A NOTE ON LINEAR COMPACTNESS

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Abstract. The linear compactness of equicontinuous submodules of linearly topologized modules of continuous multilinear mappings is studied.

Criteria for the compactness of equicontinuous subsets of certain topological modules of continuous multilinear mappings have been established in [3]. The purpose of this note is to prove a criterion for the linear compactness of equicontinuous submodules of certain linearly topologized modules of continuous multilinear mappings, as well as to derive a few consequences of it.

Throughout this note R denotes a commutative topological ring with an identity element. Let us recall ([5]; [7], §31) that: (a) E is a *linearly topologized R-module* if E is a topological R-module whose origin admits a fundamental system of neighborhoods consisting of submodules of E; (b) E is a *linearly compact R-module* if E is a separated linearly topologized R-module such that every family of cosets of closed submodules of E with the finite intersection property has a non-empty intersection; (c) a submodule of a separated linearly topologized R-module E is *linearly compact in E* if it is a linearly compact R-module under the induced topology.

If $E_1, \dots, E_m$, F are R-modules, we shall denote by $\mathcal{F}(E_1, \dots, E_m; F)$ the R-module of all mappings from $E_1 \times \cdots \times E_m$ into F and by $\mathcal{L}_a(E_1, \dots, E_m; F)$ the submodule of $\mathcal{F}(E_1, \dots, E_m; F)$ consisting of all m-linear mappings from $E_1 \times \cdots \times E_m$ into F. If $E_1, \dots, E_m, F$ are topological R-modules, we shall denote by $\mathcal{L}(E_1, \dots, E_m; F)$ the submodule of $\mathcal{L}_a(E_1, \dots, E_m; F)$ consisting of all continuous m-linear mappings from $E_1 \times \cdots \times E_m$ into F.

If $E_1, \dots, E_m$ are R-modules and F is a separated topological R-module, then $(\mathcal{F}(E_1, \dots, E_m; F), \tau_s)$ is a separated topological R-module (τ_s being the topology of pointwise convergence), which is a separated linearly topologized R-module if F is a separated linearly topologized R-module. Hence $(\mathcal{L}_a(E_1, \dots, E_m; F), \tau_s)$ (resp. $(\mathcal{L}(E_1, \dots, E_m; F), \tau_s)$) is a separated linearly topologized R-module if $E_1, \dots, E_m$ are R-modules (resp. $E_1, \dots, E_m$ are topological R-modules) and F is a separated linearly topologized R-module.

Our main objective is to prove the following

Theorem 1. Let $E_1, \dots, E_m$ be topological R-modules and F a linearly compact R-module. If $\mathfrak{X}$ is an equicontinuous submodule of $\mathcal{L}(E_1, \dots, E_m; F)$, then $\overline{\mathfrak{X}}$ is linearly compact in $(\mathcal{L}(E_1, \dots, E_m; F), \tau_s)$.

In order to prove the theorem, we shall need an auxiliary result:

Lemma 2. If $E_1, \dots, E_m$ are R-modules and F is a separated topological R-module, then $\mathcal{L}_a(E_1, \dots, E_m; F)$ is closed in $(\mathcal{F}(E_1, \dots, E_m; F), \tau_s)$.

Proof. For each $z_1 \in E_1, \dots, z_m \in E_m$, the linear mapping

$$f \in (\mathcal{F}(E_1, \dots, E_m; F), \tau_s) \mapsto f(z_1, \dots, z_m) \in F$$

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is continuous. Thus, for each $i \in \{1, \ldots, m\}$, the set

$$\mathcal{L}_i = \bigcap_{\substack{\lambda,\mu \in R, x_1 \in E_1, \ldots, \\ x_i, y_i \in E_i, \ldots, x_m \in E_m}} \{f \in \mathcal{F}(E_1, \ldots, E_m; F); f(x_1, \ldots, \lambda x_i + \mu y_i, \ldots, x_m) - \lambda f(x_1, \ldots, x_i, \ldots, x_m) - \mu f(x_1, \ldots, y_i, \ldots, x_m) = 0\}$$

is closed in $(\mathcal{F}(E_1, \ldots, E_m; F), \tau_s)$ because $\{0\}$ is closed in F. Since $\mathcal{L}_a(E_1, \ldots, E_m; F) = \bigcap_{i \in \{1,\ldots,m\}} \mathcal{L}_i$, the lemma is proved.

Proof of Theorem 1. By Theorem 31.10 of [7], $(\mathcal{F}(E_1, \ldots, E_m; F), \tau_s)$ is linearly compact. Let $\overline{\mathfrak{X}^{\mathcal{F}}}$ be the closure of $\mathfrak{X}$ in $(\mathcal{F}(E_1, \ldots, E_m; F), \tau_s)$; $\overline{\mathfrak{X}^{\mathcal{F}}}$ is a submodule of $\mathcal{F}(E_1, \ldots, E_m; F)$. By Theorem 31.9 (2) of [7], $\overline{\mathfrak{X}^{\mathcal{F}}}$ is linearly compact in $(\mathcal{F}(E_1, \ldots, E_m; F), \tau_s)$. On the other hand, by Proposition 6, p. 28 of [2], $\overline{\mathfrak{X}^{\mathcal{F}}}$ is equicontinuous. Hence, by Lemma 2, $\overline{\mathfrak{X}^{\mathcal{F}}} \subset \mathcal{L}(E_1, \ldots, E_m; F)$, and so $\overline{\mathfrak{X}^{\mathcal{F}}} = \overline{\mathfrak{X}}$. Therefore $\overline{\mathfrak{X}}$ is linearly compact in $(\mathcal{L}(E_1, \ldots, E_m; F), \tau_s)$, as was to be shown.

Before we proceed let us recall ([4], p.171) that if R is a linearly topologized ring, $\mathcal{W}$ is a fundamental system of neighborhoods of 0 in R formed by ideals of R and E is an R-module, then the set $\{WE; W \in \mathcal{W}\}$, formed by submodules of E, is a fundamental system of neighborhoods of 0 for a unique R-module topology on E (which therefore is linear). This topology is said to be deduced from that of R.

Corollary 3. Let $\mathbb{Z}_p$ be the topological ring of p-adic integers and let $E_1, \ldots, E_m$ be $\mathbb{Z}_p$-modules. If $\mathfrak{X}$ is a submodule of $\mathcal{L}_a(E_1, \ldots, E_m; \mathbb{Z}_p)$, then $\overline{\mathfrak{X}}$ is linearly compact in $(\mathcal{L}_a(E_1, \ldots, E_m; \mathbb{Z}_p), \tau_s)$.

Proof. Consider E_i endowed with the topology deduced from that of $\mathbb{Z}_p$ for $i = 1, \ldots, m$. If $\mathcal{Y}$ is an arbitrary subset of $\mathcal{L}_a(E_1, \ldots, E_m; \mathbb{Z}_p)$ and W is a neighborhood of 0 in $\mathbb{Z}_p$ which is an ideal of $\mathbb{Z}_p$, then

$$A((WE_1) \times \cdots \times (WE_m)) \subset W$$

for all $A \in \mathcal{Y}$. Hence $\mathcal{Y}$ is equicontinuous at $(0, \ldots, 0) \in E_1 \times \cdots \times E_m$. Moreover, if W_1, W_2 are arbitrary neighborhoods of 0 in $\mathbb{Z}_p$, then $W_1 W_2$ is a neighborhood of 0 in $\mathbb{Z}_p$, which implies that the product of any neighborhood of 0 in $\mathbb{Z}_p$ by any neighborhood of 0 in E_i is a neighborhood of 0 in E_i for $i = 1, \ldots, m$. By the theorem proved in [1], $\mathcal{Y}$ is equicontinuous. In particular, $\mathcal{L}_a(E_1, \ldots, E_m; \mathbb{Z}_p) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{Z}_p)$. Therefore the result follows from Theorem 1.

Corollary 4. Let $E_1, \ldots, E_m$ be commutative topological groups and F a linearly compact group. If $\mathfrak{X}$ is an equicontinuous subgroup of $\mathcal{L}(E_1, \ldots, E_m; F)$, then $\overline{\mathfrak{X}}$ is linearly compact in $(\mathcal{L}(E_1, \ldots, E_m; F), \tau_s)$.

Proof. Follows immediately from Theorem 1 since a commutative topological group is a topological $\mathbb{Z}$-module and a linearly compact group is a linearly compact $\mathbb{Z}$-module, $\mathbb{Z}$ being endowed with the discrete topology.

Corollary 5. Assume that R is linearly compact (that is, a linearly compact R-module, R being endowed with its canonical R-module structure). Let E be a topological R-module and U a neighborhood of 0 in E. Then

$$\mathfrak{X} = \{A \in \mathcal{L}(E; R);\ A(x) = 0 \text{ for all } x \in U\}$$

is linearly compact in $(\mathcal{L}(E; R), \tau_s)$.

Proof. Obviously, $\mathfrak{X}$ is an equicontinuous submodule of $\mathcal{L}(E;R)$. Moreover, $\mathfrak{X}$ is closed in $(\mathcal{L}(E;R),\tau_s)$ because

$$\mathfrak{X} = \bigcap_{x\in U}\{A \in \mathcal{L}(E;R);\ A(x) = 0\}.$$

Therefore the result is an immediate consequence of Theorem 1.

Corollary 6. Let $\mathbb{K}$ be a discrete field and E a topological vector space over $\mathbb{K}$. If U is a neighborhood of 0 in E, then

$$\mathfrak{X} = \{A \in \mathcal{L}(E;\mathbb{K});\ A(x) = 0 \text{ for all } x \in U\}$$

is linearly compact in $(\mathcal{L}(E;\mathbb{K}),\tau_s)$.

Proof. Since a discrete field is clearly linearly compact, the result is a direct consequence of Corollary 5.

Corollary 6 is already known in the special case where E is a separated linearly topologized space over $\mathbb{K}$ and U is a neighborhood of 0 in E which is a vector subspace of E; see [6], p.97.

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