WIGNER'S WEAKLY POSITIVE OPERATORS

SAICHI IZUMINO * AND MASAHIRO NAKAMURA **

Received May 25, 2006; revised July 28, 2006

ABSTRACT. Weakly positive operators was introduced by Wigner in 1968. In this note, Wigner's operators are illuminated by the light of the recent knowledge: (1) Wang's theorem which is a step to Reid's inequality is given a new simple proof with Wigner's notion, (2) the Kantorovich inequality is extended to one for weakly positive operators which is nothing but Bourin's recent norm inequality, (3) the Carlin-Noble definition of the square root of a weakly positive operator is justified by showing independence of its factorization, and (4) the Anderon-Trapp solution to the Riccati equation is considered as a consequence of the Fujii-Fujii theorem on Kubo's identity on operator means.

1. Wigner [15] defined a (bounded linear) operator T on a Hilbert space H to be *weakly positive* if T is similar to a positive invertible operator D by a positive operator X, i.e.,

$$T = XDX^{-1}. \tag{1}$$

If T satisfies (1), then we put $A = X^2$ and $Z = X^{-1}DX^{-1}$, and have a factorization

$$T = AZ. \tag{2}$$

of T by two positive invertible operators $A,\ Z > 0$. (An operator B is denoted by $B > 0$ if B is positive invertible.) Clearly this identity (2) is, conversely, equivalent to the condition for T to be a weakly positive operator. Denoting the spectrum of T by $\sigma(T)$, we see, from (1), that

$$\sigma(T) = \sigma(XDX^{-1}) = \sigma(D) \subset \mathbb{R}^+(= [0, \infty)).$$

Furthermore since $D > 0$, there are positive scalars m and M such that $mI \leq D \leq MI$, or $\sigma(T) \subset [m, M]$.

An operator T is convexoid if the closure $\bar{W}(T)$ of its numerical range $W(T)$ is equal to the convex hull $co\sigma(T)$ of $\sigma(T)$. Now we have the following:

Proposition 1. *If a weakly positive operator $T = AZ$ with $A,\ Z > 0$ is convexoid, then A and Z commute each other, and so T itself is positive.*

2000 *Mathematics Subject Classification.* 47A10, 47A12, 47A30, 47A63, Secondary 47B20.

Key words and phrases. Kantorovich inequality, weakly positive operator, numerical radius, geometric mean, Riccati equation, square root of weakly positive operator.

Proof. Since T is convexoid and $\sigma(T) \subset \mathbb{R}^+$, we see that $\bar{W}(T) = co\sigma(T) \subset \mathbb{R}^+$, Hence T is positive and $AZ = ZA$. □

We remark that in the above proposition we need not assume invertibility of A and Z, so that we need only positivity of both operators, for then we have $\sigma(T) \subset \sigma(A^{1/2}ZA^{1/2}) \cup \{0\} \subset \mathbb{R}^+$. Let us call an operator $T = AZ$ an *extended weakly positive* operator if A and Z are simply assumed to be positive.

We here give an example of a weakly positive operator T which is non-normaloid, i.e., $\|T\| > r(T)$:

$$T := \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is weakly positive, since T is a product of two positive matrices. By a calculation we have $\|T\| = 3.56... > 3.41... = r(T)$.

Reid's inequality is popular by Halmos' book [7, Problem ♯ 82]. Wang [14] gave another proof of the inequality; his proof depends on the following fact:

Theorem 2. *If $A \geq 0$ and $AB \geq 0$ for an operator B with $\|B\| \leq 1$, then $A \geq AB$.*

Proof. Let $A_\epsilon = A + \epsilon I$ $(\epsilon > 0)$ and put $T_\epsilon = (A_\epsilon)^{-1}AB$. Then T_ϵ is extended weakly positive (or weakly positive if B is invertible), so that $\sigma(T_\epsilon)$ lies in $\mathbb{R}^+$, as remarked after Proposition 1. Further, since

$$\|T_\epsilon\| \leq \|(A_\epsilon)^{-1}A\|\|B\| \leq 1,$$

we see that $\sigma(T_\epsilon) \subset [0, 1]$. Hence

$$\sigma(A_\epsilon^{-1/2}ABA_\epsilon^{-1/2}) = \sigma(T_\epsilon) \subset [0, 1].$$

This implies $A_\epsilon^{-1/2}ABA_\epsilon^{-1/2} \leq 1$, or $AB \leq A_\epsilon I$. Tending $\epsilon \to 0$, we have the desired inequality. □

2. Bourin [2] presented an interesting norm inequality for a weakly positive operator which is a substantial generalization of the celebrated Kantorovich inequality for a positive operator, as follows:

Theorem 3 ([2], [5]). *Let $T = AZ$ for positive invertible operatos A and Z with $mI \leq Z \leq MI$ for some positive scalars $m < M$. Then*

$$\|T\|^2 \leq K(m, M)r(T)^2 \quad (r(T) \text{ is the spectral radius of } T) \tag{3}$$

where $K(m, M) = \dfrac{(M + m)^2}{4Mm}$ is the Kantorovich constant of Z with respect to m and M.

We here give a simple proof of the theorem by using the Kantorovich inequality [2], [6]: Let $x \in H$ be a unit vector and let

$$\Phi(X) = (AXAx, x)/\|Ax\|^2.$$

Then by the Kantorovich inequality (say, [6, Theorem 1.32]) we have

$$\Phi(Z^2) \le K\Phi(Z)^2 \ (K = K(m, M)),$$

that is,

$$\frac{(AZ^2Ax, x)}{\|Ax\|^2} \le K\frac{(AZAx, x)^2}{\|Ax\|^4} \quad \text{or} \quad (AZ^2Ax, x) \le K\frac{(AZAx, x)^2}{\|Ax\|^2}.$$

By the identity $r(A^{1/2}ZA^{1/2}) = r(AZ)$ and Cauchy-Schwarz inequality, we have

$$(AZAx, x) = (A^{1/2}ZA^{1/2} \cdot (A^{1/2}x), A^{1/2}x) \le \|A^{1/2}ZA^{1/2}\|\|A^{1/2}x\|^2$$
$$= r(A^{1/2}ZA^{1/2})(Ax, x) \le r(AZ)\|Ax\|.$$

Hence

$$(AZ^2Ax, x) \le Kr(AZ)^2 \ \ \text{(for all unit vectors } x \in H),$$

which implies

$$\|AZ\|^2 = \|AZ^2A\| \le Kr(AZ)^2.$$

In (3), the constant $K(m, M)$ of the right hand side depends only on the factor Z of $T = AZ$, so that we may give a better (i.e., smaller) constant instead if we obtain another factorization of T. Concerning this, first we show the following:

Lemma 4. *Let $T = AZ$ be a weakly positive operator with $A,\ Z > 0$. Assume that*

$$sA + tZ^{-1} > 0, \quad \textit{or equivalently, } sI + tA^{-1/2}Z^{-1}A^{-1/2} > 0$$

for some real scalars s and t. Then if we put

$$Z_1 = (sA + tZ^{-1})^{-1} \ \textit{and} \ \ A_1 = AZZ_1^{-1},$$

we have a new represention $T = A_1Z_1$ of two positive operators A_1 and Z_1.

Proof. It suffices to show that A_1 is positive. From the definition of A_1
$A_1 = AZZ_1^{-1} = AZ(sA + tZ^{-1})$
$= sAZA + tA = A^{1/2}(sA^{1/2}ZA^{1/2} + tI)A^{1/2}$
and

$$sA^{1/2}ZA^{1/2} + tI = A^{1/2}ZA^{1/2}(sI + tA^{-1/2}Z^{-1}A^{-1/2}) > 0.$$

Hence $A_1 > 0$. ∎

Example. Let

$$T = AZ = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then we can see that $\dfrac{3-\sqrt{5}}{2} I \le Z \le \dfrac{3+\sqrt{5}}{2} I$ and $K_Z = K((3-\sqrt{5})/2, (3+\sqrt{5})/2) = 9/4$ is the Kantorovich constant to the factorization. Now to obtain a smaller Kantorovich constant, following the lemma, let ($s = 1$ and)

$$A + tZ^{-1} = \begin{bmatrix} t+1 & -t \\ -t & 2t+2 \end{bmatrix} > 0.$$

Then t has to satisfy $t > -1$. Put

$$Z_1 = (A + tZ^{-1})^{-1} = \begin{bmatrix} t+1 & -t \\ -t & 2t+2 \end{bmatrix}^{-1} = \frac{1}{t^2+4t+2} \begin{bmatrix} 2t+2 & t \\ t & t+1 \end{bmatrix},$$

$$A_1 = TZ_1^{-1} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} t+1 & -t \\ -t & 2t+2 \end{bmatrix} = \begin{bmatrix} t+2 & 2 \\ 2 & 2t+4 \end{bmatrix}.$$

Then both A_1 and Z_1 are positive invertible, and $T = A_1 Z_1$. Since the equation

$$\begin{aligned} \det(Z_1^{-1} - \lambda I) &= \det(A + tZ^{-1} - \lambda I) \\ &= \lambda^2 - (3t+3)\lambda + (t^2+4t+2) = 0 \end{aligned}$$

has solutions λ_1 and λ_2 ($\lambda_1 < \lambda_2$) such that $\lambda_1 + \lambda_2 = 3t+3$, $\lambda_1\lambda_2 = t^2+4t+2$ and $\lambda_1 I \le Z_1^{-1} \le \lambda_2 I$, we see that $\dfrac{1}{\lambda_2} I \le Z_1 \le \dfrac{1}{\lambda_1} I$, so that the Kantorovich constant of Z_1 is

$$K_{Z_1} = K(1/\lambda_2, 1/\lambda_1) = K(\lambda_1, \lambda_2) = \frac{(\lambda_1+\lambda_2)^2}{4\lambda_1\lambda_2} = \frac{(3t+3)^2}{4(t^2+4t+2)}.$$

By an elementary computation $K_{Z_1} = K_{Z_1}(t)$ ($t \ge -1$) has its minimum at $t = 0$. Hence we have $K_{Z_1} = K_{Z_1}(0) = 9/8$ as a better Kantorovich constant in such a factorization of T. (For $t = 0$, we have $T = 2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \dfrac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$).

3. Pusz and Woronowicz [12] defined the geometric operator mean of two positive operators A and B by

$$A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

Previous to them, Carlin and Noble [3] had already defined the geometric mean of positive matrices A and B by using the square root of the product $A^{-1}B$, which is well-defined in virtue of the spectral property of such a matrix, i.e., weakly positive matrix. We here give a definition of the square root of a weakly positive operator $A^{-1}B$, by which we can justify the above definition of the geometric operator (matrix) mean by Carlin and Noble. Let us suppose that $T = A^{-1}B$ is a weakly positive operator with $A,\ B > 0$. Then T is expressed as $A^{-1/2}(A^{-1/2}BA^{-1/2})A^{1/2}$, hence it is natural to ask if we can define the square root of $T = A^{-1}B$, by

$$T^{1/2} = A^{-1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}. \tag{4}$$

Now we shall show that $T^{1/2}$ is well-defined by (4), so that we have

$$A(A^{-1}B)^{1/2} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} = A\sharp B,$$

the last identity of which is nothing but the geometric mean introduced by Pusz and Woronowicz [12].

Theorem 5. *If $T = A^{-1}B = C^{-1}D$ for $A,\ B,\ C,\ D > 0$. Then $\sigma(T) = \sigma(A^{-1/2}BA^{-1/2}) = \sigma(C^{-1/2}DC^{-1/2})$ and*

$$A^{-1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} = C^{-1/2}(C^{-1/2}DC^{-1/2})^{1/2}C^{1/2}. \tag{5}$$

Moreover, generally

$$A^{-1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} = C^{-1/2}f(C^{-1/2}DC^{-1/2})C^{1/2} \tag{6}$$

for any continuous function f defined on an interval containing $\sigma(T)$. Hence, not only $T^{1/2}$ by (4) but also $f(T)$ by

$$f(T) = A^{-1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

are well-defined.

Proof. Put $X = A^{-1/2}BA^{-1/2}$ and $Y = C^{-1/2}DC^{-1/2}$, then from $A^{-1}B = C^{-1}D$, we have

$$T = A^{-1/2}XA^{1/2} = C^{-1/2}YC^{1/2},$$

so that for the spectrums we have $\sigma(T) = \sigma(X) = \sigma(Y)$. For (5) and (6), note that for all integers n

$$A^{-1/2}X^nA^{1/2} = C^{-1/2}Y^nC^{1/2}.$$

Hence also, for all polynomials p,

$$A^{-1/2}p(X)A^{1/2} = C^{-1/2}p(Y)C^{1/2}.$$

By the functional calculus we then have

$$A^{-1/2}f(X)A^{1/2} = C^{-1/2}f(Y)C^{1/2}$$

for all continuous functions defined on an interval containing $\sigma(T)(= \sigma(X) = \sigma(Y))$. In particular, for $f(t) = t^{1/2}$, we have

$$A^{-1/2}X^{1/2}A^{1/2} = C^{-1/2}Y^{1/2}C^{1/2}.$$

□

The Carlin-Noble definition of the square root of a weakly positive operator gives a simple proof of the Anderson-Trapp theorem [1] which is closely related to the Pedersen-Takesaki theorem [11].

Theorem 6. *The Riccati equation* $XA^{-1}X = B$ *where* $A,\ B > 0$ *has a unique positive solution* $X = A\sharp B$.

Proof. Put $X = A(A^{-1}B)^{1/2}$, then we have

$$XA^{-1}X = A(A^{-1}B)^{1/2}A^{-1}A(A^{-1}B)^{1/2} = A(A^{-1}B) = B,$$

hence $X = A\sharp B$ is a solution of the equation. □

4. To conclude our note we must apologize that the Anderson-Trapp paper [1] had been unavailable for us until we almost finished the note, so that we did not know that C. W. Hoe gave the justification of the definition of the square root of a weakly positive operator $T = AB$. Hence we do not state our originality to the functional calculus for weakly positive operators.

It is regretable to state that in [4], J. I. Fujii and M. Fujii have presented a theorem which essentially implies the Anderson-Trapp theorem [1] and also gives a mean-theoretic view to the celebrated Pedersen-Takesaki theorem [11] for their Radon-Nikodym theorem:

Theorem 7. ([4, Theorem 4]). *The geometric mean is characterized by the following Kubo identity: For any operator mean* m,

$$(A\sharp B)(Am^*B)^{-1}(A\sharp B) = AmB, \tag{7}$$

where m^* $(Am^*B = (B^{-1}mA^{-1})^{-1})$ *is the dual of* m *according to the Kubo-Ando theory on operator means.*

If we put $m = \sharp_r$, i.e., $AmB = A\sharp_r B = B$ and $Am^*B = A\sharp_r^* B = A$ [8], then we at once obtain the Ricatti equation from (7).

For a short proof of the theorem, note that there corresponds to an operator mean σ the (unique) representing function $f = f_\sigma$ which is defined by $f_\sigma(t)I = I\sigma(tI)$. We easily see that $f_\sharp(t) = \sqrt{t}$. Now by the transformer identity, a general property of the operator mean, i.e.,

$$C(A\sigma B)C = CAC\sigma CBC \quad (A, B, C > 0),$$

we can obtain, from (7),

$$(1\sharp A^{-1/2}BA^{-1/2})(1m^*A^{-1/2}BA^{-1/2})^{-1}(1\sharp A^{-1/2}BA^{-1/2}) = 1mA^{-1/2}BA^{-1/2},$$

by putting $C = A^{-1/2}$, or

$$(1\sharp D)(1m^*D)^{-1}(1\sharp D) = 1mD \tag{8}$$

by puttig further $D = A^{-1/2}BA^{-1/2}$. ¿From this identity we have

$$\sqrt{t}(f_{m^*}(t))^{-1}\sqrt{t} = f_m(t), \tag{9}$$

which is equivalent to (7). We now show that (9) holds, and that $g(t)(f_{m^*}(t))^{-1}g(t) = f_m(t)$ holds for all m only if $g(t) = \sqrt{t}$. From the general theory of operator means we know that

$$f_{m^*}(t) = \frac{t}{f_m(t)}, \tag{10}$$

which at once implies (9) and also the remaining assertion.

Added in Proof. 1. As stated after Proposition 1, if $T = AZ$ with positive operators A and Z, that is, T is an extended weakly positive operator, then $\sigma(T) \subset \mathbb{R}^+$. Related to this fact, in [9], Nakamoto proved the following general fact: If A or B is positive, then $\sigma(AB) \subset \bar{W}(A)\bar{W}(B)$.

2. A very short proof of Theorem 2 was presented by the referee: From $AB \geq 0$, it follows $(AB)^* = B^*A$. Hence $(AB)^2 = ABB^*A \leq A^2$ since $\|B\| \leq 1$, so that we have $AB \leq A$ by Löwner-Heinz inequality.

Acknowledgement. The authors express their hearty thanks to Professor T. Furuta who gave a copy of Trapp's introduction [13] on operator means by which the authors learned that the geometric operator mean is introduced by Carlin and Noble whose paper is not available to them.

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* Faculty of Education, Toyama University, Gofuku, Toyama 930-8555, JAPAN
E-mail address: s-izumino@@h5.dion.ne.jp

** Department of Mathematics Osaka Kyoiku University, Kashiwara, Osaka 582, JAPAN