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A RELATION ON SUBTRACTION ALGEBRAS

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Abstract. As a generalization of a subtraction homomorphism, the notion of a relation on subtraction algebras, called an SA-relation, is introduced. Some fundamental properties to subtraction algebras are discussed.

1. Introduction

B. M. Schein([7]) considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$"of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$"(and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka([8]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim([5]) showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup which is a generalization of a ring. Y. B. Jun et al.([3]) introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [2], Y. B. Jun and H. S. Kim established the ideal generated by a set and discussed related results. Y. B. Jun and K. H. Kim([4]) introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be prime/irreducible ideal. In this paper, we introduce the notion of a relation on subtraction algebras, called an SA-relation, which is a generalization of a subtraction homomorphism, and then we discuss some fundamental properties of subtraction algebras.

2. Preliminaries

A *subtraction algebra* is defined as an algebra $(X; -)$ with a binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in

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which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3,4]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Example 2.1. Let $X := \{0, a, b, c\}$ be a set with the following table:

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then X is a subtraction algebra.

A non-empty subset A of a subtraction algebra X is called a *subalgebra* of X if $x - y \in X$ for all $x, y \in X$.

Definition 2.2. ([3]) A non-empty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$,
(I2) $(\forall x \in X)(\forall y \in A)(x - y \in A \implies x \in A)$.

Lemma 2.3. *Let $(X; -)$ be a subtraction algebra. Then $(X; -)$ is a poset.*

Proof. Since $x - x = 0$ for all $x \in X$, we have $x \leq x$. Assume that $x \leq y$ and $y \leq z$ for all $x, y, z \in X$. By (a9), $y \leq z$ implies $x - z \leq x - y$ for all $z \in X$. If $x \leq y$, then $x - y = 0$. Hence $x - z \leq 0$, and so, $x \leq z$. Therefore we have $x \leq y$ and $y \leq z$ imply $x \leq z$. Suppose $x \leq y$ and $y \leq x$ for all $x, y \in X$. Then $x - y = 0$ and $y - x = 0$. Using (a2) and (S2), we have $x = x - 0 = x - (x - y) = y - (y - x) = y - 0 = y$. Hence $x \leq y$ and $y \leq x$ imply $x = y$. Thus $(X; -)$ is a partially ordered set. $\square$

Note that a subtraction algebra is equivalent to an implicative BCK-algebra and every BCK-algebra has a poset structure by defining $x \leq y \Leftrightarrow x * y = 0, \forall x, y \in X$.

Let $X := (X; -_X, \leq)$ and $Y := (Y; -_Y, \leq')$ be subtraction algebras. A mapping f from a poset $(X; \leq)$ into a poset $(Y; \leq')$ is called a *Harris map* ([6]) if for any incomparable elements $x, y \in X$, either $f(x) = f(y)$ or $f(x)$ and $f(y)$ are incomparable. We denote the fact that x and y are incomparable by $x || y$. A mapping $f : X \to Y$ is called a (*subtraction*) *homomorphism* if

(H1) $f(x -_X y) = f(x) -_Y f(y)$ for any $x, y \in X$,
(H2) f is a Harris map.
Note that $f(0_X) = 0_Y$, since $x - x = 0$. Let $Ker(f) := \{x \in X | f(x) = 0_Y\}$ be the kernel of f. If a mapping $f : X \to Y$ satisfies (H1), then it is order preserving, i.e., $x \leq y$ implies $0_Y = f(x - y) = f(x) - f(y)$, i.e., $f(x) \leq f(y)$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$.

Example 2.4. In Example 2.1, if we define a map $\varphi : X \to X$ by $\varphi(0) = 0, \varphi(a) = b, \varphi(b) = a, \varphi(c) = c$, then it is easy to verify that φ is a homomorphism.

3. SA-relations

Definition 3.1. Let X and Y be subtraction algebras. A non-empty relation $\mathcal{H} \subseteq X \times Y$ is called an *SA-relation* if it satisfies:
(R1) $(\forall x \in X)\ (\exists y \in Y)(\ x\mathcal{H}y)$,
(R2) $(\forall x, y \in X)\ (\forall a, b \in Y)\ (x\mathcal{H}a, y\mathcal{H}b \implies (x - y)\mathcal{H}(a - b))$.

We usually denote such relation by $\mathcal{H} : X \to Y$. It is clear from (R1) and (R2) that $0_X\mathcal{H}0_Y$.

Example 3.2. Let $X := \{0, a, b\}$ be a set with the following table:

-	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Then $(X; -)$ is a subtraction algebra. Define a relation $\mathcal{H} : X \to X$ as follows: $0\mathcal{H}0, 0\mathcal{H}a, 0\mathcal{H}b, a\mathcal{H}0, a\mathcal{H}a, a\mathcal{H}b, b\mathcal{H}0$. It is easy to verify that $\mathcal{H}$ is an SA-relation. A relation $\mathcal{D} : X \to X$ given by $0\mathcal{D}0, 0\mathcal{D}b, a\mathcal{D}0, a\mathcal{D}b, b\mathcal{D}b$ is an SA-relation.

Proposition 3.3. *Every homomorphism of subtraction algebras is an SA-relation.*

Proof. Suppose that $\mathcal{H} : X \to Y$ be a homomorphism of subtraction algebras. Clearly, $\mathcal{H}$ satisfies conditions (R1) and (R2). □

Note that every diagonal SA-relation on a subtraction algebra(i.e., an SA-relation satisfying $x\mathcal{H}x$ for all $x \in X$ in which $x\mathcal{H}y$ is false whenever $x \neq y$) is clearly a subtraction homomorphism. But, in general, the converse of Proposition 3.3 need not be true as seen in the following example.

Example 3.4. In Example 3.2, the SA-relations $\mathcal{H}$ and $\mathcal{D}$ are not homomorphisms.

Let $\mathcal{H} : X \to Y$ be an SA-relation. For any $x \in X$ and any $y \in Y$,

$$\mathcal{H}[x] := \{y \in Y : x\mathcal{H}y\}, \quad \text{and} \quad \mathcal{H}^{-1}[y] := \{x \in X : x\mathcal{H}y\}.$$

Note that $\mathcal{H}[x]$ and $\mathcal{H}^{-1}[y]$ are not subalgebras of Y and X, respectively as seen in the following example.

Example 3.5. Let $X := \{0, a, b, c\}$, $Y := \{0, 1, 2, 3\}$ be sets with the following table, respectively:

$-_X$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

$-_Y$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Define a relation $\mathcal{H} : X \to Y$ as follows: $0\mathcal{H}0, a\mathcal{H}0, b\mathcal{H}1, c\mathcal{H}2$. It is easy to verify that $\mathcal{H}$ is an SA-relation, but $\mathcal{H}^{-1}[2] = \{c\}$ (resp., $\mathcal{H}[c] = \{2\}$) is not a subalgebra of X (resp., Y).

Theorem 3.6. *For any SA-relation $\mathcal{H} : X \to Y$, we have*
(i) *$\mathcal{H}[0_X]$, called the zero image of $\mathcal{H}$, is a subalgebra of Y.*
(ii) *$\mathcal{H}^{-1}[0_Y]$, called the kernel of $\mathcal{H}$, and denoted by $Ker\mathcal{H}$, is a subalgebra of X.*
(iii) *$Ker\mathcal{H}$ is an ideal of X.*

Proof. (i) Since $0_X\mathcal{H}0_Y$, $\mathcal{H}[0_X] \neq \emptyset$. Let $y_1, y_2 \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}y_1$ and $0_X\mathcal{H}y_2$. It follows from (R2) and $x - x = 0$ for all $x \in X$ that $0_X\mathcal{H}(y_1 - y_2)$, that is, $y_1 - y_2 \in \mathcal{H}[0_X]$.
(ii) Since $0_X\mathcal{H}0_Y$, $0_X \in Ker\mathcal{H}$. Let $x_1, x_2 \in Ker\mathcal{H}$. Then $x_1\mathcal{H}0_Y$ and $x_2\mathcal{H}0_Y$. By using (R2) and $x - x = 0$ for all $x \in X$, we obtain $(x_1 - x_2)\mathcal{H}0_Y$ and $x_1 - x_2 \in Ker\mathcal{H}$.
(iii) Since $0_X\mathcal{H}0_Y$, $0_X \in Ker\mathcal{H}$. Let $a - b, b \in Ker\mathcal{H}$ for any $a \in X$. Then $(a - b)\mathcal{H}0_Y$ and $b\mathcal{H}0_Y$. For such an $a \in X$, there exists $\alpha \in Y$ such that $a\mathcal{H}\alpha$, since $\mathcal{H}$ is an SA-relation. Hence $(a - b)\mathcal{H}(\alpha - 0_Y)$, i.e., $(a - b)\mathcal{H}\alpha$. Since $(a - b)\mathcal{H}\alpha$ and $(a - b)\mathcal{H}0_Y$, we have $((a - b) - (a - b))\mathcal{H}(\alpha - 0_Y)$, i.e., $0_X\mathcal{H}\alpha$ and so $(a - 0_X)\mathcal{H}(\alpha - \alpha)$, i.e., $a\mathcal{H}0_Y$. Therefore $a \in Ker\mathcal{H}$. Thus $Ker\mathcal{H}$ is an ideal of X. This completes the proof. □

Proposition 3.7. *Let $\mathcal{H} : X \to Y$ be an SA-relation and $a, b \in X$, $u, v \in Y$.*
(i) *If $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$, then $a - b \in Ker\mathcal{H}$.*
(ii) *If $\mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v] \neq \emptyset$, then $u - v \in \mathcal{H}[0_X]$.*

Proof. (i) Let $a, b \in X$ be such that $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$. Taking $y \in \mathcal{H}[a] \cap \mathcal{H}[b]$, we have $a\mathcal{H}y$ and $b\mathcal{H}y$. It follows from (R2) and $x - x = 0$ for all $x \in X$ that $(a - b)\mathcal{H}(y - y) = (a - b)\mathcal{H}0_Y$ so that $a - b \in Ker\mathcal{H}$.
(ii) Let $x \in \mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v]$. Then $x\mathcal{H}u$ and $x\mathcal{H}v$. Using (R2) and $x - x = 0$ for all $x \in X$, we obtain $(x - x)\mathcal{H}(u - v) = 0_X\mathcal{H}(u - v)$, i.e., $u - v \in \mathcal{H}[0_X]$. This completes the proof. □

Theorem 3.8. *Let $\mathcal{H} : X \to Y$ be an SA-relation and let S be a subalgebra of X. Then*

$$\mathcal{H}[S] := \{y \in Y | x\mathcal{H}y \quad for\ some\ x \in S\}$$

is a subalgebra of Y.

Proof. Clearly, $\mathcal{H}[S] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $y_1, y_2 \in \mathcal{H}[S]$. Then $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$ for some $x_1, x_2 \in S$. Using (R2), we obtain $(x_1 - x_2)\mathcal{H}(y_1 - y_2)$ which implies that $y_1 - y_2 \in \mathcal{H}[S]$ since $x_1 - x_2 \in S$. Therefore $\mathcal{H}[S]$ is a subalgebra of $\mathcal{H}[X]$. □

Corollary 3.9. *Let $\mathcal{H} : X \to Y$ be an SA-relation. Then*
(i) *$\mathcal{H}[X]$ is a subalgebra of Y.*
(ii) *$\mathcal{H}[X] = \cup_{x \in X}\mathcal{H}[x]$.*
(iii) *The zero image of $\mathcal{H}$ is a subalgebra of $\mathcal{H}[X]$.*

Proof. (i) and (ii) are straightforward.
(iii) Let $a, b \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}a$ and $0_X\mathcal{H}b$, and hence $0_X\mathcal{H}(a - b)$, i.e., $a - b \in \mathcal{H}[0_X]$. Therefore $\mathcal{H}$ is a subalgebra of $\mathcal{H}[X]$. □

Theorem 3.10. *Let $\mathcal{H} : X \to Y$ be an SA-relation and T be a subalgebra of Y. Then*

$$\mathcal{H}^{-1}[T] := \{x \in X | x\mathcal{H}y \text{ for some } y \in T\}$$

is a subalgebra of X.

Proof. Obviously, $\mathcal{H}^{-1}[T] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $x_1, x_2 \in \mathcal{H}^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$. Note that $y_1 - y_2 \in T$ since T is a subalgebra of Y. It follows from (R2) that $(x_1 - x_2)\mathcal{H}(y_1 - y_2)$ so that $x_1 - x_2 \in \mathcal{H}^{-1}[T]$. Hence $\mathcal{H}^{-1}[T]$ is a subalgebra of X. □

Corollary 3.11. *Let $\mathcal{H} : X \to Y$ be an SA-relation. Then*
(i) *$\mathcal{H}^{-1}[Y]$ is a subalgebra of X,*
(ii) *$\mathcal{H}^{-1}[Y] := \cup_{y \in Y}\mathcal{H}^{-1}[y]$,*
(iii) *The kernel of $\mathcal{H}$ is a subalgebra of $\mathcal{H}^{-1}[Y]$.*

Proof. (i) and (ii) are straightforward.
(iii) Let $x, y \in Ker\mathcal{H}$. Then $x\mathcal{H}0_Y$ and $y\mathcal{H}0_Y$. It follows from (R2) and $x - x = 0$ for all $x \in X$ that $(x - y)\mathcal{H}(0_Y - 0_Y) = (x - y)\mathcal{H}0_Y$ so that $x - y \in Ker\mathcal{H}$. Hence $\mathcal{H}$ is a subalgebra of $\mathcal{H}^{-1}[Y]$. This completes the proof. □

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