N-FLUID VARIETIES

Dedicated to the memory of Dietmar Schweigert

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Abstract. If a variety V contains all algebras derived by hypersubstitutions, then it is called *solid*. If the only derived algebras obtained from an algebra $\mathcal{A}$ in V are those isomorphic to $\mathcal{A}$, then the variety V is called *fluid*. We generalize this concept to the concepts of n-fluid, weakly fluid and $\aleph_0$-fluid varieties, prove some general properties and determine all n-fluid varieties of bands.

1 Preliminaries

Let $W_\tau(X)$ be the set of all terms built up from operation symbols from an indexed set $(f_i)_{i\in I}$ and from variables of an alphabet X. Hypersubstitutions map operation symbols to terms and preserve arities. Any hypersubstitution $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following inductive definition

(i) $\hat{\sigma}[x] := x$ for every variable $x \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$ for compound terms $f_i(t_1, \dots, t_{n_i})$.

Using this extension we may define a multiplication $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ of hypersubstitutions where $\circ$ is the usual composition of mappings. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. Using the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, \dots, x_{n_i})$ for all $i \in I$ one obtains the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all hypersubstitutions of type τ. An identity $s \approx t$ in a variety V of algebras of type τ is said to be a *hyperidentity* in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V for any $\sigma \in Hyp(\tau)$. A variety V is called *solid* if each of its identities is a hyperidentity. Throughout we will use the following notation: Let IdV be the set of all identities satisfied in the variety V, and let $\mathcal{A} \models s \approx t$ mean that the equation $s \approx t$ is an identity in the algebra $\mathcal{A}$. Let $P(V)$ be the set of all hypersubstitutions of $Hyp(\tau)$ which preserve all identities in the variety V, i.e. $P(V) := \{\sigma \mid \forall s \approx t \in IdV(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV)\}$. Hypersubstitutions from $P(V)$ are called *V-proper* ([6]). It is easy to see that $P(V)$ forms a submonoid of $Hyp(\tau)$. The variety V is solid if and only if $P(V) = Hyp(\tau)$. J. Płonka defined in [6] the following binary relation on $Hyp(\tau)$:

$$\sigma_1 \sim_V \sigma_2 :\Longleftrightarrow \forall i \in I(\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV).$$

Clearly, the relation $\sim_V$ is an equivalence relation on $Hyp(\tau)$ but in general it is not a congruence relation. Further, it was proved that the set $P(V)$ is a union of full equivalence classes with respect to $\sim_V$ (i.e. $P(V)$ is $\sim_V$-saturated), i.e. if V is a variety of type τ, if $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$. The elements of $P_0(V) := [\sigma_{id}]_{\sim_V}$ are called *inner* hypersubstitutions. Clearly, the set of inner hypersubstitutions forms a submonoid of $P(V)$ (see e.g. [2], [3]).

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A hypersubstitution can also be applied to an algebra $\mathcal{A}$. The algebra $\sigma(\mathcal{A})$ derived from a hypersubstitution σ is the algebra $\sigma(\mathcal{A}) := (A; (\sigma(f_i)^{\mathcal{A}})_{i\in I})$. Here $\sigma(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma(f_i)$. For a variety V of algebras of type τ let $\sigma(V)$ be the class of all derived algebras $\sigma(\mathcal{A})$, for $\mathcal{A} \in V$. Using the concept of a derived algebra in [4] the following binary relation $\sim_{V-iso}$ on $Hyp(\tau)$ was defined:

$$\sigma_1 \sim_{V-iso} \sigma_2 :\Longleftrightarrow \forall \mathcal{A} \in V(\sigma_1(\mathcal{A}) \cong \sigma_2(\mathcal{A})).$$

Since $\sigma_1 \sim_V \sigma_2$ is equivalent to $\sigma_1(\mathcal{A}) = \sigma_2(\mathcal{A})$, the relation $\sim_V$ is a subrelation of $\sim_{V-iso}$. The relation $\sim_{V-iso}$ is in general not a congruence relation on $Hyp(\tau)$. As in the case of $\sim_V$, the set $P(V)$ of all proper hypersubstitutions is a union of equivalence classes with respect to $\sim_{V-iso}$ (i.e. $P(V)$ is $\sim_{V-iso}$-saturated). In some cases, for instance for varieties of bands, the relations $\sim_V$ and $\sim_{V-iso}$ collapse. In general we have $\sim_V \subseteq \sim_{V-iso}$. The cardinality $d_p(V) := |P(V)/\sim_V|$ which is called the *degree* of proper hypersubstitutions of V measures the degree of invariance of a variety V under application of hypersubstitutions. In [4] we introduced the degree $d_p(V)$ of proper hypersubstitutions as well as $isd_p(V) := |P(V)/\sim_{V-iso}|$, the isomorphism degree of proper hypersubstitutions. The inclusion $\sim_V \subseteq \sim_{V-iso}$ implies $d_p(V) \geq isd_p(V)$. In [5] (see also [1]) D. Schweigert introduced the concept of a fluid variety.

Definition 1.1. A variety V is called *fluid* if the following implication is satisfied for every algebra $\mathcal{A} \in V$ and every $\sigma \in Hyp(\tau)$:

$$\sigma(\mathcal{A}) \in V \Longrightarrow \sigma(\mathcal{A}) \cong \mathcal{A}. \qquad (F_1)$$

This means that a fluid variety contains no proper derived algebra. The following simple consequences are easy to prove:

Proposition 1.2. ([5],[4])

(i) *Let V be a fluid variety of type τ. Then $P(V) = [\sigma_{id}]_{\sim_{V-iso}}$, i.e. $isd_p(V) = 1$.*

(ii) *Let V be a solid variety of type τ. Then V is fluid if and only if $P(V) = Hyp(\tau) = [\sigma_{id}]_{\sim_{V-iso}}$, i.e. if $isd_p(V) = 1$.*

In [5] the following was proved:

Theorem 1.3. *Every subvariety of a fluid variety is fluid.*

In the next sections we want to generalize the concept of a fluid variety in several directions.

2 N-Fluid Varieties

We define n-fluid varieties as follows:

Definition 2.1. Let $n \geq 1$ be a natural number. A variety V of type τ is called *n-fluid* if there are hypersubstitutions $\sigma_1, \ldots, \sigma_n \in P(V)$ with $\sigma_i \not\sim_{V-iso} \sigma_j$ for $1 \leq i < j \leq n$ such that for all $\mathcal{A} \in V$ and for all $\sigma \in Hyp(\tau)$ the following implication holds:

$$\sigma(\mathcal{A}) \in V \Longrightarrow \exists k \in \{1, \ldots, n\} \text{ such that } \sigma(\mathcal{A}) \cong \sigma_k(\mathcal{A}). \qquad (Fl)$$

Fluid varieties are 1-fluid. Therefore this new concept generalizes that of a fluid variety. By definition, for any n-fluid variety V we have $isd_p(V) \geq n$. But we get:

Proposition 2.2. *For any n-fluid variety V with $d_p(V) = n$ we have $isd_p(V) = n$.*

Proof. We have $n = d_p(V) \geq isd_p(V) \geq n$ and thus $isd_p(V) = n$. □

The converse of Proposition 2.2 is in general not true. But for a solid variety we have:

Proposition 2.3. *If V is a solid variety with $isd_p(V) = n$, then V is n-fluid.*

Proof. Because of $isd_p(V) = n$ there are hypersubstitutions $\sigma_1, \ldots, \sigma_n$ such that $P(V) = \bigcup_{i=1}^{n} [\sigma_i]_{\sim_{V-iso}}$ and $\sigma_i \not\sim_{V-iso} \sigma_j$ for $1 \leq i < j \leq n$. Since V is solid, we have $P(V) = Hyp(\tau) = \bigcup_{i=1}^{n} [\sigma_i]_{\sim_{V-iso}}$. Therefore for every hypersubstitution $\sigma \in Hyp(\tau)$ there is an integer $k \in \{1, \ldots, n\}$ such that $\sigma \sim_{V-iso} \sigma_k$. Thus, for every $\mathcal{A} \in V$ we have $\sigma(\mathcal{A}) \cong \sigma_k(\mathcal{A})$. Since $\sigma(\mathcal{A}) \in V$ for every $\mathcal{A} \in V$ when V is solid, the implication (Fl) is satisfied. This shows that V is n-fluid. □

Now we want to prove that the proper hypersubstitutions from Definition 2.1 may be taken to be hypersubstitutions of special kind. A projection hypersubstitution is a mapping which maps each operation symbol of the type to a variable, i.e. for every $i \in I$ there is an integer j with $1 \leq j \leq n_i$ such that $\sigma(f_i) = x_j$. Non-trivial algebras derived by projection hypersubstitutions are called *projection* algebras. All fundamental operations of a projection algebra are projections, i.e. mappings $e_j^{n_i,A} : A^{n_i} \longrightarrow A$ satisfying $e_j^{n_i,A}(a_1, \ldots, a_{n_i}) = a_j$ for $1 \leq j \leq n_i$. Let RA_τ be the variety of type τ generated by the set of all projection algebras of this type. It is well-known (see e.g. [2]) that RA_τ (which is called the variety of *rectangular* algebras) is the least non-trivial solid variety of type τ.
If our type is $(1, \ldots, 1)$ then there is exactly one projection hypersubstitution for the type. Therefore we now consider types with at least one symbol of arity at least two, where more than one projection hypersubstitution exist. Then we have

Lemma 2.4. *Let V be a non-trivial variety of type $\tau = (n_i)_{i \in I}$ with at least one operation symbol of arity > 1 and let σ, σ' be distinct projection hypersubstitutions of this type. Then $\sigma \not\sim_{V-iso} \sigma'$.*

Proof. Since σ and σ' are distinct projection hypersubstitutions, we have $\sigma(f_j) = x_{k(j)} \neq x_{l(j)} = \sigma'(f_j)$ for at least one $j \in I$ with $n_j > 1$ and $k(j), l(j) \in \{1, \ldots, n_j\}$. Since V is non-trivial, it contains a non-trivial algebra $\mathcal{B} \in V$, and $\sigma(\mathcal{B})$, and $\sigma'(\mathcal{B})$ are non-trivial projection algebras. Since the set B has cardinality greater than one, we can choose some elements $a \neq b \in B$. Suppose that $\sigma \sim_{V-iso} \sigma'$. Then there is an isomorphism h from $\sigma(\mathcal{B})$ onto $\sigma'(\mathcal{B})$. Consider the n_j-tuple $(a, \ldots, a, b, a, \ldots, a)$ where $a_i = a$ for all $i \in \{1, \ldots, n_j\} \setminus \{l(j)\}$ and $a_{l(j)} = b$. Then

$$\begin{aligned} h(a) &= h(e_{k(j)}^{n_j,B}(a, \ldots, a, b, a, \ldots, a)) \\ &= h(\sigma(f_j)^B(a, \ldots, a, b, a, \ldots, a)) \\ &= \sigma'(f_j)^B(h(a), \ldots, h(a), h(b), h(a), \ldots, h(a)) \\ &= e_{l(j)}^{n_j,B}(h(a), \ldots, h(a), h(b), h(a), \ldots, h(a)) \\ &= h(b) \end{aligned}$$

which is a contradiction since h is bijective. Therefore $\sigma \not\sim_{V-iso} \sigma'$. □

We show next that if V is a non-trivial solid variety, then the identity hypersubstitution σ_{id} is not $\sim_{V-iso}$-related to any projection hypersubstitution.

Proposition 2.5. *Let V be a non-trivial solid variety of type $\tau = (n_i)_{i \in I}$ with at least one operation symbol of arity > 1 and let σ be a projection hypersubstitution. Then $\sigma \not\sim_{V-iso} \sigma_{id}$.*

Proof. Since there is a $j \in I$ such that $n_j > 1$, there is a projection hypersubstitution σ' which is different from the projection hypersubstitution σ. This means that there are integers $k(j), l(j) \in \{1, \dots, n_j\}, k(j) \neq l(j)$, such that $\sigma(f_j) = x_{k(j)} \neq x_{l(j)} = \sigma'(f_j)$. Since V is solid, we obtain $\sigma, \sigma' \in P(V)$. Suppose that $\sigma \sim_{V-iso} \sigma_{id}$. Then $\sigma(\mathcal{A}) \cong \mathcal{A}$ for each $\mathcal{A} \in V$ implies that every algebra $\mathcal{A} \in V$ is isomorphic to a projection algebra and for every $i \in I$ we have $\mathcal{A} \models f_i(x_1, \dots, x_{n_i}) \approx x_{k(i)}$ and thus $f_i(x_1, \dots, x_{n_i}) \approx x_{k(i)} \in IdV$. Since σ' is proper, we get $\hat{\sigma}'[f_i(x_1, \dots, x_{n_i})] \approx \hat{\sigma}'[x_{k(i)}] \in IdV$ and then also $x_{l(j)} \approx \hat{\sigma}'[f_j(x_1, \dots, x_{n_j})] \approx \hat{\sigma}'[x_{k(j)}] = x_{k(j)} \in IdV$, a contradiction since V is non-trivial. □

One more property of an n-fluid variety can be expressed by the following proposition.

Proposition 2.6. *Let V be a variety of type τ containing pairwise different projection algebras $\mathcal{A}_1, \dots, \mathcal{A}_m$, i.e. with different sequences of projections as fundamental operations and assume that $isd_p(V) = m + 1$. Then V is $(m+1)$-fluid if and only if V is not k-fluid for all $k \leq m$ and there are projection hypersubstitutions $\sigma_1, \dots, \sigma_m \in P(V)$ such that for any $\mathcal{A} \in V$ and $\sigma \in Hyp(\tau)$:*

$$\sigma(\mathcal{A}) \in V \Longrightarrow \sigma(\mathcal{A}) \cong \sigma_k(\mathcal{A}) \qquad (*)$$

for some $k \in \{0, 1, \dots, m\}$ where $\sigma_0 := \sigma_{id}$.

Proof. Since V contains m pairwise different projection algebras, we have $m \geq 2$. Therefore type τ has at least one operation symbol of arity > 1. Assume that V is not k-fluid for all $k \leq m$ and that there are projection hypersubstitutions $\sigma_1, \dots, \sigma_m \in P(V)$ which satisfy condition $(*)$ for all $\mathcal{A} \in V$ and for all $\sigma \in Hyp(\tau)$. Then Lemma 2.4 implies $\sigma_i \not\sim_{V-iso} \sigma_j$ for all $i, j \in \{1, \dots, m\}$ with $i \neq j$. Since $m \geq 2$, there are at least two different V-proper projection hypersubstitutions. This shows that $\sigma_i \not\sim_{V-iso} \sigma_{id}$ for all $i \in \{1, \dots, m\}$. Therefore $\sigma_i \not\sim_{V-iso} \sigma_j$ for all $i, j \in \{0, 1, \dots, m\}$ such that $i \neq j$. Because of $(*)$ the variety V is $(m+1)$-fluid.
Assume now that V is $(m+1)$-fluid. Since $\mathcal{A}_1, \dots, \mathcal{A}_m$ are projection algebras, for each $i \in I$, there are $k_1(i), \dots, k_m(i) \in \{1, \dots, m\}$ such that $f_i^{\mathcal{A}_l}(a_1, \dots, a_{n_i}) = a_{k_l(i)}$ for all $a_1, \dots, a_{n_i} \in A_l$ and $1 \leq l \leq m$. Clearly, $Mod\{f_i(x_1, \dots, x_{n_i}) \approx x_{k_l(i)} \mid i \in I\} = ModId\mathcal{A}_l \subseteq V$ for all $l \in \{1, \dots, m\}$. Let $l \in \{1, \dots, m\}$ and let σ_{k_l} be the hypersubstitution defined by $\sigma_{k_l}(f_i) = x_{k_l(i)}$ for all $i \in I$. Then $\sigma_{k_l}(\mathcal{B}) \in Mod\{f_i(x_1, \dots, x_{n_i}) \approx x_{k_l(i)} \mid i \in I\}$ for all $\mathcal{B} \in V$. Therefore $\sigma_{k_l}(\mathcal{B}) \in V$ for all $\mathcal{B} \in V$ and $\sigma_{k_l} \in P(V)$. Since $\mathcal{A}_1, \dots, \mathcal{A}_m$ are pairwise different projection algebras, the variety V is non-trivial and the hypersubstitutions $\sigma_{k_p}, \sigma_{k_q}$ are different for all $p, q \in \{1, \dots, m\}$ with $p \neq q$. Lemma 2.4 implies $\sigma_{k_p} \not\sim_{V-iso} \sigma_{k_q}$ for all $p, q \in \{1, \dots, m\}$ with $p \neq q$. Next we will show that $ModId\mathcal{A}_l \subset V$ for all $l \in \{1, \dots, m\}$. Suppose that there is an integer $l \in \{1, \dots, m\}$ such that $Mod\{f_i(x_1, \dots, x_{n_i}) \approx x_{k_l(i)} \mid i \in I\} = ModId\mathcal{A}_l = V$. Then $f_i(x_1, \dots, x_{n_i}) \approx x_{k_l(i)} \in IdV$ for all $i \in I$. For $p \in \{1, \dots, m\} \setminus \{l\}$ there is an integer $j \in I$ such that $\sigma_{k_p}(f_j) = x_{k_p(j)} \neq x_{k_l(j)} = \sigma_{k_l}(f_j)$. Since V is non-trivial, this implies that $\hat{\sigma}_{k_p}[f_j(x_1, \dots, x_{n_j})] \approx \hat{\sigma}_{k_p}[x_{k_l(j)}] \notin IdV$. So $\sigma_{k_p} \notin P(V)$, a contradiction. Therefore, $ModId\mathcal{A}_l \subset V$ for all $l \in \{1, \dots, m\}$. This means that there are algebras $\mathcal{B}_l \in V \setminus ModId\mathcal{A}_l$ for all $l \in \{1, \dots, m\}$. We have $\sigma_{k_l}(\mathcal{B}_l) \ncong \mathcal{B}_l = \sigma_{id}(\mathcal{B}_l)$ for all $l \in \{1, \dots, m\}$. Therefore $\sigma_{k_l} \not\sim_{V-iso} \sigma_{id}$. Since $isd_p(V) = m + 1$, we obtain that $P(V) = [\sigma_{id}]_{\sim_{V-iso}} \cup [\sigma_{k_1}]_{\sim_{V-iso}} \cup \ldots \cup [\sigma_{k_m}]_{\sim_{V-iso}}$ and we may take the set M consisting of these $m+1$ different hypersubstitutions to be a set of representatives of the $m+1$ classes with respect to $\sim_{V-iso}$. Since V is $(m+1)$-fluid, M satisfies the implication (Fl). This shows that condition (*) is satisfied. □

For minimal varieties we have:

Proposition 2.7. *If V is a minimal variety and $d_p(V) = n$, then there is a natural number m with $m \leq n$ such that V is m-fluid.*

Proof. We consider the following two cases for hypersubstitutions.
case 1: $\sigma \notin P(V)$. Then there is an identity $s \approx t \in IdV$ such that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin IdV$. Since V is minimal, so $IdV = Id\mathcal{A}$ for all non-trivial algebras $\mathcal{A} \in V$. Therefore $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{A}$ for all non-trivial algebras $\mathcal{A} \in V$. So $\sigma(\mathcal{A}) \notin V$ for all non-trivial algebras $\mathcal{A} \in V$. If $\mathcal{A}$ is trivial, then $\sigma(\mathcal{A})$ is also trivial and $\sigma(\mathcal{A}) \cong \mathcal{A}$. This means that the implication (Fl) is satisfied.
case 2: $\sigma \in P(V)$. From $d_p(V) = n$ we get that there are $\sigma_1, \dots, \sigma_n \in P(V)$ with $|P(V)/\!\sim_V| = |\{[\sigma_1]_{\sim_V}, \dots, [\sigma_n]_{\sim_V}\}| = n$. But $\sim_V \subseteq \sim_{V-iso}$, so $|P(V)/\!\sim_{V-iso}| \le |P(V)/\!\sim_V|$; i.e. there is an $m \in \mathbb{N}$ with $m \le n$ and $|P(V)/\!\sim_{V-iso}| = m$. Hence, there are $\sigma'_1, \dots, \sigma'_m \in \{\sigma_1, \dots, \sigma_n\}$ with $P(V)/\!\sim_{V-iso} = \{[\sigma'_1]_{\sim_{V-iso}}, \dots, [\sigma'_m]_{\sim_{V-iso}}\}$. From $\sigma \in P(V)$ we get that there is a $k \in \{1, \dots, m\}$ with $\sigma \sim_{V-iso} \sigma'_k$. So $\sigma(\mathcal{A}) \cong \sigma'_k(\mathcal{A})$ for all $\mathcal{A} \in V$. This means that the implication (Fl) is satisfied. □

We define the following generalization of n-fluidity.

Definition 2.8. A variety V of type τ is *weakly fluid* if for all $\mathcal{A} \in V$ and for all $\sigma \in Hyp(\tau)$ the following implication holds:

$$\sigma(\mathcal{A}) \in V \Longrightarrow \exists \sigma' \in P(V)\ (\sigma(\mathcal{A}) \cong \sigma'(\mathcal{A})).$$

Clearly every solid variety is weakly fluid. If V is n-fluid, $(n \ge 1)$, then it is also weakly fluid.

Proposition 2.9. *Let V be a weakly fluid variety of type $\tau = (n_i)_{i \in I}$ with at least one operation symbol of arity > 1 and $P(V) \subseteq \{\sigma_{id}\} \cup \{\sigma \in Hyp(\tau) \mid \sigma$ is a projection hypersubstitution$\}$. Then V is n-fluid iff $d_p(V) = n$.*

Proof. Clearly, if $d_p(V) = n$, then V is n-fluid since V is a weakly fluid variety and from $P(V) \subseteq \{\sigma_{id}\} \cup \{\sigma \in Hyp(\tau) \mid \sigma$ is a projection hypersubstitution $\}$, we have

$$|P(V)/\!\sim_{V-iso}| = |P(V)/\!\sim_V|$$

because the projection hypersubstitutions are all pairwise non-equivalent with respect to $\sim_{V-iso}$. Otherwise, if $\sigma \sim_{V-iso} \sigma_{id}$ where σ is a projection hypersubstitution defined by $\sigma(f_i) = x_{k(i)}$ for all $i \in I$, then $\mathcal{A} \cong \sigma(\mathcal{A})$ for all $\mathcal{A} \in V$ and $\sigma(\mathcal{A}) \models f_i(x_1, \dots, x_{n_i}) \approx x_{k(i)}$, so $f_i(x_1, \dots, x_{n_i}) \approx x_{k(i)} \in IdV$ there follows $\hat{\sigma}_{id}[f_i(x_1, \dots, x_{n_i})] = f_i(x_1, \dots, x_{n_i}) \approx x_{k(i)} = \hat{\sigma}[f_i(x_1, \dots, x_{n_i})]$. Therefore $\sigma \sim_V \sigma_{id}$. Conversely, assume that V is n-fluid. Then $isd_p(V) \ge n$ and there are $\sigma_1, \dots, \sigma_n \in P(V)$ which satisfy the condition (Fl). Now we show that $d_p(V) = n$.
case 1: $n = 1$. This means that σ_1 satisfies the implication (Fl). Let $\sigma \in P(V)$. Then $\sigma(\mathcal{A}) \in V$ for all $\mathcal{A} \in V$ implies $\sigma(\mathcal{A}) \cong \sigma_1(\mathcal{A})$ by (Fl). So $\sigma \sim_{V-iso} \sigma_1$. Therefore $P(V)/\!\sim_{V-iso} = \{[\sigma_1]_{\sim_{V-iso}}\}$, i.e. $d_p(V) = 1$.
case 2: $n \ne 1$. Now we show that $\sigma_{id} \in \{\sigma_1, \dots, \sigma_n\}$. Suppose that $\sigma_{id} \notin \{\sigma_1, \dots, \sigma_n\}$. From $P(V) \subseteq \{\sigma_{id}\} \cup \{\sigma \in Hyp(\tau) \mid \sigma$ is a projection hypersubstitution $\}$, we get that σ_i, $i \in \{1, \dots, n\}$ is a projection hypersubstitution. Since $\sigma_{id} \notin \{\sigma_1, \dots, \sigma_n\}$, for each $\mathcal{A} \in V$, $\mathcal{A} = \sigma_{id}(\mathcal{A}) \cong \sigma_i(\mathcal{A})$ for some $i \in \{1, \dots, n\}$ by (Fl). Because σ_i and σ_j, $i \ne j$ are different projection hypersubstitutions, we know that V cannot be 1-fluid since different projection algebras are not isomorphic. This means that V is a variety consisting only of projection algebras and at least two elements of V are projection algebras which have different sets of projections as fundamental operations, a contradiction because direct products of different projection algebras are not projection algebras. Therefore $\sigma_{id} \in \{\sigma_1, \dots, \sigma_n\}$. Now we show that $|P(V)/\!\sim_{V-iso}| = n$. Suppose that $|P(V)/\!\sim_{V-iso}| > n$. Then there is a projection hypersubstitution $\sigma \in P(V) \setminus \{\sigma_1, \dots, \sigma_n\}$ which satisfies the following condition:

$$\forall \mathcal{A} \in V \; \exists i \in \{1, \ldots, n\} \; (\sigma(\mathcal{A}) \cong \sigma_i(\mathcal{A})).$$

From $\sigma \not\sim_{V-iso} \sigma_{id}$ we get that there is a non-trivial algebra $\mathcal{A} \in V$ with $\sigma(\mathcal{A}) \ncong \sigma_{id}(\mathcal{A})$. There follows that there is a projection hypersubstitution σ' with $\sigma(\mathcal{A}) \cong \sigma'(\mathcal{A})$, a contradiction because $\sigma(\mathcal{A}), \sigma'(\mathcal{A})$ are distinct projection algebras. Therefore $|P(V)/\sim_{V-iso}| = n$ implies that $|P(V)/\sim_V| = n$, i.e. $d_p(V) = n$. □

3 N-Fluid Varieties of Bands

In this section for each variety V of bands we want to determine the integer n for which V is n-fluid.

First of all we show that the variety of all semigroups is not weakly fluid. Consider the variety $NB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1^2 \approx x_1,\ x_1x_2x_3x_1 \approx x_1x_3x_2x_1\}$ of all normal bands. The two-generated free algebra over NB contains precisely the blocks $[x_1]_{IdNB}$, $[x_2]_{IdNB}$, $[x_1x_2]_{IdNB}$, $[x_2x_1]_{IdNB}$, $[x_1x_2x_1]_{IdNB}$ and $[x_2x_1x_2]_{IdNB}$. We determine the algebra derived from $F_{NB}(\{x_1, x_2\})$ by the hypersubstitution $\sigma_{x_1x_2x_1}$ which maps the binary operation symbol to the binary term $x_1x_2x_1$. Since the variety NB is solid, we have $\sigma(F_{NB}(\{x_1, x_2\})) \in NB$ for every $\sigma \in Hyp(\tau)$. Therefore $\sigma_{x_1x_2x_1}(F_{NB}(\{x_1, x_2\}))$ belongs to the variety of all semigroups. Obviously, there are exactly the following proper hypersubstitutions over the variety of all semigroups: $\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_2x_1}$. It is easy to see that $\sigma_{x_1x_2x_1}(F_{NB}(\{x_1, x_2\}))$ cannot be isomorphic to one of the algebras derived from $F_{NB}(\{x_1, x_2\})$ by one of the proper hypersubstitutions $\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{x_2x_1}$.
We mentioned already that in general the relation $\sim_V$ is included in $\sim_{V-iso}$. If we have equality, then we obtain

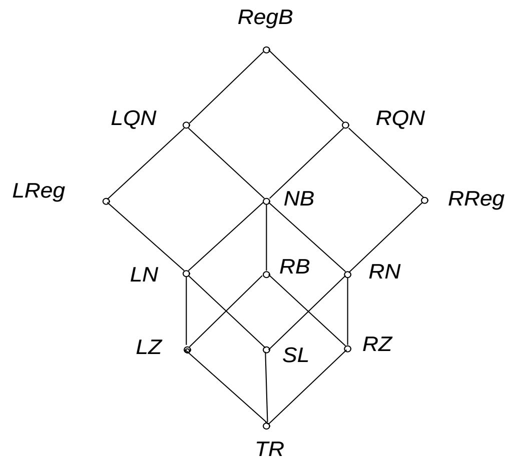
Proposition 3.1. *Let V be a weakly fluid variety of type τ and $\sim_V = \sim_{V-iso}$. If $d_p(V) = n$, then V is n-fluid.*

Proof. Assume that $d_p(V) = n$. Then there are $\sigma_1, \ldots, \sigma_n \in P(V)$ with $\sigma_i \not\sim_V \sigma_j$ for $1 \leq i < j \leq n$ and by $\sim_V = \sim_{V-iso}$ we get $\sigma_i \not\sim_{V-iso} \sigma_j$ for $1 \leq i < j \leq n$. Because of $d_p(V) = n$ and $\sim_V = \sim_{V-iso}$, the quotient set $P(V)/\sim_V$ can be written as $P(V)/\sim_{V-iso} = \{[\sigma_i]_{\sim_{V-iso}} \mid 1 \leq i \leq n\}$. Let $\mathcal{A} \in V$ and $\sigma \in Hyp(\tau)$ such that $\sigma(\mathcal{A}) \in V$. Since V is weakly fluid, there is a $\sigma' \in P(V)$ such that $\sigma(\mathcal{A}) \cong \sigma'(\mathcal{A})$. But σ' belongs to one of the n blocks of $P(V)$ with respect to $\sim_{V-iso}$. Therefore, there is an i with $1 \leq i \leq n$ such that $\sigma' \sim_{V-iso} \sigma_i$ and this means $\sigma'(\mathcal{A}) \cong \sigma_i(\mathcal{A})$. By transitivity we have $\sigma(\mathcal{A}) \cong \sigma_i(\mathcal{A})$. This shows that V is n-fluid. □

In [4] the following was proved:

Proposition 3.2. *Let V be a variety of bands. Then $\sim_V = \sim_{V-iso}$.*

Now we want to find out which varieties of bands satisfy also the second assumption of Proposition 3.1, i.e. which varieties of bands are weakly fluid. For the following investigations we need the lower part of the lattice of all varieties of bands which is pictured below



$TR = Mod\{x_1 \approx x_2\}$,
$LZ = Mod\{x_1x_2 \approx x_1\}$,
$RZ = Mod\{x_1x_2 \approx x_2\}$,
$SL = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_2x_1\}$,
$RB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3, {x_1}^2 \approx x_1\}$,
$NB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$,
$RegB = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_1x_3x_1$
$\approx x_1x_2x_3x_1\}$,
$LN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2\}$,
$RN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_2x_1x_3\}$,
$LReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_1x_2x_1\}$,
$RReg = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2 \approx x_2x_1x_2\}$,
$LQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_2x_1x_3\}$,
$RQN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, {x_1}^2 \approx x_1, x_1x_2x_3 \approx x_1x_3x_2x_3\}$.

Proposition 3.3. *A variety V of bands is weakly fluid iff $V \subseteq NB$ or $V \in \{LReg, RReg, RegB\}$.*

Proof. The trivial variety TR and the varieties RB, NB and $RegB$ are solid (see [1]) and therefore weakly fluid. It is easy to check that the varieties LZ, RZ and SL are weakly fluid. We consider the cases $V \nsubseteq RegB$ and $V \subseteq RegB$. Assume that $V \nsubseteq RegB$. Let $F_{NB}(\{x_1, x_2\})$ be the two-generated free algebra over NB. Since NB is solid, the derived algebra $\sigma_{x_1x_2x_1}(F_{NB}(\{x_1, x_2\}))$ where $\sigma_{x_1x_2x_1}$ is the hypersubstitution mapping the binary operation symbol to the term $x_1x_2x_1$, belongs to NB. Since $V \nsubseteq RegB$, we have $NB \subseteq V$, i.e. $\sigma_{x_1x_2x_1}(F_{NB}(\{x_1, x_2\})) \in V$. We have $P(V)/\sim_V \subseteq \{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_2x_1}]_{\sim_V}\}$ (see [4]). Moreover, the algebra $\sigma_{x_1x_2x_1}(F_{NB}(\{x_1, x_2\}))$ is not isomorphic to one of the algebras $\sigma_{x_1}(F_{NB}(\{x_1, x_2\})), \sigma_{x_2}(F_{NB}(\{x_1, x_2\})), \sigma_{x_1x_2}(F_{NB}(\{x_1, x_2\})), \sigma_{x_2x_1}(F_{NB}(\{x_1, x_2\}))$. This shows that V is not weakly fluid. Now we suppose that $V \subseteq RegB$ and that V is different from a solid and from a fluid variety (see [4]). Then $V \in \{LN, RN, LReg, RReg, LQN, RQN\}$. If $V = LQN$ or $V = RQN$, then $NB \subseteq V$ and $F_{NB}(\{x_1, x_2\})\} \in V$. Since NB is solid, the algebra $\sigma_{x_2x_1}(F_{NB}(\{x_1, x_2\}))$ belongs to $NB \subseteq V$. In this case we have $P(V)/\sim_V = \{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_1x_2x_1}]_{\sim_V}\}$

for $V = LQN$ and $P(V)/\sim_V = \{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_2x_1x_2}]_{\sim_V}\}$ for $V = RQN$. Then we have $\sigma_{x_2x_1}(F_{NB}(\{x_1,x_2\})) \ncong \sigma_t(F_{NB}(\{x_1,x_2\}))$ for $t \in \{x_1, x_2, x_1x_2, x_1x_2x_1\}$ and $t \in \{x_1, x_2, x_1x_2, x_2x_1x_2\}$ in the second case. This shows that V is not weakly fluid. Suppose that $V = LN$ or $V = LReg$. Let $\sigma \in Hyp(2)$ and $\mathcal{A} \in V$ with $\sigma(\mathcal{A}) \in V$. We have $Hyp(2)/\sim_V = \{[\sigma_{x_1}]_{\sim_V}, [\sigma_{x_2}]_{\sim_V}, [\sigma_{x_1x_2}]_{\sim_V}, [\sigma_{x_2x_1}]_{\sim_V}\}$. If $\sigma \sim_V \sigma_{x_1}$ or $\sigma \sim_V \sigma_{x_1x_2} \sim_V \sigma_{x_1x_2x_1}$, then $\sigma(\mathcal{A}) \cong \sigma_{x_1}(\mathcal{A})$ or $\sigma(\mathcal{A}) \cong \sigma_{x_1x_2}(\mathcal{A}) \cong \sigma_{x_1x_2x_1}(\mathcal{A})$ where both hypersubstitutions σ_{x_1} and $\sigma_{x_1x_2}$ belong to $P(V)$. If $\sigma \sim_V \sigma_{x_2}$, then $\sigma(\mathcal{A}) \cong \sigma_{x_2}(\mathcal{A}) \in RZ$, i.e. $\sigma_{x_2}(\mathcal{A}) \in RZ \cap V = TR$. Thus $\mathcal{A}$ is the trivial algebra and then $\sigma_{x_2}(\mathcal{A}) \cong \sigma_{x_1x_2}(\mathcal{A})$, i.e. $\sigma(\mathcal{A}) \cong \sigma_{x_1x_2}(\mathcal{A})$. If $\sigma \sim_V \sigma_{x_2x_1} \sim_V \sigma_{x_2x_1x_2}$, then $\sigma(\mathcal{A}) \cong \sigma_{x_2x_1}(\mathcal{A})$, i.e. $\sigma_{x_2x_1}(\mathcal{A}) \in V$. Since $x_1x_2 \approx x_1x_2x_1 \in IdV$, this equation is also an identity in $\mathcal{A}$ and then $\hat{\sigma}_{x_2x_1}[x_1x_2] \approx \hat{\sigma}_{x_2x_1}[x_1x_2x_1] \in Id\mathcal{A}$. Thus $x_1x_2x_1 \approx x_2x_1 \in Id\mathcal{A}$ and $\mathcal{A} \in RReg$. This shows that $\mathcal{A} \in RReg \cap LReg = SL$. Since $\sigma_{x_2x_1} \sim_{SL} \sigma_{x_1x_2}$, we have $\sigma_{x_2x_1}(\mathcal{A}) \cong \sigma_{x_1x_2}(\mathcal{A})$ and consequently, $\sigma(\mathcal{A}) \cong \sigma_{x_1x_2}(\mathcal{A})$. This shows that V is weakly fluid. Dually we can show that both varieties RN and $RReg$ are weakly fluid. □

By Proposition 3.3 precisely the following varieties of bands are weakly fluid: $TR, LZ, RZ, SL, RB, LN, RN, NB, LReg, RReg$ and $RegB$.
In [4] the following theorem was proved:

Theorem 3.4. *Let V be a variety of bands. Then*
$d_p(V) = 1$ *iff* $V \in \{TR, LZ, RZ, SL\}$,
$d_p(V) = 2$ *iff* $V \in \{LN, RN, LReg, RReg\}$,
$d_p(V) = 3$ *iff V is not dual solid and* $V \notin \{LZ, RZ, LN, RN, LReg, RReg, LQN, RQN\}$,
$d_p(V) = 4$ *iff V is dual solid, and either* $V \notin \{TR, SL, NB, RegB\}$ *or* $V \in \{LQN, RQN\}$,
$d_p(V) = 6$ *iff* $V \in \{NB, RegB\}$.

Then we obtain the following result.

Theorem 3.5. *The varieties TR, LZ, RZ and SL are fluid. $LN, RN, LReg$, $RReg$ are 2-fluid. $NB, RegB$ are 6-fluid. For all other varieties V of bands (i.e. if $V \nsubseteq RegB$ or $V \in \{LQN, RQN\}$) there is no natural number $n \geq 1$ such that V is n-fluid.*

4 Some Generalizations of N-Fluidity

In section 2 we defined weakly fluid varieties as a generalization of the concept of an n-fluid variety. If instead of $P(V)$ in the definition of a weakly fluid variety we use a set Σ of hypersubstitutions, then we get the concept of a Σ-fluid variety.

Definition 4.1. Let Σ be a set of hypersubstitutions of type τ. A variety V of type τ is called Σ-*fluid* if for all $\mathcal{A} \in V$ and all $\sigma \in Hyp(\tau)$ the implication

$$\sigma(\mathcal{A}) \in V \Longrightarrow \sigma(\mathcal{A}) \cong \sigma'(\mathcal{A}) \text{ for some } \sigma' \in \Sigma$$

holds.

Clearly the concept of a Σ-fluid variety generalizes that of a fluid variety since fluid varieties are precisely $\{\sigma_{id}\}$-fluid. Weakly fluid varieties are Σ-fluid for $\Sigma = P(V)$ and n-fluid varieties are Σ-fluid for an n-element subset $\{\sigma_1, \ldots, \sigma_n\} \subseteq P(V)$. If a variety V is Σ-fluid and if $W \subseteq V$ is a subvariety, then W is also Σ-fluid. For this, let $\sigma(\mathcal{A}) \in W$, then $\sigma(\mathcal{A}) \in V$ and there is a hypersubstitution $\sigma' \in \Sigma$ such that $\sigma(\mathcal{A}) \cong \sigma'(\mathcal{A})$.

Proposition 4.2. *Let V be a variety of type τ and let σ' be a projection hypersubstitution of type τ. If V is $\{\sigma_{id}, \sigma'\}$-fluid, but not fluid, then V is 2-fluid.*

Proof. Since V is not fluid, it is not $\{\sigma_{id}\}$-fluid and there is an algebra $\mathcal{A} \in V$ and a hypersubstitution $\sigma \in Hyp(\tau)$ with $\sigma(\mathcal{A}) \in V$, but $\sigma_{id}(\mathcal{A}) \ncong \sigma(\mathcal{A})$, i.e. with $\sigma \not\sim_{V-iso} \sigma_{id}$. Because of $\sim_V \subseteq \sim_{V-iso}$ we get $\sigma \not\sim_V \sigma_{id}$. Since V is $\{\sigma_{id}, \sigma'\}$-fluid and $\sigma_{id}(\mathcal{A}) \ncong \sigma(\mathcal{A})$, we get $\sigma(\mathcal{A}) \cong \sigma'(\mathcal{A})$. But then $\sigma'(\mathcal{A}) \in V$. Let $\mathcal{B} \in V$ with the cardinality a. Then we consider a direct product $\mathcal{C}$ of the algebra $\sigma'(\mathcal{A})$ with cardinality $\geq a$. There is a surjective mapping $\alpha : C \to B$ and for $i \in I$ there is a $k_i \in \{1, \ldots, n_i\}$ with $\alpha(f_i^C(a_1, \ldots, a_{n_i})) = \alpha(a_{k_i}) = f_i^{\sigma'(\mathcal{B})}(\alpha(a_1), \ldots, \alpha(a_{n_i}))$. Thus $\alpha : \mathcal{C} \to \sigma'(\mathcal{B})$ is a homomorphism, i.e. $\sigma'(\mathcal{B}) \in V$. This shows $\sigma' \in P(V)$. Consequently, V is 2-fluid. □

As a corollary of Proposition 2.6 for $m = 1$ we get:

Proposition 4.3. *Let V be a 2-fluid variety of type $\tau = (n_i)_{i \in I}$. Assume that V contains a non-trivial projection algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$. Then V is $\{\sigma_{id}, \sigma'\}$-fluid for some projection hypersubstitution σ'.*

Not every weakly fluid variety must be n-fluid for some natural number $n \geq 1$ since $P(V)$ need not be finite.

Definition 4.4. Let τ be a finite type and let $\aleph_0$ be the cardinal number of the set of natural numbers. A weakly fluid variety of type τ is called $\aleph_0$-*fluid* if it is not n-fluid for any natural number $n \geq 1$.

With this definition we have:
A variety V of type τ is weakly fluid iff it is k-fluid for some $k \in \mathbb{N}^* \cup \{\aleph_0\}$ where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. It is well-known and easy to see ([4]) that for a finite type $|Hyp(\tau)| \leq \aleph_0$. Then $|P(V)| \leq \aleph_0$.
We will give an example of a $\aleph_0$-fluid variety.

Proposition 4.5. *Let $\tau = (n_i)_{i \in I}$ be a finite type with $n_i \geq 2$ for some $i \in I$. Then the variety $Alg(\tau)$ of all algebras of type τ is $\aleph_0$-fluid.*

Proof. Since $Alg(\tau)$ is a solid variety, it is weakly fluid. In order to prove that $Alg(\tau)$ is $\aleph_0$-fluid, we have to check that there is no natural number $n \in \mathbb{N}^*$ such that $Alg(\tau)$ is n-fluid. For $k \in \mathbb{N}^*$ we define a hypersubstitution $\sigma_k \in Hyp(\tau)$ in the following way:
Let $x_2^{(n_i-1)r+1}$ be defined inductively by $x_2^{(n_i-1)0+1} := x_2$, $x_2^{(n_i-1)r+1} := f_i(x_2^{(n_i-1)(r-1)+1}, x_2, \ldots, x_2)$ for $r \geq 1, r \in \mathbb{N}$. Then σ_k is defined by $\sigma_k(f_i) := f_i(x_1, x_2^{(n_i-1)(k-1)+1}, x_2, \ldots, x_2), \sigma_k(f_j) := x_1$ for all $j \in I \setminus \{i\}$. Let V be the variety of type τ defined by the following identities: $f_j(x_1, \ldots, x_{n_j}) \approx x_1$ for $j \in I \setminus \{i\}$ and $f_i(x_1, \ldots, f_i(x_r, \ldots, x_{r+n_i-1}), \ldots, x_{2n_i-1}) \approx f_i(x_1, \ldots, f_i(x_s, \ldots, x_{s+n_i-1}), \ldots, x_{2n_i-1})$ for $1 \leq r < s \leq n_i$. Let $\mathcal{S}$ be the free one-generated algebra with respect to V, freely generated by the element a. Assume that $Alg(\tau)$ is n-fluid for some natural number $n \geq 1$. Then there are $1 \leq k < l \in \mathbb{N}$ such that $\sigma_k(\mathcal{S}) \cong \sigma_l(\mathcal{S})$ since $\{\sigma_i(\mathcal{S}) \mid 1 \leq i \in \mathbb{N}\} \subseteq Alg(\tau)$. So, there is an isomorphism $h : \sigma_k(\mathcal{S}) \longrightarrow \sigma_l(\mathcal{S})$ with $a^{p_m} := h(a^{(n_i-1)m+1})$ for $0 \leq m < k$ where $a^{(n_i-1)m+1}$ is defined as $x^{(n_i-1)m+1}$ before $0 \leq m < k$. By induction on $n \in \mathbb{N}$ we show that for $0 \leq m < k$ there holds:

$$h(a^{n(n_i-1)k+1+m(n_i-1)}) = a^{p_m + n(n_i-1)lp_0}.$$

Indeed, for $n = 0$ we have $h(a^{1+m(n_i-1)}) = a^{p_m}$ by definition. Suppose, the statement holds for $n = p$. Then for $n = p+1$ we have $h(a^{(p+1)(n_i-1)k+1+m(n_i-1)})$

$$\begin{aligned}
&= h(\sigma_k(f_i)(a^{p(n_i-1)k+1+m(n_i-1)}, a, \ldots, a) \\
&= \sigma_l(f_i)(h(a^{p(n_i-1)k+1+m(n_i-1)}), h(a), \ldots, h(a)) \\
&= \sigma_l(f_i)(a^{p_m+p(n_i-1)lp_0}, a^{p_0}, \ldots, a^{p_0}) \\
&= a^{p_m+p(n_i-1)lp_0+(n_i-1)lp_0} \\
&= a^{p_m+(p+1)(n_i-1)lp_0}.
\end{aligned}$$

This shows that $h(S) = \{a^{p_m+n(n_i-1)lp_0} \mid 0 \leq m < k, n \in \mathbb{N}\}$. Since $h(S) = S$, we have $\{a^1, a^{(n_i-1)+1}, \ldots, a^{l(n_i-1)+1}\} \subseteq S$. Since $p_m + n(n_i-1)lp_0 > (n_i-1)l + 1$ for all $m \in \{0, \ldots, k-1\}$ and any $n \in \mathbb{N}^*$ we have $\{a^1, a^{(n_i-1)+1}, \ldots, a^{l(n_i-1)+1}\} \subseteq \{a^{p_0}, \ldots, a^{p_{k-1}}\}$, i.e. $l \leq k$, a contradiction. This shows $\sigma_k(\mathcal{S}) \ncong \sigma_l(\mathcal{S})$, a contradiction. □

References

[1] Denecke, K., Koppitz, J., *Fluid, unsolid, and completely unsolid varieties*, Algebra Colloquium 7:4(2000), 381-390.

[2] Denecke, K., Wismath, S. L., *Hyperidentities and Clones*, Gordon and Breach Science Publishers, 2000.

[3] Denecke, K., Wismath, S. L., *Universal Algebra and Applications in Theoretical Computer Science*. Boca Raton, London, Washington, D.C.: Chapman & Hall/CRC 2002.

[4] Denecke, K., Koppitz, J., Srithus, R., *The Degree of Proper Hypersubstitutions*, preprint 2005.

[5] Schweigert, D., *On derived varieties*, Discuss. Math. Algebra Stochastic Methods, 18(1998), no. 1,17–26.

[6] Płonka, J., *Proper and inner hypersubstitutions of varieties*, Proceedings of the International Conference Sommer School on General Algebra and Ordered Sets, Olomouc 1994, 106-116.

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