Scientiae Mathematicae Japonicae Online, **e-2006**, 1035–1043

NOTE ON ASYMPTOTICS OF WHITTLE ESTIMATORS FOR SQUARE TRANSFORMED ARCH(∞) MODELS

Tomoyuki AMANO

Received June 22, 2006; revised July 12, 2006

Abstract. Whittle estimators are important and fundamental in time series estimation. We apply Whittle estimation to the square transformed ARCH(∞) models, which can be expressed as linear processes. Whittle estimators for linear processes are known to be asymptotically normal with asymptotic variance $V_W = V_2 + V_4$, where V_2 is written in terms of the second-order spectra only, and V_4 includes the fourth-order cumulant spectra. This note gives a useful and explicit expression of V_4, and shows that there exists a case of $V_4 < 0$. Since V_2 can be regarded as the inverse of Fisher information F^{-1} in terms of the second-order spectra, the result implies that there is a case when $V_W < F^{-1}$. For ARCH models with various innovation distributions, we evaluate V_W, V_2 and V_4 numerically. The numerical studies elucidate some interesting features of the Whittle estimators.

1. Introduction

ARCH model arises frequently in economic time series, which was introduced by Engle (1982). This model assumes the dependence of the one period forecast variance on a finite number of passed variables. Robinson (1991) extended this model to ARCH(∞) model, the one period forecast variance depends on an infinite number of passed variables. Giraitis et al. (2000) have derived sufficient conditions for the existence of a stationary solution of ARCH(∞) model. The square transformed ARCH(∞) models have representations as linear processes.

To estimate a parameter θ of linear process, Whittle estimation is widely used. Recently, Whittle estimators for a class of parametric ARCH(∞) models are shown to be asymptotically normal in Giraitis and Robinson (2001). For a general class of linear processes, Hosoya and Taniguchi (1982) introduced a Whittle estimator, which is obtained by minimizing $\int_{-\pi}^{\pi} \left\{ \log f_\theta(\lambda) + \frac{I_x(\lambda)}{f_\theta(\lambda)} \right\} d\lambda$, where $I_x(\lambda)$ is the periodogram and $f_\theta(\lambda)$ is the spectral density of the process concerned, and derived the asymptotic variance, $V_W = V_2 + V_4$, where V_2 is written in terms of the second-order spectra only, and V_4 includes the fourth-order cumulant spectra. V_2 is known to be the inverse of time series Fisher information F^{-1}. In this note we apply the Whittle estimators to the squared ARCH(∞) models, and investigate behavior of V_W. Then it is shown that there is a case when $V_W < F^{-1}$. Numerical evaluation for V_W, V_2 and V_4 are also provided.

This note is organized as follows. Section 2 describes Whittle estimators for the square transformed ARCH(∞) models and gives results of these asymptotics and provides a useful and explicit representation of V_4. We also give two examples satisfying $V_W = F^{-1}$ and $V_W < F^{-1}$. Section 3 provides numerical studies of V_W, V_2 and V_4. The results elucidate some interesting features of the asymptotics of the Whittle estimator for the parameter of ARCH(∞) model. Proof is relegated to Section 4.

2000 *Mathematics Subject Classification.* Primary 62M10, 62M15; Secondary 62-07, 62-09.
Key words and phrases. ARCH(∞) model, Whittle estimator, Asymptotic efficiency, Spectral density.

2. Estimation and Asymptotics

Throughout this paper we deal with the following ARCH(∞) model.

$$X_t = u_t \sqrt{a_0 + \sum_{j=1}^{\infty} a_j X_{t-j}^2}$$

where $a_0 > 0$, $a_j \geq 0$, $j = 1, \cdots$, and $\{u_t\}$ is a sequence of i.i.d random variables with mean 0, variance 1. Let $Y_t \equiv X_t^2$, $\xi_t \equiv u_t^2$, $\sigma_t^2 = a_0 + a_1 Y_{t-1} + \cdots$, then we may write $Y_t = \sigma_t^2 \xi_t$ and $E[\xi_t] = 1$. If we define $\epsilon_t \equiv Y_t - a_0 - a_1 Y_{t-1} - \cdots = \sigma_t^2(\xi_t - 1)$, then ϵ_t is an uncorrelated process. So Y_t is an autoregressive process. We impose the following assumption for the estimation of the parameter of $\{X_t\}$.

Assumption 1.
(i)$E[u_t^4]^{\frac{1}{2}} \sum_{j=1}^{\infty} a_j < 1$
(ii)$E u_t^8 < \infty$.
(iii)a_0 and a_j's are functions of an unknown parameter $\theta = (\theta_1, \theta_2, \cdots, \theta_q)$.
(iv)$a_j = a_j(\theta)$'s are differentiable with respect to θ.

The assumption (i) implies the second order stationarity of $\{Y_t\}$ (see, Giraitis et al (2000)). Hence $\{\epsilon_t\}$ is second order stationary. Henceforth we denote the spectral densities of $\{Y_t\}$ and $\{\epsilon_t\}$ by $f_{Y,\theta}$ and $f_{\epsilon,\theta}$, respectively.

Hosoya and Taniguchi (1982) introduced a Whittle estimator for a linear process in the case when the innovation variance depends on θ. We estimate θ by use of the Whittle likelihood for the square-transformed stretch $Y_1, \cdots, Y_n$. That is,

$$\hat{\theta}_n^W \equiv \text{argmin}_\theta \int_{-\pi}^{\pi} \left\{ \log f_{Y,\theta}(\lambda) + \frac{I_Y(\lambda)}{f_{Y,\theta}(\lambda)} \right\} d\lambda$$

where $I_Y(\lambda)$ is the periodogram i.e., $I_Y(\lambda) = \frac{1}{2\pi n} |\sum_{t=1}^{n} Y_t e^{it\lambda}|^2$.
To describe the asymptotics of $\hat{\theta}_n^W$, we need the following assumption.

Assumption 2.
(i) $f_{Y,\theta}$ is square-integrable with respect to λ.
(ii) Let

$$\sum_{l_1,l_2,l_3=-\infty}^{\infty} |C_{\epsilon,\theta}(l_1, l_2, l_3)| < \infty,$$

where $C_{\epsilon,\theta}(l_1, l_2, l_3)$ is the fourth cumulant of ϵ_t.
(iii)

$$M(\theta) = \int_{-\pi}^{\pi} \left[f_Y^2(\lambda) \frac{\partial}{\partial \theta} (f_{Y,\theta}(\lambda))^{-1} \frac{\partial}{\partial \theta'} (f_{Y,\theta}(\lambda))^{-1} \right] d\lambda$$

is a nonsingular matrix.

Under Assumptions 1 and 2, from Hosoya and Taniguchi (1982) we obtain

$$\sqrt{n}(\hat{\theta}_n^W - \theta) \xrightarrow{d} N(0, M(\theta)^{-1} V(\theta) M(\theta)^{-1})$$

where

$$V(\theta) = 4\pi M(\theta) + 2\pi \int\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial\theta}(f_{Y,\theta}^{-1}(\lambda_1)) \ \frac{\partial}{\partial\theta'}(f_{Y,\theta}^{-1}(\lambda_2))\right] f_{Y,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2,$$

$$f_{Y,\theta}(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{(2\pi)^3} \sum_{t_1,t_2,t_3=-\infty}^{\infty} \exp\{-i(\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3)\} C_{Y,\theta}(t_1, t_2, t_3)$$

and $C_{Y,\theta}(t_1, t_2, t_3)$ is the fourth order cumulant of $Y(t), Y(t+t_1), Y(t+t_2), Y(t+t_3)$. Further

$$2\pi \int\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial\theta}(f_{Y,\theta}^{-1}(\lambda_1)) \ \frac{\partial}{\partial\theta'}(f_{Y,\theta}^{-1}(\lambda_2))\right] f_Y(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2$$

$$(1) \qquad = \frac{(2\pi)^2}{(m-1)^2} \left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right) \ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right] (2\pi f_{\eta,\theta}(0) - 2(E(\eta_0)))^2)$$

where $\eta_t = \epsilon_t^2$, $f_{\eta,\theta}$ is the spectral density of η_t and $m = E[u_t^4]$. The proof of (1) is placed in Section 4.

Let

$$V_2(\theta) = 4\pi M(\theta)^{-1},$$

$$V_4(\theta) = 2\pi M(\theta)^{-1} \int\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial\theta}(f_{Y,\theta}^{-1}(\lambda_1)) \ \frac{\partial}{\partial\theta'}(f_{Y,\theta}^{-1}(\lambda_2))\right] f_{Y,\theta}(-\lambda_1, \lambda_2, -\lambda_2) d\lambda_1 d\lambda_2 M(\theta)^{-1}.$$

Then the asymptotic variance $V_W(\theta)$ of $\sqrt{n}(\hat{\theta}_n^W - \theta)$ is written as

$$V_W(\theta) = V_2(\theta) + V_4(\theta).$$

Note that $V_2(\theta)$ is known to be the inverse of time series Fisher information $F(\theta)^{-1}$ in terms of the second order spectra. If the asymptotic variance satisfies $V_W(\theta) = F(\theta)^{-1}$, we say that $\hat{\theta}_n^W$ is asymptotically efficient in the sense of second order spectra. ¿From the above discussion, if

$$(2) \qquad 2\pi f_{\eta,\theta}(0) - 2(E(\eta_0)))^2 \leq 0,$$

then (2) implies

$$V_W(\theta) \leq F(\theta)^{-1}.$$

Examples satisfying $V_W(\theta) = F(\theta)^{-1}$ and $V_W(\theta) < F(\theta)^{-1}$ are given as follows.
(i)Let $P(u_t = 1) = P(u_t = -1) = \frac{1}{2}$ then $V_4(\theta) = 0$, that is the Whittle estimator is asymptotically efficient in the sense of second order spectra.
(ii)Let $P(u_t = 0) = \frac{1}{2}$, $P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{4}$ and $a_j = 0$ $j \geq 1$ then $V_4(\theta)$ is negative, that is, $V_W(\theta)$ is smaller than $F(\theta)^{-1}$.

3. Numerical examples

In this section we evaluate the asymptotic variance numerically. Let us consider the following ARCH(1) models.

$$(3) \qquad X_t = \sqrt{a_0 + aX_{t-1}^2} u_t \quad (\theta = a)$$

We examine the asymptotic variance $V_W = V_W(a)$, $V_2 = V_2(a)$ and $V_4 = V_4(a)$ of the Whittle estimator for $\theta = a$. Since the values of V_W, V_2 and V_4 are not affected by a_0, we set $a_0 = 1$.

In Figures 1-5, we plotted V_W, V_2 and V_4 $(0 \leq a < 0.1)$ for the case of (1) $u_t \sim N(0,1)$, (2) $u_t \sim$Logistic, (3) $u_t \sim$T-distribution with degrees of freedom 60, (4) $u_t \sim$T-distribution with degrees of freedom 30, (5) $u_t \sim$T-distribution with degrees of freedom 10, respectively.

Figures 1-5 are about here.

We can see that, V_W becomes large as the tail of the distribution becomes heavy, V_W and V_4 are much larger than V_2, and that V_2 goes down as a increases.

In Figure 6, we plotted V_W,V_2 and V_4 for (3) with $P(u_t = 1) = P(u_t = -1) = \frac{1}{2}$, $0 \leq a \leq 0.1$.

Figure 6 is about here.

We can see that, $V_W = V_2$ and V_2 goes down as a increases.

In Figure 7, we plotted V_W,V_2 and V_4 for (3) with $P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{4}$, $P(u_t = 0) = \frac{1}{2}$, $0 \leq a \leq 0.4$.

Figure 7 is about here.

We can see that, $V_W < V_2$, and V_2 goes down as a increases.

4. Appendix.

Proof of (1).

Since the spectral density of $\{\epsilon_t\}$ is given by

$$f_{\epsilon,\theta}(\lambda) = \frac{E[u_t^4 - 1]E[\sigma_t^4]}{2\pi}$$

and ϵ_t takes the form

$$\epsilon_t = \sum_{j=0}^{\infty} \beta_j V_{t-j}$$

where $\beta_0 = 1$, $\beta_j = -a_j (j \geq 1)$, $\alpha \sum_{j=0}^{\infty} \beta_j = a_0$ and $V_t = Y_t - \alpha$. The transfer function $B(\lambda)$ and the spectral density of Y are obtained by

$$B(\lambda) = \sum_{j=0}^{\infty} \beta_j \exp(-ij\lambda)$$

$$f_{Y,\theta}(\lambda) = \frac{E[u_t^4 - 1]E[\sigma_t^4]}{2\pi} \times \frac{1}{|B(\lambda)|^2}.$$

Noting Remark 3.1 of Hosoya and Taniguchi (1982), we have

$$
\begin{aligned}
&2\pi \int\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial\theta}(f_{Y,\theta}^{-1}(\lambda_1))\ \frac{\partial}{\partial\theta'}(f_{Y,\theta}^{-1}(\lambda_2))\right] f_{Y,\theta}(-\lambda_1,\lambda_2,-\lambda_2)d\lambda_1 d\lambda_2 \\
&= \frac{(2\pi)^3}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right] \\
&\qquad\qquad \times \int\int_{-\pi}^{\pi} B(-\lambda_1)B(\lambda_2)B(-\lambda_2)B(\lambda_1) f_{Y,\theta}(-\lambda_1,\lambda_2,-\lambda_2)d\lambda_1\lambda_2 \\
&= \frac{(2\pi)^3}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right]\int\int_{-\pi}^{\pi} f_{\epsilon,\theta}(-\lambda_1,\lambda_2,-\lambda_2)d\lambda_1\lambda_2 \\
&= \frac{1}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right] \\
&\qquad\qquad \times \sum_{l_1,l_2,l_3=-\infty}^{\infty} C_{\epsilon,\theta}(l_1,l_2,l_3)\int_{-\pi}^{\pi}\exp(il_1\lambda_1)d\lambda_1 \int_{-\pi}^{\pi}\exp i(l_3-l_2)\lambda_2 d\lambda_2 \\
&= \frac{(2\pi)^2}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right]\sum_{l=-\infty}^{\infty}\mathrm{Cum}(\epsilon_0,\epsilon_0,\epsilon_l,\epsilon_l) \\
&= \frac{(2\pi)^2}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right](E[\eta_0^2]-3(E(\eta_0)))^2+\sum_{l\neq 0}R_\eta(l)) \\
&= \frac{(2\pi)^2}{(m-1)^2}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{E(\sigma_t^4)}\right)\ \frac{\partial}{\partial\theta'}\left(\frac{1}{E(\sigma_t^4)}\right)\right](2\pi f_{\eta,\theta}(0)-2(E(\eta_0)))^2)
\end{aligned}
$$

where $f_{\epsilon,\theta}(\lambda_1,\lambda_2,\lambda_3)$ is a fourth cumulant spectrum of ϵ_t and $R_\eta(l)$ is an autocovariance function of η_t.

Acknowledgments

The author would like to express his sincere thanks to Professor Masanobu Taniguchi for his instructive advice.

References

[1] Engle, R., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica **50**, 987–1007.

[2] Giraitis, L., Kokoszka, P., Leipus, R., 2000. Stationary ARCH models: dependence structure and central limit theorem. Econometric Theory **16**, 3–22.

[3] Giraitis, L., Robinson, P.M., 2001. Whittle estimation of arch models. Econometric Theory. **17**, 608-631.

[4] Hosoya, Y., Taniguchi, M., 1982. A central limit theorem for stationary processes and the parameter estimation of linear processes. Ann. Statist. **10**, 132–153.

[5] Robinson, P.M., 1991. Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. J. Econometrics **47**, 67-84.

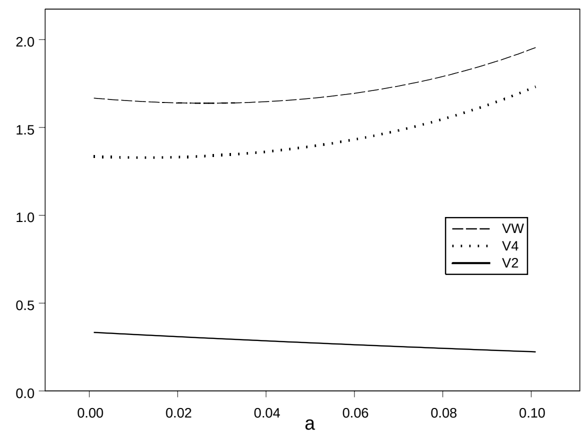

Figure 1: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with $u_t \sim N(0,1)$, $0 \leq a \leq 0.1$.

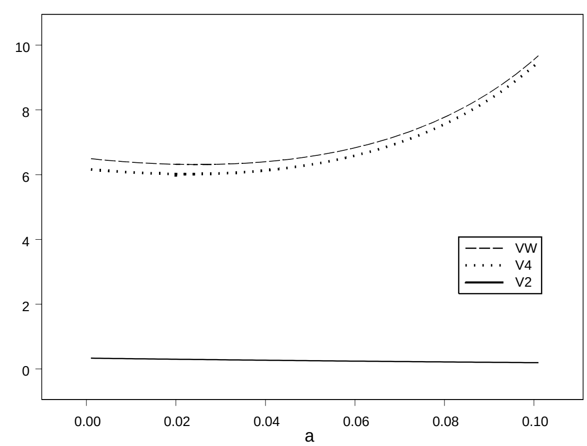

Figure 2: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with $u_t \sim$Logistic, $0 \leq a \leq 0.1$.

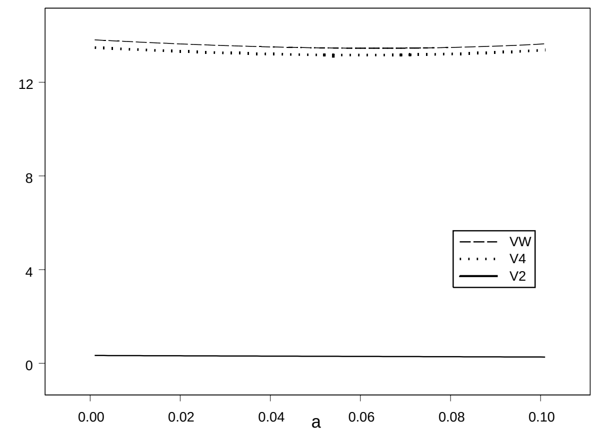

Figure 3: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with u_t ∼T-distribution (degrees of freedoms is 60), $0 \le a \le 0.1$.

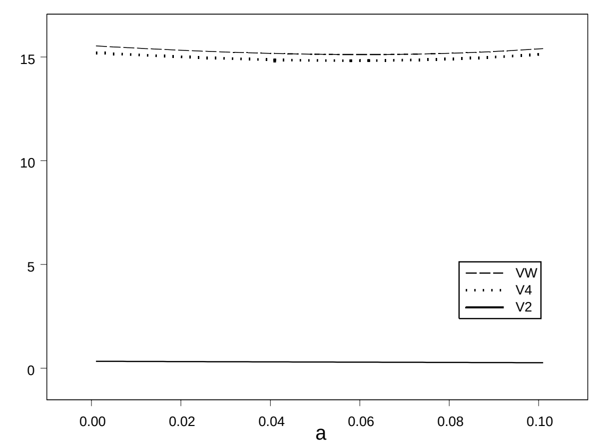

Figure 4: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with u_t ∼T-distribution (degrees of freedoms is 30), $0 \le a \le 0.1$.

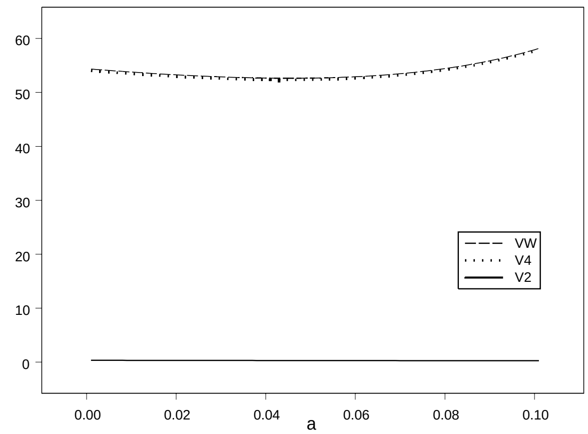

Figure 5: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2} u_t$) with u_t ~T-distribution (degrees of freedoms is 10), $0 \le a \le 0.1$.

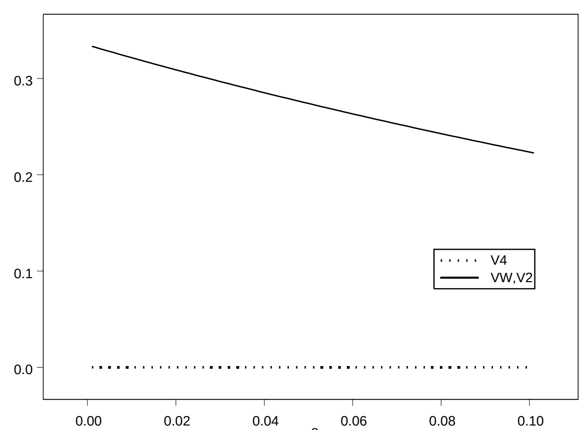

Figure 6: V_W, V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2} u_t$) with $P(u_t = 1) = P(u_t = -1) = \frac{1}{2}$, $0 \le a \le 0.1$.

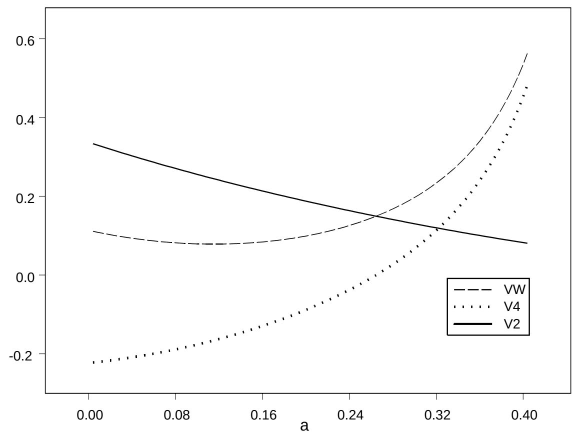

Figure 7: V_W (dashed line), V_2 (solid line) and V_4 (dotted line) of the Whittle estimators for the parameter of ARCH(1) models ($X_t = \sqrt{a_0 + aX_{t-1}^2}u_t$) with $P(u_t = \sqrt{2}) = P(u_t = -\sqrt{2}) = \frac{1}{4}$, $P(u_t = 0) = \frac{1}{2}$, $0 \leq a \leq 0.4$.

Tomoyuki.AMANO
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
3-4-1 Okubo Shinjuku-ku
Tokyo 169-8555
Japan
E-mail: tomtochami@ruri.waseda.jp