilon_t = Y_t - a_0 - a_1 Y_{t-1} - \cdots = \sigma_t^2 (\xi_t - 1),\) then \(\epsilon_t\) is an uncorrelated process. So \(Y_t\) is an autoregressive process. We impose the following assumption for the estimation of the parameter of \(\{X_t\}\).

**Assumption 1.**

(i) \(E[u_t^4] \sum_{j=1}^{\infty} a_j < 1\)

(ii) \(E u_t^8 < \infty.\)

(iii) \(a_0\) and \(a_j\)'s are functions of an unknown parameter \(\theta = (\theta_1, \theta_2, \ldots, \theta_q).\)

(iv) \(a_j = a_j(\theta)\)'s are differentiable with respect to \(\theta.\)

The assumption (i) implies the second order stationarity of \(\{Y_t\}\) (see, Giraitis et al (2000)). Hence \(\{\epsilon_t\}\) is second order stationary. Henceforth we denote the spectral densities of \(\{Y_t\}\) and \(\{\epsilon_t\}\) by \(f_{Y,\theta}\) and \(f_{\epsilon,\theta},\) respectively.

Hosoya and Taniguchi (1982) introduced a Whittle estimator for a linear process in the case when the innovation variance depends on \(\theta.\) We estimate \(\theta\) by use of the Whittle likelihood for the square-transformed stretch \(Y_1, \ldots, Y_n.\) That is,

\[ \hat{\theta}_n^W = \arg\min_\theta \int_{-\pi}^{\pi} \left\{ \log f_{Y,\theta}(\lambda) + \frac{I_Y(\lambda)}{f_{Y,\theta}(\lambda)} \right\} d\lambda \]

where \(I_Y(\lambda)\) is the periodogram i.e., \(I_Y(\lambda) = \frac{1}{2\pi n} \sum_{i=1}^{n} Y_i e^{it\lambda^2}.\)

To describe the asymptotics of \(\hat{\theta}_n^W,\) we need the following assumption.

**Assumption 2.**

(i) \(f_{Y,\theta}\) is square-integrable with respect to \(\lambda.\)

(ii) Let

\[ \sum_{l_1,l_2,l_3=-\infty}^{\infty} |C_{\epsilon,\theta}(l_1,l_2,l_3)| < \infty, \]

where \(C_{\epsilon,\theta}(l_1,l_2,l_3)\) is the fourth cumulant of \(\epsilon_t.\)

(iii)

\[ M(\theta) = \int_{-\pi}^{\pi} \left[ f_Y^2(\lambda) \frac{\partial}{\partial\theta} (f_{Y,\theta}(\lambda))^{-1} \frac{\partial}{\partial\theta'} (f_{Y,\theta}(\lambda))^{-1} \right] d\lambda \]

is a nonsingular matrix.

Under Assumptions 1 and 2, from Hosoya and Taniguchi (1982) we obtain

\[ \sqrt{n}(\hat{\theta}_n^W - \theta) \xrightarrow{d} N(0, M(\theta)^{-1} V(\theta) M(\theta)^{-1}) \]
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where

\[ V(\theta) = 4\pi M(\theta) + 2\pi \int \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} (f_{\theta}^{-1}(\lambda_1)) \frac{\partial}{\partial \theta'} (f_{\theta}^{-1}(\lambda_2)) \right] f_{Y,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2, \]

\[ f_{Y,\theta}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{(2\pi)^3} \sum_{t_1, t_2, t_3 = -\infty}^{\infty} \exp \{-i(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3)\} C_{Y,\theta}(t_1, t_2, t_3) \]

and \( C_{Y,\theta}(t_1, t_2, t_3) \) is the fourth order cumulant of \( Y(t), Y(t+t_1), Y(t+t_2), Y(t+t_3) \). Further

\[ \begin{align*}
2\pi \int \int_{-\pi}^{\pi} & \left[ \frac{\partial}{\partial \theta} (f_{\theta}^{-1}(\lambda_1)) \frac{\partial}{\partial \theta'} (f_{\theta}^{-1}(\lambda_2)) \right] f_{Y,-\theta}(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 \\
= & \frac{(2\pi)^2}{(m-1)^2} \left[ \frac{\partial}{\partial \theta} \left( \frac{1}{E(\sigma_1^2)} \right) \frac{\partial}{\partial \theta'} \left( \frac{1}{E(\sigma_1^2)} \right) \right] (2\pi f_{\eta,\theta}(0) - 2(E(\eta_0))^2)
\end{align*} \]

where \( \eta_t = c_t^2, f_{\eta,\theta} \) is the spectral density of \( \eta_t \) and \( m = E[u_1^4] \). The proof of (1) is placed in Section 4.

Let

\[ V_2(\theta) = 4\pi M(\theta)^{-1}, \]

\[ V_4(\theta) = 2\pi M(\theta)^{-1} \int \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} (f_{\theta}^{-1}(\lambda_1)) \frac{\partial}{\partial \theta'} (f_{\theta}^{-1}(\lambda_2)) \right] f_{\theta,-\theta}(\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 M(\theta)^{-1}. \]

Then the asymptotic variance \( V_W(\theta) \) of \( \sqrt{n}(\hat{\theta}_n^W - \theta) \) is written as

\[ V_W(\theta) = V_2(\theta) + V_4(\theta). \]

Note that \( V_2(\theta) \) is known to be the inverse of time series Fisher information \( F(\theta)^{-1} \) in terms of the second order spectra. If the asymptotic variance satisfies \( V_W(\theta) = F(\theta)^{-1} \), we say that \( \hat{\theta}_n^W \) is asymptotically efficient in the sense of second order spectra. From the above discussion, if

\[ 2\pi f_{\eta,\theta}(0) - 2(E(\eta_0))^2 \leq 0, \]

then (2) implies

\[ V_W(\theta) \leq F(\theta)^{-1}. \]

Examples satisfying \( V_W(\theta) = F(\theta)^{-1} \) and \( V_W(\theta) < F(\theta)^{-1} \) are given as follows.

(i) Let \( P(u_t = 1) = P(u_t = -1) = \frac{1}{2} \) then \( V_4(\theta) = 0 \), that is the Whittle estimator is asymptotically efficient in the sense of second order spectra.

(ii) Let \( P(u_t = 0) = \frac{1}{2}, \ P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{4} \) and \( a_j = 0 \ j \geq 1 \) then \( V_4(\theta) \) is negative, that is, \( V_W(\theta) \) is smaller than \( F(\theta)^{-1} \).

3. Numerical examples

In this section we evaluate the asymptotic variance numerically. Let us consider the following ARCH(1) models.

\[ X_t = \sqrt{a_0 + aX_{t-1}^2} u_t \quad (\theta = a) \]
We examine the asymptotic variance $V_W = V_W(a)$, $V_2 = V_2(a)$ and $V_4 = V_4(a)$ of the Whittle estimator for $\theta = a$. Since the values of $V_W$, $V_2$ and $V_4$ are not affected by $a_0$, we set $a_0 = 1$.

In Figures 1-5, we plotted $V_W$, $V_2$ and $V_4$ ($0 \leq a < 0.1$) for the case of (1) $u_t \sim N(0, 1)$, (2) $u_t \sim$ Logistic, (3) $u_t \sim$ T-distribution with degrees of freedom 60, (4) $u_t \sim$ T-distribution with degrees of freedom 30, (5) $u_t \sim$ T-distribution with degrees of freedom 10, respectively.

**Figures 1-5 are about here.**

We can see that, $V_W$ becomes large as the tail of the distribution becomes heavy, $V_2$ and $V_4$ are much larger than $V_2$, and that $V_2$ goes down as $a$ increases.

In Figure 6, we plotted $V_W, V_2$ and $V_4$ for (3) with $P(u_t = 1) = P(u_t = -1) = \frac{1}{7}$, $0 \leq a \leq 0.1$.

**Figure 6 is about here.**

We can see that, $V_W = V_2$ and $V_2$ goes down as $a$ increases.

In Figure 7, we plotted $V_W, V_2$ and $V_4$ for (3) with $P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{7}$, $P(u_t = 0) = \frac{1}{2}$, $0 \leq a \leq 0.4$.

**Figure 7 is about here.**

We can see that, $V_W < V_2$, and $V_2$ goes down as $a$ increases.

4. Appendix.

**Proof of (1).**

Since the spectral density of $\{\epsilon_t\}$ is given by

$$f_{\epsilon, \theta}(\lambda) = \frac{E[u_t^4 - 1]E[\sigma_t^4]}{2\pi}$$

and $\epsilon_t$ takes the form

$$\epsilon_t = \sum_{j=0}^{\infty} \beta_j V_{t-j}$$

where $\beta_0 = 1$, $\beta_j = -a_j (j \geq 1)$, $\alpha \sum_{j=0}^{\infty} \beta_j = a_0$ and $V_t = Y_t - \alpha$. The transfer function $B(\lambda)$ and the spectral density of $Y$ are obtained by

$$B(\lambda) = \sum_{j=0}^{\infty} \beta_j \exp(-ij\lambda)$$

$$f_{Y, \theta}(\lambda) = \frac{E[u_t^4 - 1]E[\sigma_t^4]}{2\pi} \times \frac{1}{|B(\lambda)|^2}.$$
Noting Remark 3.1 of Hosoya and Taniguchi (1982), we have

\[
2\pi \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \phi}(f_{Y,\theta}(\lambda_1)) \frac{\partial}{\partial \phi'}(f_{Y,\theta}^{-1}(\lambda_2)) \right] f_{Y,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 \\
= \frac{(2\pi)^3}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] \\
\times \int_{-\pi}^{\pi} B(-\lambda_1) B(-\lambda_2) B(\lambda_1) f_{Y,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 \\
= \frac{(2\pi)^3}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] \int_{-\pi}^{\pi} f_{\epsilon,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 \\
= \frac{1}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] \\
\times \sum_{l_1, l_2, l_3} \sum_{l=-\infty}^{\infty} C_{\epsilon,\theta}(l_1, l_2, l_3) \int_{-\pi}^{\pi} \exp(i\lambda_1 l_1) d\lambda_1 \int_{-\pi}^{\pi} \exp(i\lambda_2 l_2) d\lambda_2 \\
= \frac{(2\pi)^2}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] \sum_{l=-\infty}^{\infty} \text{Cum}(\epsilon_0, \epsilon_0, \epsilon_1, \epsilon_1) \\
= \frac{(2\pi)^2}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] (E[\epsilon_0^2] - 3(E[\epsilon_0]))^2 + \sum_{l\neq 0} R_\eta(l) \\
= \frac{(2\pi)^2}{(m-1)^2} \left[ \frac{\partial}{\partial \phi} \left( \frac{1}{E(\sigma_1^4)} \right) \frac{\partial}{\partial \phi'} \left( \frac{1}{E(\sigma_1^4)} \right) \right] (2\pi f_{\eta,\theta}(0) - 2(E[\eta_0])^2)
\]

where \( f_{\epsilon,\theta}(-\lambda_1, \lambda_2, \lambda_3) \) is a fourth cumulant spectrum of \( \epsilon_t \) and \( R_\eta(l) \) is an autocovariance function of \( \eta_t \).

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**References**

Figure 1: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with $u_t \sim N(0,1)$, $0 \leq a \leq 0.1$.

Figure 2: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with $u_t \sim \text{Logistic}$, $0 \leq a \leq 0.1$. 
Figure 3: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2u_t}$) with $u_t \sim T$-distribution (degrees of freedoms is 60), $0 \leq a \leq 0.1$.

Figure 4: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2u_t}$) with $u_t \sim T$-distribution (degrees of freedoms is 30), $0 \leq a \leq 0.1$. 
Figure 5: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX^2_{t-1}u_t}$) with $u_t \sim T$-distribution (degrees of freedoms is 10), $0 \leq a \leq 0.1$.

Figure 6: $V_W$, $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX^2_{t-1}u_t}$) with $P(u_t = 1) = P(u_t = -1) = \frac{1}{2}$, $0 \leq a \leq 0.1$. 
Figure 7: $V_W$ (dashed line), $V_2$ (solid line) and $V_4$ (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2u_t}$) with $P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{4}, P(u_t = 0) = \frac{1}{2}, 0 \leq a \leq 0.4$.

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