DEFAULTABLE FORWARD RATE MODEL WITH JUMP RISK UNDER VASICEK-TYPE HAZARD RATE

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Abstract. In this paper, we construct the forward rate model with both the default risk and the jump risk to clarify the effect of the default risk through the implied volatility. For tractability, the Vasicek-type hazard rate model is applied to the default risk instead of the traditional point process. As well known that the model with jump risk can express the smile or the skew, this paper is shows that the implied volatility including the hazard rate can also illustrate smile or skew through a numerical example. Furthermore, it is clarified that there are differences of influence caused by default risk according to the exercise price and the time interval.

1 Introduction Consider the term structure model for forward rates with both the default and the jump risks. Duffie and Singleton [10] introduced the hazard rate model in order to include the default risk of the underlying assets: their model was concerned with the term structure of defaultable bond issued by the defaultable issuer and with the fractional loss rate at the time of default. However, it accompanies the difficulties in the estimation of parameters of the hazard rate process without enough data. Therefore the hazard rate process is described by a stochastic differential equation with positive values (e.g. CIR model). Since the probability described by some stochastic processes is out of range $[0, 1]$ occasionally, it is necessary for setting the parameters to be cautious. Davis and Mavroidis [9] applied the Gaussian model with the deterministic drift which depends on time so as to model the hazard rate dynamics and valuated credit defaultable swaps. On the other hand, Aonuma and Nakagawa [2] also valuated credit defaultable swaps with the Vasicek model in place of Gaussian type hazard rate model. Vasicek(1977)[22] dealt with a term structure model with the property of both mean-reverting and tractability. It is pointed out that the probability described by Vasicek type hazard rate can be also negative.

Moreover, term-structure models for interest rates have been explored to amend the consistency with actual behaviors (cf. the classification of term structure models: Brigo and Mercurio [6] and Chan, Karolyi, Longstaff and Sanders [7], et al). Furthermore, Brace, Gatarek and Musiela [5] gave an interest model with forward rate measure so that there might be no arbitrage opportunity in the interest rate market. It was formulated that the stochastic differential equation of forward rate might be expressed as one of spot rates and there is no arbitrage opportunity over the whole term. Moreover, it was shown that the distribution of forward rate follows the log normal. And the proof of the invariant nature of the equivalent martingale measure is given in terms of Krylov-Bogoliubov Theorem (cf. [8]).

As a literature using LIBOR market model, Glasserman and Kou [12] deal the interest rate model with the jump process for the evaluation of cap, floor, etc. In [12], for the closed

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form solution they assume that the intensity of the jump process is defined as the function that is the multiplication of the intensity, the distribution function and the differential of the mark, therefore it is the sub-class interest rate model because of the restriction on the intensity. Moreover, it is shown that the conditions of the arbitrage-free are fulfilled using Theorem 3.13 of Björk, Kabanov and Runggaldier [4]. The theorem is that under the existence of jump risks, there is a martingale measure if and only if two conditions are satisfied. The first condition is that the intensity under the Q-measure can be expressed as the multiplication of the intensity under the P-measure and probability distribution. The second condition is that there is the equivalence between the drifts of bond price and the forward rate.

From the practical view point, the cause of using the model with the jump risk is that it can generate the implied volatility with smile or skew which is observed in the market. In [12], they illustrated the flexibility of Jump process by which the implied volatility had 'smile or skew'. Moreover, there are many papers which deal with the smile or skew by Poisson density [12], [18], etc. By Constant Elasticity Variance (CEV) process, e.g. [1].

In this paper, our consideration is on the interest rate model with jump and default risks in LIBOR rate market. And particularly, using the implied volatility we analyze the effect of the default risk. LIBOR is the rate for the inter bank (Ranking of AA or more, [21]) debt and is well used as a standard of the floating rate. However, as an interest rate related to a derivative, it may be treated as default-free interest rates.

The martingale measure of the forward rate model is based on the forward measure such as in BGM model. This forward measure based on LIBOR rate is assumed to be the default-free, and it is used to actually price the derivative. But Schönbucher [20] ([3]) is in the thing coping with the existence of the default risk of the LIBOR. Unlike forward measure of BGM model, in [20], the survival forward measure under the defaultable forward rate is used. In this paper, we construct the defaultable forward rate with jump risk. And it is considered using defaultable forward measure (survival forward measure). Then our model is based on [12] for adding the jump risk to BGM model using this defaultable forward measure and as Aonuma and Nakagawa [2], the hazard rate is assumed to follow the Vasicek model in order to consider the possibility of default risk.

The composition of this paper is as follows: In section 2: we give preliminaries and LIBOR market model. And it is confirmed that the conditions of arbitrage-free of forward rate model comes from Theorem 3.13 of Björk, Kabanov and Runggaldier [4]. In section 3, we derive an approximate caplet price that the model with the default risk defined by the hazard rate model with Vasicek type. Furthermore under some assumptions, we confirm the boundedness of difference between the caplet price and the approximate caplet price. In section 4, we propose the model with both the default and jump risks. A numerical example which illustrates the influence caused by the default risk is shown in section 5 . Finally a conclusion of this paper is summarized.

2 Preliminaries

2.1 Notation For the financial market, define the stochastic model on the filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}, \mathbb{P})$ where $\mathbf{F} = \{\mathscr{F}_t\}_{t \geqslant 0}$ is the natural filtration generated by a one-dimensional Brownian Motion W_t. Let $\mathscr{G}$ be a filtration such that $\bigcap_{s>t}(\mathscr{F}_s \vee \sigma\{\tau \wedge s\})$ where τ is a stopping time. For jump components, $\mu(dt, dx)$ is a marked point process measure on Blackwell space $(E, \mathscr{E})$ and $v(dt, dx)$ is the compensator of the measure. Moreover $\mu'(dt, dx)$ and $v'(dt, dx)$ are a marked point (one-point) process measure on $(E', \mathscr{E}')$ and it's compensator for the default component. r_t, $B(t, T)$ and $f(t, T)$ denote the short rate, the bond price and the (instantaneous) forward rate, respectively. These dynamics are as follows:

Assumption 2.1
Let $dB(t,T)$ and $df(t,T)$ be stochastic processes of the bond price and the forward rate without jump and default risks, respectively.

$$dB(t,T) = B(t,T)\{b^B(t,T)dt + v^B(t,T)dW_t\}, \tag{1}$$
$$df(t,T) = b^f(t,T)dt + v^f(t,T)dW_t, \tag{2}$$

where W_t is a Brownian Motion in $\mathbb{R}$ defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

Moreover when these dynamics have jump risks given by the marked point process, assumption 2.1 is rewritten as follows[1].

Assumption 2.2
Let $dB^j(t,T)$ and $df^j(t,T)$ be stochastic processes of the bond price and the forward rate with jump risks and without default risk, respectively.

$$\begin{aligned} dB^j(t,T) &= B^j(t-,T)\Big\{b^B(t,T)dt + v^B(t,T)dW_t \\ &\quad + \int_E q^{B,j}(t,x,T)\mu(dt,dx)\Big\}, \end{aligned} \tag{3}$$
$$df^j(t,T) = b^f(t,T)dt + v^f(t,T)dW_t + \int_E q^{f,j}(t,x,T)\mu(dt,dx). \tag{4}$$

Furthermore if the dynamics of assumption 2.2 has the default risk, these dynamics are defined as below.

Assumption 2.3
Let $dB^{j,d}(t,T)$ and $df^{j,d}(t,T)$ be the stochastic processes of the bond price and the forward rate with jump and default risks, respectively.

$$\begin{aligned} dB^{j,d}(t,T) &= B^{j,d}(t-,T)\Big\{b^B(t,T)dt + v^B(t,T)dW_t \\ &\quad + \int_E q^{B,j}(t,x,T)\mu(dt,dx) + \int_{E'} q^{B,d}(t,x,T)\mu(dt,dx)\Big\}, \end{aligned} \tag{5}$$

$$\begin{aligned} df^{j,d}(t,T) &= b^f(t,T)dt + v^f(t,T)dW_t + \int_E q^{f,j}(t,x,T)\mu(dt,dx) \\ &\quad + \int_{E'} q^{f,d}(t,x,T)\mu'(dt,dx). \end{aligned} \tag{6}$$

Using these assumptions 2.2 and 2.3, define the spread yield or the hazard rate, equivalently.

[1]This type is considered in Björk, Kabanov and Runggaldier [4].

Definition 2.1
Under assumption 2.3, let the forward rate with jump risk, and one with both jump and default risk be given by the equation (5) and (6). Then the hazard rate (or the yield spread, the intensity) $\lambda(t,s)$ is defined as

$$\int_t^T f^{j,d}(t,s)ds + \int_t^T \lambda(t,s)ds = \int_t^T f^j(t,s)ds, \tag{7}$$

or equivalently

$$f^{j,d}(t,s) + \lambda(t,s) = f^j(t,s). \tag{8}$$

Furthermore define some rates and derive some connections between them. Let $F(t,T)$, $F^j(t,T)$ and $F^{j,d}(t,T)$ denote the forward rate, the forward rate with jump risks, and the forward rate with jump and default risks, respectively. Let $S^{j-(j,d)}(t,T)$ be the yield spread between $F^j(t,T)$ and $F^{j,d}(t,T)$. As mentioned above, $\lambda^f(t,T)$ is the hazard rate of the default. The application of the hazard rate is found in Duffie and Singleton [10]. As they consider the default risk of the bond, the hazard rate is used as the term structure for expressing the default risk. And the yield spread is defined by the spread of the rate between the defaultable bond and the default-free bond. Here, we treat the spread of the forward rates between the jump risk and the jump-default risks as the hazard rate. The relations of these rates followed.

From the HJM framework [14], for a positive constant δ the forward rate at t without jump and default risks over $[T, T+\delta]$ is given by

$$F(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) = \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f(t,s)ds\right\} - 1\right), \tag{9}$$

and the forward rate with jump and the forward rate with jump and default risks are

$$F^j(t,T) = \frac{1}{\delta}\left(\frac{B^j(t,T)}{B^j(t,T+\delta)} - 1\right) = \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f^j(t,s)ds\right\} - 1\right), \tag{10}$$

and

$$\begin{aligned} F^{j,d}(t,T) &= \frac{1}{\delta}\left(\frac{B^{j,d}(t,T)}{B^{j,d}(t,T+\delta)} - 1\right) \\ &:= \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds\right\} - \exp\left\{-\int_T^{T+\delta} \lambda_s^f ds\right\}\right). \end{aligned} \tag{11}$$

Equation (11) is assumed to obtain the convenient form of the expected defaultable forward rate and to reduce the effect of the default risk. Since the expectation of $exp\{-\int_T^{T+\delta}\lambda_s^f ds\}$ is the conditional survival probability, the conditional probability is close to 1 as long as the degree of the change rate of the survival probability with respect to time is not large. Setting $exp\{-\int_T^{T+\delta}\lambda_s^f ds\} = 1$, the following lemma 2.1 is expressed as

$$\mathbb{E}\left[F^{j,d}(t,T)\middle|\mathscr{G}_t\right] = \mathbb{P}(\tau > T+\delta|\tau > T)\mathbb{E}\left[F^j(t,T)\middle|\mathscr{G}_t\right] + \frac{1}{\delta}(\mathbb{P}(\tau > T+\delta|\tau > T) - 1).$$

Note that (11) is equivalent to set the second term 0 and to decrease the effect of default risk because $\mathbb{P}(\tau > T+\delta|\tau > T) \leqslant 1$. In section 5, it will be shown that the conditional probability is in $[0.995705, 0.996478]$ by numerical example. Though the difference between 1 and the conditional probability seems to be small, it has the large effect to the caplet price because $\mathbb{P}(\tau > T+\delta|\tau > T)\mathbb{E}\left[F^j(t,T)\middle|\mathscr{G}_t\right]$ is $0.06\mathbb{P}(\tau > T+\delta|\tau > T)$ in the caplet formula. Therefore the reduction of the effect of default risks is achieved by setting (11).

Equation (11) is used to obtain the explicit form of the forward rate as in lemma 2.1. Notice that so as to obtain no-arbitrage conditions bellow, the foward rate is assumed to be $F^{j,d}(t,T) = \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds\right\} - 1\right)$ in place of (11).

In definition 2.1, the hazard rate $\lambda(t,s)$ expresses the component of the default risk. The following definition is concerned with the relation between the default risk (the survival probability) and the hazard rate.

Definition 2.2
Under definition 2.1, the survival probability is defined as

$$\mathbb{P}\{\tau > T|\tau > t\} = \mathbb{E}\left[\exp\left\{-\int_t^T \lambda(t,s)ds\right\}\middle|\mathscr{G}_t\right] \tag{12}$$

$$= \mathbb{E}\left[1_{\{\tau>T\}}\middle|\mathscr{G}_t\right], \tag{13}$$

where τ is the default time and 1 is the indicator function.

Remark 2.1
In definition 2.2, the survival probability is defined by using the intensity (i.e the hazard rate). The left hand side of (12) can be evaluated under the assumption on the dynamics of hazard rate and in [17] they assume the hazard rate process is affine-type. Moreover in [2], under the hazard rate in Vasicek model, the default risk is considered. In this paper, we assume the hazard rate to be Vasicek type as in [2]. So (12) is expressed explicitly in section 3 .

Definition 2.1 and 2.2 lead to the following relation for forward rates.

Lemma 2.1
Let instantaneous forward rates $f^j(t,s)$ and $f^{j,d}(t,s)$ be in assumption 2.2 and 2.3. Assume the default risk and the instantaneous forward rate are independent. Then forward rates with and without default risk: $F^j(t,T)$ and $F^{j,d}(t,T)$ are given as follows:

$$\mathbb{E}\left[\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds\right\}\middle|\mathscr{G}_t\right] = \mathbb{P}(\tau > T+\delta|\tau > T)\mathbb{E}\left[\exp\left\{\int_T^{T+\delta} f^j(t,s)ds\right\}\middle|\mathscr{G}_t\right] \tag{14}$$

and

$$\mathbb{E}\left[F^{j,d}(t,T)\middle|\mathscr{G}_t\right] = \mathbb{P}(\tau > T+\delta|\tau > T)\mathbb{E}\left[F^j(t,T)\middle|\mathscr{G}_t\right]. \tag{15}$$

Proof. By equation (7),

$$\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds + \int_T^{T+\delta} \lambda_s^f ds\right\} = \exp\left\{\int_T^{T+\delta} f^j(t,s)ds\right\},$$

where λ_s^f denotes $\lambda(t,s)$ of the forward rate for convenience. And taking the expectation, (14) holds.

Moreover, by (11) and (7), we have

$$F^{j,d}(t,T) = \frac{1}{\delta}\left(\exp\left\{-\int_T^{T+\delta}\lambda_s^f ds\right\}\exp\left\{\int_T^{T+\delta} f^j ds\right\} - \exp\left\{-\int_T^{T+\delta}\lambda_s^f ds\right\}\right)$$

and taking expectation,

$$\mathbb{E}\left[F^{j,d}(t,T)\middle|\mathscr{G}_t\right] = \mathbb{P}(\tau > T+\delta|\tau > T)\mathbb{E}\left[F^j(t,T)\middle|\mathscr{G}_t\right].$$

□

Moreover, the spread yield in definition 2.1 is satisfied under the arbitrage-free in the markets i.e. the absence of the arbitrage opportunities between the bond and forward rate markets. The condition is important so as to construct the forward rate models. In this paper, we derive the conditions according to [4] [5] et al. in section 2.3.

In [20], the spread yield is defined as the difference between $F(t,T)$ and $F^d(t,T)$. But in this paper we define it as $F^d(t,T)/F(t,T)$ i.e. $\exp\{-\int \lambda_s^f ds\}$. The relation of the spread yield between [20] and our definition is described as the following properties.

Proposition 2.1
Let the spread yield be defined as in lemma 2.1 and the credit spread $S(T,T+\delta)$ in [20] be $F^{j,d}(t,T) - F^j(t,T)$. Then lemma 2.1 holds under the intensity $H(T,T+\delta)$ or the credit spread $S(T,T+\delta)$ in [20].

Proof. From the proof of lemma 2.1 and equation (10),

$$\begin{aligned} F^{j,d}(t,T) &= \exp\left\{-\int_T^{T+\delta}\lambda_s^f ds\right\}\frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f^j(t,s)ds\right\} - 1\right) \\ &= \frac{1_{\{\tau>T+\delta\}}}{1_{\{\tau>T\}}}\frac{1}{\delta}\left(\left(\delta F^j(t,T)+1\right)-1\right) \\ &= \frac{1_{\{\tau>T+\delta\}}}{1_{\{\tau>T\}}}F^j(t,T) \qquad (16) \end{aligned}$$

On the other hand, following the definition of the credit spread in [20], the credit spread with jump risks is defined as

$$S(T,T+\delta) := F^{j,d}(t,T) - F^j(t,T)$$

and the default intensity is defined as

$$\begin{aligned} &H(T,T+\delta) \\ &:= \frac{1}{\delta}\left\{\frac{B^{d,j}(t,T)}{B^{d,j}(t,T+\delta)}\frac{B^j(t,T+\delta)}{B^j(t,T)} - 1\right\} \\ &= \frac{1}{\delta}\left\{\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds\right\}\exp\left\{-\int_T^{T+\delta} f^j(t,s)ds\right\} - 1\right\} \\ &= \frac{1}{\delta}\left\{\frac{1_{\{\tau>T\}}}{1_{\{\tau>T+\delta\}}}\exp\left\{\int_T^{T+\delta} f^j(t,s)ds\right\}\exp\left\{-\int_T^{T+\delta} f^j(t,s)ds\right\} - 1\right\} \\ &= \frac{1}{\delta}\left\{\frac{1_{\{\tau>T\}}}{1_{\{\tau>T+\delta\}}} - 1\right\}. \end{aligned}$$

So from these equations and (16), we obtain

$$\begin{aligned}\exp\left\{-\int_T^{T+\delta}\lambda^f(t,s)ds\right\} &= \frac{F^{j,d}(t,T)}{F^j(t,T)}\\ &= \frac{1}{\delta H(T,T+\delta)+1}. \qquad (17)\end{aligned}$$

Furthermore, using the definition of $H(T,T+\delta)$, $S(T,T+\delta)$ in [20],

$$H(T,T+\delta) = \frac{S(T,T+\delta)}{1+\delta F^j(t,T)}. \qquad (18)$$

Therefore by (17) and (18), we have

$$\exp\left\{-\int_T^{T+\delta}\lambda^f(t,s)ds\right\} = \frac{1+\delta F^j(t,T)}{\delta S(T,T+\delta)+\delta F^j(t,T)+1}. \qquad (19)$$

Since the yield spread $\exp\left\{-\int_T^{T+\delta}\lambda^f(t,s)ds\right\}$ can be expressed in terms of the intensity $H(T,T+\delta)$, or the credit spread $S(T,T+\delta)$, lemma 2.1 holds. □

Finally we define the hazard rate process. In Aonuma and Nakagawa [2], they assume that the hazard rate process is the Vasicek process. Under their assumption, this paper defines $dh^f(t,s)$ as the stochastic process of Vasicek-type.

Assumption 2.4
Let $d\lambda(t,s)$ be the hazard process in definition 2.1. We assume $d\lambda(t,s)$ is the following process.

$$d\lambda(t,s) = c(m-\lambda(t,s))ds + \sigma d\tilde{W}_s, \quad \lambda(t,t)>0, t\leqslant s \qquad (20)$$

where c, m, and σ are positive constants, $\lambda(t,t)$ is the initial value at time t. $\tilde{W}_t$ is the standard Brownian motion in $\mathbb{R}$ on $(\Omega,\mathscr{F},\mathbf{F},\mathbb{P})$ and independent of W_t which is consistent with the independence of the default risks and the forward rate in lemma 2.1 [2].

2.2 Relation between the bond price and the forward rate In the interest rate markets, there exist some connections between them. Brace Gatarek, and Musiela [5] give the relation between the bond price and forward rate which corresponds to assumption 2.1 and Glasserman and Kou [12] and Björk, Kabanov and Runggaldier [4] discuss the same problem under assumption 2.2. We need the arbitrage-free condition on assumption 2.3. The following two propositions give the relation.

Proposition 2.2
Let the stochastic processes of rates be on assumption 2.1. If the forward rate is under assumption 2.2, then the bond price is provided by

$$\begin{aligned}\frac{dB(t,T)}{B(t,T)} &= \left\{r_s - \int_t^T b^f(t,s)ds + \frac{1}{2}\left(\int_t^T v^f(t,s)ds\right)^2\right\}dt\\ &\quad - \int_t^T v^f(t,s)dsdW_t. \qquad (21)\end{aligned}$$

[2]If there exists the correlation, then by the Cholesky decomposition, we can construct the independence of them.

Proof. See Appendix A.1. □

If there exists the jump risk, then the bond price is given by proposition 2.2 in Björk, Kabanov and Runggaldier [4]. With jump and default risk, as bellow.

Proposition 2.3
Let the stochastic processes of rates be on assumption 2.3. If the forward rate is under assumption 2.3 and 2.4, then the dynamics of the bond price is expressed as follows:

$$\begin{aligned}\frac{dB^{j,d}(t,T)}{B^{j,d}(t-,T)} &= \frac{dB(t,T)}{B(t,T)} + \int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\mu(dt,dx) \\ &\quad + \int_{E'} \left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\} - 1\right]\mu'(dt,dx). \end{aligned} \tag{22}$$

Proof. See Appendix A.1. □

2.3 Arbitrage-Free and Girsanov In this subsection, we describe the relationship between the arbitrage-free and the Girsanov theorem under jump and default risks. In [4], they give the Girsanov theorem with the existence of the point process. Their contribution leads to the extension of forward rate models. One of them, Schönbucher [20] considers the forward rate with the default risk. They assume the marked point process as default time. Moreover, Glasserman and Kou [12] present a simplified forward model with the jump risk. In their model, the marked point process is defined as the jump of the forward rate. We need to include jump and default risks in the forward rate.

Before discussing the Girsanov theorem, some assumptions in addition to assumption 2.2 [3] are given as follows (cf. [4],[15]).

Assumption 2.5
The filtration is such that:

$$\mathbf{F} = \sigma\{W_t, \mu([0,t]\times A), B; 0 \leqslant t \leqslant T+\delta, A \in \mathscr{E}, B \in \mathscr{N}\},$$

where $\mathscr{N}$ is the collection of P null sets from $\mathscr{F}$.

Assumption 2.6
There is a predictable intensity process $\lambda(t,dx)$ such that:

$$v(dt,dx) = \lambda(t,dx)dt,$$

where $v(dt,dx)$ is a compensator.

Thus there is a marked point process in the forward rate and it is necessary to change the measure with marked point process. Since there exists the measure, all martingale M can be expressed as

$$M = M_0 + H(\mu(dt,dx) - v(dt,dx)),$$

where H is a predictable and Stieltjes-integrable process with respect to $\mu(dt,dx)$ and $v(dt,dx)$. (See Jacod and Shiryaev [15])

[3] Under assumption 2.3, the filtration $\mathbf{F}'$ is $\sigma\{W_t, \mu([0,t]\times A), \mu'([0,t]\times A'), B; 0 \leqslant t \leqslant T+\delta, A \in \mathscr{E}, A' \in \mathscr{E}', B \in \mathscr{N}\}$ and the compensator $v'(dt,dx)$ is $\lambda'(t,dx)dt$.

Theorem 2.1 (Girsanov)
Assume assumption 2.6. Let θ_t be the predictable process and $\Phi(s,x)$ be a nonnegative predictable function such that

$$\int_0^t \theta_s^2 ds < \infty, \qquad \int_0^t \int_E \Phi(s,x)\lambda(s,dx)ds < \infty,$$

for all finite time t. Define the process L_t by

$$\begin{aligned}\log L_t &= \int_0^t \theta_s dWs - \frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \int_E \log\Phi(s,x)\mu(ds,dx) \\ &\quad + \int_0^t \int_E (1-\Phi(s,x))v(ds,dx),\end{aligned}$$

equivalently,

$$dL_t = L_{t-}\theta_t dW_t + L_{t-} - \int_E (1-\Phi(s,x))(\mu(ds,dx) - v(ds,dx)),$$

with $L_0 = 1$, and suppose $\mathbb{E}^P[L_t] = 1$ for all finite time t.

Then, there exists a probability measure Q that is equivalent to P on $\mathscr{F}$: $dP_t = L_t dQ_t$ such that $dW_t = \theta_t dt + dW_t'$ and $\lambda_Q(t,dx) = \Phi(t,x)\lambda(t,dx)$, where W' is Q-Winer process and λ_Q is Q-intensity.

By using Girsanov, we obtain the arbitrage-free condition between the forward rate and the bond price.

Theorem 2.2 (Arbitrage-free condition)
Let the stochastic process of the forward rate be (6) and the bond price be (5). Assume the assumption 2.5, 2.6 are satisfied. And the hazard process is assumed to be a Vasicek model. Then there exists a martingale measure if and only if for all $T > 0$, the following arbitrage-free condition is satisfied

$$\begin{aligned}\int_t^T b^f(t,s)ds &= \frac{1}{2}\left(\int_t^T v^f(t,s)ds\right)^2 - \theta_t \int_t^T v^f(t,s)ds \\ &\quad - \int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\Phi(t,x)\lambda(t,dx) \\ &\quad - \int_{E'} \left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\} - 1\right]\Phi'(t,x)\lambda'(t,dx)\end{aligned}$$

or equivalently

$$\begin{aligned}b^f(t,s) &= v^f(t,s)\left(\int_t^T v^f(t,s)ds\right) - \theta_t v^f(t,s) \\ &\quad -\frac{\partial}{\partial s}\int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\Phi(t,x)\lambda(t,dx) \\ &\quad -\frac{\partial}{\partial s}\int_{E'} \left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\} - 1\right]\Phi'(t,x)\lambda'(t,dx).\end{aligned}$$

Proof. See Appendix A.2. □

By the arbitrage-free condition of theorem 2.2, we obtain the bond price under the arbitrage-free as follows:

Corollary 2.1
Let the dynamics of the bond price be given in proposition 2.3. If the arbitrage-free condition in theorem 2.2 is satisfied, then the bond price is

$$\begin{aligned} B^{j,d}(t,T) &= \exp\left\{\left\{\int_t^T \left(r_u - \frac{1}{2}\theta_u^2\right) du - \int_t^T \theta_u dW_u\right\} - \int_t^T \int_E \left(q^{f,j}(u,x,s)ds\right)\right. \\ &\quad \mu(du,dx) - \int_t^T \int_{E'} \left(q^{f,d}(u,x,s)ds\right) \mu'(du,dx) \\ &\quad + \int_t^T \int_E \left(1 - \exp\left\{-\int_t^T q^{f,j}(u,x,s)ds\right\}\right) v(du,dx) \\ &\quad \left. + \int_t^T \int_{E'} \left(1 - \exp\left\{-\int_t^T q^{f,d}(u,x,s)ds\right\}\right) v'(du,dx)\right\}, \end{aligned} \tag{23}$$

where $\theta_u = \int_t^T v^f(u,s)ds$.

Proof. Substituting $\int_t^T b^f(t,s)ds$ in theorem 2.2 into (22), (23) is derived. □

2.4 Extended forward rate model The BGM model is shown by Brace, Gatarek and Musiela [5]. They consider the arbitrage-free interest rate over the whole term by using the forward rate measure. And the forward rate model given by them is extended to two cases. One of them is the model with jump risk in [12]. In order to treat the derivatives, they assume the model with jump term. To find the closed form solution of the price of these derivatives, they assume that

$$\lambda_Q(t,dx) = \lambda g(x)dx,$$

where λ is a constant and $g(\cdot)$ is a density. Therefore the jump process is reduced to a compound Poisson process with arrival rate λ under this assumption. By dint of this simplification, a closed form solution is given.

In addition to this assumption, suppose that there is no jump risk in theorem 2.2, and it implies the same condition for arbitrage-free.

On the other hand, Schönbucher [20] considers the default risk of underlying bond in forward rate model to construct the survival measure. We describe their model briefly. First, the defaultable time τ is defined as the point process. And the T-forward (default-free) measure is such that

$$p' = \frac{p(t)}{B(0,T)} = \mathbb{E}^P\left[\frac{e^{-\int_0^T r_s ds}B(T,T)}{B(0,T)}X\right] = E^Q[X],$$

where $p(t)$ is the price at time t that is paid the amount X at time T. So, $p'$is the discounted value of $p(t)$ at time 0. The Radon-Nikodym density process is

$$L(t) := \frac{e^{-\int_0^t r_s ds}B(t,T)}{B(0,T)} = \frac{dQ}{dP}.$$

This discount factor $B(0,T)$ has no default risk. When the discount factor has the default risk, $P(dP)$ is the T-survival measure and

$$p' = \frac{p(t)}{B^d(0,T)} = \mathbb{E}^{P^d}\left[\frac{e^{-\int_0^T r_s ds} B^d(T,T) 1_{\tau>T}}{B^d(0,T)} X\right] = E^{Q^d}[X],$$

and

$$L^d(t) := \frac{e^{-\int_0^t r_s ds} B^d(t,T) 1_{\tau>T}}{B^d(0,T)} = \frac{dQ^d}{dP^d}.$$

In this Radon-Nikodym, $e^{-\int_0^T r_s ds} 1_{\tau>T} X$ is the discounted payment (at time T) of the obligor . So in theorem 2.2, if we assume no jump risk and the default risk of the marked point process, then the forward martingale measure is same as [20].

Moreover, our consideration is the default risk of the bond or equivalently the forward rate. For example, the object is to price the forward rate derivative settled with some exercise rate at the maturity [4]. Therefore if the default risk of the obligor is included (e.g. some contract with the default risk at the promised time.), then we must consider the default risk of the obligor (not bond issuer) and the correlation between the obligor and the bond issuer.

3 Forward Rate Model with Default Risk and Vasicek-type Hazard Rate In this section, we are concerned with the hazard rate. Particularly, for tractability, Vasicek-type stochastic process which is given in Vasicek [22]. The process has the desirable properties: mean-reverting, tractability. Aonuma and Nakagawa [2] use Vasicek model to estimate the hazard rate, however it also has undesirable property: it comes into negative value. Moreover the solution of the generalized hazard rate with the affine model is given in [17].

3.1 Vasicek-type Hazard Rate Model In [17], they give the solution of the hazard rate with affine type model. So our concerning is Vasicek type model and we give the solution for the estimation as [2].

Theorem 3.1
If h(t,T) is the solution of the stochastic differential equation (Vasicek model) of assumption 2.4, then the probability of the default time $\tau > t$ is given by

$$\begin{aligned} \mathbb{P}(\tau > t) &= \mathbb{E}\left[\exp\left\{-\int_0^t h(0,s)ds\right\}\right] \\ &= \exp\left[\frac{1}{c}\left(e^{-ct}-1\right)h(0,0) - \frac{1}{c}\left(e^{-ct}-1\right)\left(m + \frac{\sigma^2}{4c^2}\left(e^{-ct}-3\right)\right)\right. \\ &\qquad \left. -t\left(m - \frac{\sigma^2}{2c^2}\right)\right]. \end{aligned} \tag{24}$$

Proof. See Proposition 2 in [17]. □

Corollary 3.1
Let h(t,T) be same as in Theorem 3.1. The conditional probability $\mathbb{P}(\tau > T+\delta | \tau > T)$ is represented as

$$\mathbb{P}(\tau > T+\delta|\tau > T) = \exp\left[\frac{1}{c}\left\{\left(e^{-c(T+\delta)} - e^{-c(T)}\right)h(0,0) + \left(e^{-c(T+\delta)} - 1\right)\right.\right.$$

[4] In [12], Pricing interest rate derivatives.

$$\left(m+\frac{\sigma^2}{4c^2}\left(e^{-c(T+\delta)}-3\right)\right)-\left(e^{-cT}-1\right)\left(m+\frac{\sigma^2}{4c^2}\left(e^{-cT}-3\right)\right)\Bigg\}-\delta\left(m-\frac{\sigma^2}{2c^2}\right)\Bigg].$$

Proof. Using Theorem 3.1 with the definition of the conditional probability,

$$\begin{aligned}
\mathbb{P}(\tau > T+\delta|\tau > T) &= \frac{\mathbb{P}(\tau > T+\delta)}{\mathbb{P}(\tau > T)} = \frac{\mathbb{E}\left[\exp\left\{-\int_0^{T+\delta} h_s ds\right\}\right]}{\mathbb{E}\left[\exp\left\{-\int_0^{T} h_s ds\right\}\right]} \\
&= \exp\Bigg[\frac{1}{c}\Bigg\{\left(e^{-c(T+\delta)}-e^{-c(T)}\right)h(0,0)+\left(e^{-c(T+\delta)}-1\right) \\
&\quad \left(m+\frac{\sigma^2}{4c^2}\left(e^{-c(T+\delta)}-3\right)\right)-\left(e^{-cT}-1\right)\left(m+\frac{\sigma^2}{4c^2}\left(e^{-cT}-3\right)\right)\Bigg\} \\
&\quad -\delta\left(m-\frac{\sigma^2}{2c^2}\right)\Bigg].
\end{aligned}$$

□

3.2 Forward Rate Model with Vasicek-type Hazard Rate The forward rate model with the default risk is given in [20] as mentioned above. The default risk is assumed to follow the point process. By contrast, our model is that the default risk is constructed by the hazard rate and the probability of the default is given in theorem 3.1. Using the stochastic process of the hazard rate with Vasicek type, first we construct the forward rate model without the jump risk. On the other hand, we consider the model with jump and default risks in the next section.

In the case of no jump risk, the arbitrage-free condition of theorem 2.2 is modified as

$$\begin{aligned}
b^f(t,s) &= v^f(t,s)\left(\int_t^T v^f(t,s)ds\right)-\theta_t v^f(t,s) \\
&\quad -\frac{\partial}{\partial s}\int_{E'}\left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\}-1\right]\Phi'(t,x)\lambda'(t,dx).
\end{aligned} \tag{25}$$

And the dynamics of the forward rate is provided by

$$df^d(t,T) = b^f(t,T)dt + v^f(t,T)dW_t + \int_{E'} q^{f,d}(t,x,s)\mu'(dt,dx). \tag{26}$$

Assume the default risk follows the Vasicek-type hazard rate process in this subsection. Then, we give the arbitrage-free condition (25) and (26) under the Vasicek-type hazard rate.

Proposition 3.1
Assume the condition of theorem 2.2 is satisfied and there is no jump risk. Moreover, assume the default risk follows the Vasicek-type hazard rate process (20). Then the arbitrage free condition is given by

$$\begin{aligned}
&\int_t^T (b^f(t,s)-c(m-\lambda(t,s)))ds \\
&= \frac{1}{2}\left(\int_t^T \sqrt{v^f(t,s)^2+\sigma^2}ds\right)^2 - \theta_t\left(\int_t^T \sqrt{v^f(t,s)^2+\sigma^2}ds\right)
\end{aligned} \tag{27}$$

Proof. Denote the forward rate with default risks by $F^d(t,T)$, i.e. $F^{j,d}(t,T)$ for case that $q^{f,j}(t,x,T)=0$ in (2.3). Since the arbitrage-free condition can be obtained from Appendix A.1 and A.2, we give the proof briefly. Using (11) with $q^{f,j}(t,x,T)=0$, (20) and $v^f(u,s)dW_u+\sigma d\bar{W}_u=\sqrt{v^f(u,s)^2+\sigma^2}dW'_u$,

$$\begin{aligned} F^d(t,T) &= \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta}\left[(f(0,s)+\lambda(0,s))+\int_0^t (b^f(u,s)+c(m-\lambda(u,s)))du\right.\right.\right. \\ &\quad \left.\left.\left. +\int_0^t \sqrt{v^f(u,s)^2+\sigma^2}dW'_u\right]ds\right\}\right). \end{aligned}$$

And consider the range of integration as $[t,T]$, by $r_s=f^d(0,s)+\int_t^T(b^f(u,s)-c(m-\lambda(u,s)))du+\int_t^T\sqrt{v^f(u,s)^2+\sigma^2}dW'_u$,

$$\begin{aligned} \int_t^T f^d(t,s)ds &= \int_t^T f^d(0,s)ds-\int_t^T r_s ds \\ &+\int_0^t\int_t^T (b^f(u,s)-c(m-\lambda(u,s)))dsdu+\int_0^t\int_t^T\sqrt{v^f(u,s)^2+\sigma^2}dsdW'_u. \end{aligned} \tag{28}$$

Applying Itô formula to $B^d(t,T)$ with the differential form of (28), we have

$$\begin{aligned} \frac{dB^d(t,T)}{B^d(t,T)} &= \left\{r_t-\int_t^T (b^f(t,s)-c(m-\lambda(t,s)))ds\right. \\ &\quad \left.+\frac{1}{2}\left(\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds\right)^2\right\}dt-\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}dsdW'_t. \end{aligned} \tag{29}$$

Thus from $dB(t,T+\delta)=r_tB(t,T+\delta)dt+dM^Q$ and $dW=\theta_t dt+dW'_t$, the arbitrage free condition is represented by

$$\begin{aligned} 0 &= -\int_t^T (b^f(t,s)-c(m-\lambda(t,s)))ds+\frac{1}{2}\left(\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds\right)^2 \\ &\quad -\theta_t\left(\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds\right). \end{aligned}$$

Thus the proof is complete. □

As $b^f(t,s)$ satisfies the arbitrage-free condition in proposition 3.1, the forward rate dynamics is

$$\begin{aligned} df^d(t,s) &= \left\{\sqrt{v^f(t,s)^2+\sigma^2}\left(\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds\right)\right. \\ &\quad \left.-\theta_t\sqrt{v^f(t,s)^2+\sigma^2}\right\}dt+\sqrt{v^f(t,s)^2+\sigma^2}dW_t. \end{aligned} \tag{30}$$

Thus, the forward rate (not instantaneous) is expressed as

$$F^d(t,T)=\frac{1}{\delta}\left[\exp\left\{\int_T^{T+\delta} f^d(t,s)ds\right\}-\exp\left\{\int_T^{T+\delta}\lambda_s^f ds\right\}\right]. \tag{31}$$

Applying Itô formula to $F^d(t,T)$[5], we obtain

$$
\begin{aligned}
dF^d(t,T) &= \frac{1}{\delta}\exp\left\{\int_T^{T+\delta} f^d(t,s)ds\right\} d\left\{\int_T^{T+\delta} f^d(t,s)ds\right\} \\
&\quad +\frac{1}{2\delta}\exp\left\{\int_T^{T+\delta} f^d(t,s)ds\right\}\left[d\left\{\int_T^{T+\delta} f^d(t,s)ds\right\}\right]^2 \qquad (32)
\end{aligned}
$$

where, using (30),

$$
\begin{aligned}
d\left\{\int_T^{T+\delta} f^d(t,s)ds\right\} &= \int_T^{T+\delta} df^d(t,s)ds \\
&= \left\{\left(\int_T^{T+\delta}\sqrt{v^f(t,s)^2+\sigma^2}ds\right)\left(\int_t^{T}\sqrt{v^f(t,s)^2+\sigma^2}ds\right)\right. \\
&\quad \left. -\theta_t\int_T^{T+\delta}\sqrt{v^f(t,s)^2+\sigma^2}ds\right\}dt+\int_T^{T+\delta}\sqrt{v^f(t,s)^2+\sigma^2}dsdW_t. \qquad (33)
\end{aligned}
$$

From (31), (32) and (33),

$$
\frac{\delta dF^d(t,T)}{1+\delta F^d(t,T)} = \left(A\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds-\theta A+\frac{1}{2}A^2\right)dt+AdW_t, \qquad (34)
$$

where $A=\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds$.

The object is to obtain the forward rate dynamics $dF^d(t,T)$. We assume as follows:

$$
\frac{dF^d(t,T)}{F^d(t,T)} := \xi_t dt+\zeta_t dW'_t. \qquad (35)
$$

By (34) and (35), we obtain

$$
\xi_t=\frac{1+\delta F^d(t,T)}{\delta F^d(t,T)}\left(A\int_t^T\sqrt{v^f(t,s)^2+\sigma^2}ds+\theta A+\frac{1}{2}A^2\right) \qquad (36)
$$

$$
\zeta_t=\frac{1+\delta F^d(t,T)}{\delta F^d(t,T)}A \qquad (37)
$$

Under $M_{T_{n+1}}$-measure [6], (35) satisfies

$$
\frac{dF^d(t,T)}{F^d(t,T)}=\zeta_t dW_t^{M_{T_{n+1}}}. \qquad (38)
$$

Then,

$$
F^d(t,T)=\exp\left\{-\frac{1}{2}\int_{T_n}^{T_{n+1}}\zeta_u^2du+\int_{T_n}^{T_{n+1}}\zeta_u dW_u^{M_{T_{n+1}}}\right\}. \qquad (39)
$$

[5] We assume $\exp\left\{\int_T^{T+\delta}\lambda_s^f ds\right\}$ is constant implicitly.

[6] $\zeta_t dW_t^{M_{T_{n+1}}}=\xi_t dt+\zeta_t dW'_t$

Let us price the caplet. Denote the price of the caplet as $C(t,T_n)$ at time t. Then under the risk neutral measure P,

$$C(t,T_n) = \delta \mathbb{E}_P \left[\exp\left\{ -\int_t^{T_{n+1}} r_s ds \right\} C(T_n, T_{n+1}) \middle| \mathscr{G}_t \right] \tag{40}$$

And under T_{n+1}-survival measure,

$$\begin{aligned} C(t,T_n) &= \delta B^d(t,T_{n+1}) \mathbb{E}_{T_{n+1}} \left[\frac{C(T_n,T_{n+1})}{B^d(T_n,T_{n+1})} \middle| \mathscr{G}_t \right] \\ &= \delta B^d(t,T_{n+1}) \mathbb{E}_{T_{n+1}} \left[\left(F^d(T_n,T_{n+1}) - K \right)^+ \middle| \mathscr{G}_t \right]. \end{aligned}$$

From this equation and Black Scholes formula,

$$C(t,T_n) = \delta B^d(t,T_{n+1}) \left[F^d(t,T_n) \mathbf{N}(d_1) - K \mathbf{N}(d_2) \right], \tag{41}$$

where $\mathbf{N}$ denotes the cumulative normal distribution function and

$$\begin{aligned} d_1 &= \frac{\log(F^d(t,T_n)/K) + 1/2\zeta_t^2 (T_n - t)}{\zeta_t \sqrt{T_n - t}}, \\ d_2 &= d_1 - \zeta_t \sqrt{T_n - t} \\ \zeta_t &= \frac{1 + \delta F^d(t,T)}{\delta F^d(t,T)} \int_t^{T_n} \sqrt{v^f(t,s)^2 + \sigma^2} ds. \end{aligned}$$

Taking expectation in (41) and using the conditional survival probability, we obtain the caplet price:

$$\begin{aligned} \mathbb{E}_t[C(t,T_n)] &= \delta \mathbb{P}(\tau > T_{n+1} | \tau > t)^{-1} B(t,T_{n+1}) \left[\mathbb{P}(\tau > T_{n+1} | \tau > T_n) \right. \\ &\qquad \left. F(t,T_n) \mathbb{E}_t[\mathbf{N}(d_1)] - K \mathbb{E}_t[\mathbf{N}(d_2)] \right], \end{aligned} \tag{42}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathscr{G}_t]$.

We define the approximate caplet formula which is expressed by the conditional survival probability in the cumulative normal distribution.

$$\begin{aligned} C(t,T_n)^e &:= \delta \mathbb{P}(\tau > T_{n+1} | \tau > t)^{-1} B(t,T_{n+1}) \left[\mathbb{P}(\tau > T_{n+1} | \tau > T_n) \right. \\ &\qquad \left. F(t,T_n) \mathbf{N}(d_1') - K \mathbf{N}(d_2') \right], \end{aligned} \tag{43}$$

where $B(t,T) = \exp\{-\int_t^T f(t,u) du\}$ and

$$\begin{aligned} d_1' &= \frac{\log(\mathbb{P}(\tau > T_{n+1} | \tau > T_n) F(t,T_n)/K) + 1/2 \hat{\zeta}_t^2 (T_n - t)}{\hat{\zeta}_t \sqrt{T_n - t}}, \\ d_2' &= d_1 - \hat{\zeta}_t \sqrt{T_n - t} \\ \hat{\zeta}_t &= \frac{1 + \delta \mathbb{P}(\tau > T_{n+1} | \tau > T_n) F(t,T)}{\delta \mathbb{P}(\tau > T_{n+1} | \tau > T_n) F(t,T)} \int_t^{T_n} \sqrt{v^f(t,s)^2 + \sigma^2} ds. \end{aligned}$$

Since the cumulative normal distribution function is convex or concave, the difference between $\mathbb{E}[C(t,T_n)]$ and $C(t,T_n)^e$ depends on d_1 and d_2. We investigate the functions that compose the approximate caplet price.

Firstly we give two propositions about the boundedness of the difference between $\mathbb{E}(\mathbf{N}(y))$ and $\mathbf{N}(\mathbb{E}[y])$ when y is positive or negative, respectively.

Proposition 3.2
Let $\mathbf{N}$: $\mathbb{R} \mapsto [0,1]$ *be the cumulative normal distribution function and* x *be a positive random variable such that there exists a positive constant* $\underline{c}$ *satisfying* $x \geqslant \underline{c}$. *Assume* $\underline{c} \neq 0$. *If* $\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \geqslant 0$, *then the difference* $\mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x])$ *is bounded by*

$$\int_0^x -\frac{1}{2\underline{c}}\left[\mathbf{N}(u)'' - \mathbf{N}(\mathbb{E}[u])''\right] u^2 e^{u/\underline{c}} du. \tag{44}$$

Moreover if $\mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' - \mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' \geqslant 0$, *then the difference* $\mathbf{N}(\mathbb{E}[x]) - \mathbb{E}[\mathbf{N}(x)]$ *is bounded:*

$$\int_0^x -\frac{1}{2\underline{c}}\left[\mathbf{N}(\mathbb{E}[u])'' - \mathbf{N}(u)''\right] u^2 e^{u/\underline{c}} du. \tag{45}$$

In (44) and (45), $\mathbf{N}(a)'' = \sqrt{2\pi}^{-1}(-a)\exp(-a^2/2)$.

Proof. Applying Taylar expansion to $\mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x])$ and by $x \geqslant \underline{c}$,

$$\begin{aligned}
& \mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x]) \\
= \; & \mathbb{E}[\mathbf{N}(0)] + \mathbb{E}[\mathbf{N}(x)]'x + \frac{1}{2}\left\{\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)'^2 + \mathbb{E}[\mathbf{N}(x)]''\mathbf{N}(x)''\right\} x^2 \\
& + o_1(x^3) - \mathbf{N}(\mathbb{E}[0]) - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]'x - \frac{1}{2}\left\{\mathbf{N}(\mathbb{E}[x])''\mathbb{E}[x]'^2\right. \\
& \left. + \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]''\right\} x^2 - o_2(x^3) \\
\cong \; & (\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]')x + \frac{1}{2}\left[\mathbf{N}(x)'' - \mathbf{N}(\mathbb{E}[x])''\right] x^2 \\
\geqslant \; & (\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]')\underline{c} + \frac{1}{2}\left[\mathbf{N}(x)'' - \mathbf{N}(\mathbb{E}[x])''\right] x^2, \qquad (46)
\end{aligned}$$

since $\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \geqslant 0$.

Therefore, the following inequality holds.

$$\begin{aligned}
0 \; & \leqslant \; \mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \\
& \leqslant \; \frac{1}{\underline{c}}(\mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x])) - \frac{1}{2\underline{c}}\left[\mathbf{N}(x)'' - \mathbf{N}(\mathbb{E}[x])''\right] x^2 \qquad (47)
\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \leqslant M(x, \mathbb{E}[x])e^{\frac{x}{\underline{c}}} \; < \; \infty, \tag{48}$$

where $M(x, \mathbb{E}[x]) = -\frac{1}{2\underline{c}}\left[\mathbf{N}(x)'' - \mathbf{N}(\mathbb{E}[x])''\right] x^2$.

Integrating (48) from 0 to x and using the following relation,

$$\begin{aligned}
& \int_0^x \mathbb{E}[\mathbf{N}(s)]'\mathbf{N}(s)' - \mathbf{N}(\mathbb{E}[s])'\mathbb{E}[s]' ds \\
& = \int_{-\infty}^x \mathbb{E}[\mathbf{N}(s)]'\mathbf{N}(s)' - \mathbf{N}(\mathbb{E}[s])'\mathbb{E}[s]' ds \\
& - \int_{-\infty}^0 \mathbb{E}[\mathbf{N}(s)]'\mathbf{N}(s)' - \mathbf{N}(\mathbb{E}[s])'\mathbb{E}[s]' ds \\
& = \mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x]) - (\mathbb{E}[\mathbf{N}(0)] - \mathbf{N}(\mathbb{E}[0])) \\
& = \mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x]),
\end{aligned}$$

the estimate (44) is obtained. Furthermore for $\mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' - \mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' \geqslant 0$, we can obtain (45) by the same procedure. $\square$

Remark 3.1
Let us consider (44) in proposition 3.2. By Lebesgue convergence theorem,

$$\begin{aligned}
&\lim_{\underline{c}\to\infty}\int_0^x -\frac{1}{2\underline{c}}\left[\mathbf{N}(u)'' - \mathbf{N}(\mathbb{E}[u])''\right]u^2 e^{u/\underline{c}}du \\
(49)\qquad &= \int_0^x \lim_{\underline{c}\to\infty} -\frac{1}{2\underline{c}}\left[\mathbf{N}(u)'' - \mathbf{N}(\mathbb{E}[u])''\right]u^2 e^{u/\underline{c}}du \to 0.
\end{aligned}$$

The difference between $\mathbb{E}[\mathbf{N}(x)]$ and $\mathbf{N}(\mathbb{E}[x])$ converges to 0. This is caused by the independence of the change of the value of the cumulative distribution function according to the change in x for enough large x. And this means that in terms of the caplet price, the approximate caplet price given by the conditional survival probability becomes better approximate price for the caplet (42). The same implication is also concluded with respect to (45), (50) and (51) in following proposition 3.3.

Proposition 3.3
Let $\mathbf{N}$ be same as in proposition 3.2 and x be a negative random variable such that there exists a positive constant $\underline{c}$ satisfying $x \leqslant -\underline{c}$. If $\underline{c} \neq 0$. If $\mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \geqslant 0$, then the difference $\mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x])$ is bounded:

$$\mathbb{E}[\mathbf{N}(x)] - \mathbf{N}(\mathbb{E}[x]) \leqslant \int_0^{-x} -\frac{1}{2\underline{c}}\{\mathbf{N}(\mathbb{E}[u])'' - \mathbf{N}(u)''\}u^2 e^{\frac{u}{\underline{c}}}du. \tag{50}$$

Moreover, if $\mathbf{N}(\mathbb{E}[x])' - \mathbb{E}[\mathbf{N}(x)'] \geqslant 0$, then the difference $\mathbf{N}(\mathbb{E}[x]) - \mathbb{E}[\mathbf{N}(x)]$ is bounded:

$$\mathbf{N}(\mathbb{E}[x]) - \mathbb{E}[\mathbf{N}(x)] \leqslant \int_0^{-x} -\frac{1}{2\underline{c}}\{\mathbf{N}(u)'' - \mathbf{N}(\mathbb{E}[u])''\}u^2 e^{\frac{u}{\underline{c}}}du. \tag{51}$$

Proof. We can obtain (50) and (51) by the same procedure as the proof of proposition 3.2. Since x is negative, we use the relation:

$$\begin{aligned}
\mathbf{N}(\mathbb{E}[x]) - \mathbb{E}[\mathbf{N}(x)] &= \int_{-\infty}^x \mathbf{N}(\mathbb{E}[u])'\mathbb{E}[u]' - \mathbb{E}[\mathbf{N}(u)]'\mathbf{N}(u)'du \\
&= -\int_0^s \mathbf{N}(\mathbb{E}[u])'\mathbb{E}[u]' - \mathbb{E}[\mathbf{N}(u)]'\mathbf{N}(u)'du \\
&= \int_0^s \mathbb{E}[\mathbf{N}(u)]'\mathbf{N}(u)' - \mathbf{N}(\mathbb{E}[u])'\mathbb{E}[u]'du,
\end{aligned} \tag{52}$$

where we denote $s = -x$ for $s \geqslant \underline{c}$. For $0 \leqslant \mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]'$, it is

$$\begin{aligned}
0 &\leqslant \mathbb{E}[\mathbf{N}(x)]'\mathbf{N}(x)' - \mathbf{N}(\mathbb{E}[x])'\mathbb{E}[x]' \\
&= \mathbf{N}(\mathbb{E}[s])'\mathbb{E}[s]' - \mathbb{E}[\mathbf{N}(s)]'\mathbf{N}(s)' \\
&\leqslant \frac{1}{\underline{c}}(\mathbf{N}(\mathbb{E}[s]) - \mathbb{E}[\mathbf{N}(s)]) - \frac{1}{2\underline{c}}\{\mathbf{N}(\mathbb{E}[s])'' - \mathbf{N}(s)''\}s^2
\end{aligned} \tag{53}$$

By Gronwall's inequality,

$$\mathbb{E}[\mathbf{N}(s)]'\mathbf{N}(s)' - \mathbf{N}(\mathbb{E}[s])'\mathbb{E}[s]' \leqslant \bar{M}(s, \mathbb{E}[s])e^{\frac{s}{\underline{c}}}, \tag{54}$$

where $\bar{M}(s, \mathbb{E}[s]) = -\frac{1}{2\underline{\mathrm{C}}}\{\mathbf{N}(\mathbb{E}[s])'' - \mathbf{N}(s)''\}s^2$. Integrating from 0 to $-x$, we obtain (50), i.e.:

$$\begin{aligned}
\mathbf{N}(\mathbb{E}[x]) - \mathbb{E}[\mathbf{N}(x)] &= \int_0^{-x} \mathbb{E}[\mathbf{N}(u)]'\mathbf{N}(u)' - \mathbf{N}(\mathbb{E}[u])'\mathbb{E}[u]' du \\
&\leqslant \int_0^{-x} \bar{M}(u, \mathbb{E}[u]) e^{\frac{u}{\underline{\mathrm{C}}}} du. \qquad (55)
\end{aligned}$$

(51) is obtained similarly. □

Remark 3.2
If the equations d_1 and d_2 in (41) satisfy the assumption in proposition 3.2 and 3.3, the difference between $\mathbb{E}_t[C(t, T_n)]$ and $C(t, T_n)^e$ is bounded by estimates given in these proposition. But for the case of d_1 and d_2 in $\mathbb{R}$, since the difference can not be bounded by their estimates, it is necessary to notice the value of d_1 and d_2.

We can confirm that between $\mathbb{E}_t[C(t, T_n)]$ and $\mathbb{E}[C(t, T_n)]^e$ is bounded by proposition 3.2 and 3.3 under the boundedness of d_i and d'_i. But we cannot discriminate positive and negative of the value of the difference. The following proposition gives the condition for the boundedness of d_i, $i = 1, 2$ in (41).

Proposition 3.4
Let d_i (i=1,2) be in (41) and denote d_i as $f_i(y)$ where y corresponds to F^d. Assume the nonnegative random variable y such that $\bar{c'} \geqslant y \geqslant \underline{c'} > 0$ and $v^f(\cdot, \cdot)^2 + \sigma^2, y < \infty$ and $T_n > t$ are satisfied. Then for any $y \in [\underline{c'}, \bar{c'}]$, $f_1(y)$ and $f_2(y)$ exist, respectively.

Proof. It is enough to verify that each term in $f_1(y)$ exists. Since $f_1(y)$ is given by

$$\begin{aligned}
f_1(y) &= (\log y - \log K)\left(1 - \frac{1}{1 + \delta y}\right)\left(\sqrt{T_n - t}\int_t^{T_n} \sqrt{v^f(t, s)^2 + \sigma^2 ds}\right)^{-1} \\
&\quad + \frac{1}{2}\left(\sqrt{T_n - t}\int_t^{T_n} \sqrt{v^f(t, s)^2 + \sigma^2 ds}\right)\left(\frac{1}{y} + \delta\right),
\end{aligned}$$

it is sufficient for the existence of $f_1(y)$ that $v^f(\cdot, \cdot)^2 + \sigma^2, y < \infty$ and $T_n > t$. Moreover, it is the same for the existence of $f_2(y)$. Thus, this proposition is proved. □

Remark 3.3
The defaultable forward rate F^d is a random variable that depends on the stochastic hazard rate. Therefore if the conditional survival probability is 1, F^d is a constant $F > 0$. And if the conditional survival probability is 0, F^d is 0. Thus F^d is in [0, F]. In proposition 3.4, we assumed $y \in [\underline{c'}, \bar{c'}]$. This assumption is fulfilled except for $F^d = 0$. When $F^d = 0$, $f_1(F^d) = \infty$ and $f_2(F^d) = -\infty$. As mentioned in Remark 3.1, when F is enough small, i.e. $\underline{c}$ in (49) is large, the difference between $\mathbb{E}_t[C(t, T_{n+1})]$ and $C(t, T_{n+1})^e$ vanishes.

4 Defaultable Forward Rate Model with Jump Risk In subsection 3.2, with the survival probability we have obtained the approximate caplet price. The option price is calculated by the probability of default $F^d(t, T)$ which is the function of the hazard rate that the component of default in Schönbucher [20] is represented. In this section, using the survival probability, we drive the approximate caplet price with both jump and default risk as following the procedure in subsection 3.2.

Firstly assume that the point process of the jump risk is a compound Poisson process to drive the explicit formula. In [16], a Poisson process is assumed to follow the jump process. Moreover in [12], to obtain the closed form solution, they assume the jump process as a compound Poisson process [7].

Assumption 4.1
Let the dynamics of the instantaneous forward rate be (8). And assume the point process of the jump risk in (8) to be a compound Poisson process.

Moreover, we give the arbitrage-free condition under the assumption that the jump risk follows a marked point process and the default risk is presented in proposition 3.1. The proof of the following proposition 4.1 is omitted because it can be obtained as well as proposition 3.1, Appendix A.1 and A.2.

Proposition 4.1
Assume the forward rate dynamics is given in assumption 2.3 with the Vasicek-type hazard rate process for the default risk instead of the point process. Then the arbitrage-free condition is described as

$$\begin{aligned} 0 &= -\int_t^T (b^f(t,s) - c(m-\lambda(t,s)))ds + \frac{1}{2}A^2 \\ &\quad -\theta_t A - \frac{\partial}{\partial s}\int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\Phi(t,x)\lambda(dx,t), \end{aligned} \tag{56}$$

where $A = \int_t^T \sqrt{v^f(t,s)^2 + \sigma^2}ds$.

By assumption 4.1, the forward rate dynamics $dF^{j,d}(t,T)/F^{j,d}(t,T)$ is under $M_{T_{n+1}}$-measure

$$\frac{dF^{j,d}(t,T)}{F^{j,d}(t,T)} = \zeta_t dW_t^{M_{T_{n+1}}} + \sum_{i=1}^{N_t}(Y_i - 1) - \lambda_n^j m_n dt, \tag{57}$$

where $\zeta_t dW_t^{M_{T_{n+1}}} = \xi_t dt + \zeta_t dW'_t$ and ξ_t, ζ_t are given by (36) and (37). $(Y-1)$ is an impulse function with mean m_n and s_n^2 denotes the variance of $\log(Y_n)$. Moreover, N_t is a Poisson process with the intensity λ^j [8].

As following [12], the option price is given by

$$\begin{aligned} &C(t,T_n) \\ &= \delta B^{j,d}(t,T_{n+1})\mathbb{E}_P\left[\left.\frac{\exp\left\{-\int_{T_n}^{T_{n+1}} r_s ds\right\} B^{j,d}(T_{n+1},T_{n+1})}{B^{j,d}(T_n,T_{n+1})} C(T_n,T_{n+1})\right| \mathscr{G}_t\right] \\ &= \delta B^{j,d}(t,T_{n+1})\mathbb{E}_{T'_{n+1}}\left[(F^{d,j}(T_n,T_{n+1}) - K)^+ \middle| \mathscr{G}_t\right] \\ &= \delta\sum_{i=0}^{\infty} e^{-\lambda_n^j(T_n-t)}\frac{(\lambda_n^j(T_n-t))^i}{i!}B^{j,d}(t,T_{n+1})\left[F^{d,f}(t,T_n)\right. \\ &\quad \left. e^{-\lambda_n^j(T_n-t)}(1+m_n)^i\mathbf{N}(d_1) - K\mathbf{N}(d_2)\right], \end{aligned} \tag{58}$$

---

[7] They give the forward rate process with the jump process to be the marked point process (Theorem 3.1 in [12]).

[8] See Glasserman and Kou [12]

where

$$\begin{aligned} d_1 &= \frac{\log(\frac{F(t,T_n)^d e^{-\lambda_n^j(T_n-t)}(1+m_n)^i}{K} + 1/2(\zeta_t^2 + is_n^2)}{\sqrt{\zeta_t^2 + is_n^2}}, \\ d_2 &= d_1 - \sqrt{\zeta_t^2 + is_n^2}, \\ \zeta_t &= \frac{1+\delta F^{j,d}(t,T_n)}{\delta F^{j,d}(t,T_n)} \int_t^{T_n} \sqrt{v^f(t,s)^2 + \sigma^2} ds. \end{aligned}$$

Furthermore, taking expectation for the conditional survival probablity, we obtain the caplet price:

$$\begin{aligned} &\mathbb{E}_t[C(t,T_n)] \\ &= \delta \sum_{i=0}^{\infty} e^{-\lambda_n^j(T_n-t)} \frac{(\lambda_n^j(T_n-t))^i}{i!} \mathbb{P}(\tau > T_{n+1}|\tau > t) B^j(t,T_{n+1}) \\ &\quad \left[\mathbb{P}(\tau > T_{n+1}|\tau > T_n) F^j(t,T_n) e^{-\lambda_n^j(T_n-t)}(1+m_n)^i \mathbb{E}_t[\mathbf{N}(d_1)] - K\mathbb{E}_t[\mathbf{N}(d_2)]\right]. \end{aligned} \tag{59}$$

And the approximate caplet price is defined as follows:

$$\begin{aligned} &C(t,T_n)^e \\ &:= \delta \sum_{i=0}^{\infty} e^{-\lambda_n^j(T_n-t)} \frac{(\lambda_n^j(T_n-t))^i}{i!} \mathbb{P}(\tau > T_{n+1}|\tau > t)^{-1} B^j(t,T_{n+1}) \\ &\quad \left[\mathbb{P}(\tau > T_{n+1}|\tau > T_n) F^j(t,T_n) e^{-\lambda_n^j(T_n-t)}(1+m_n)^i \mathbf{N}(d_1') - K\mathbf{N}(d_2')\right], \end{aligned} \tag{60}$$

where

$$\begin{aligned} d_1' &= \frac{\log(\mathbb{P}(\tau > T_{n+1}|\tau > T_n)\frac{F^j(t,T_n)e^{-\lambda_n^j(T_n-t)}(1+m_n)^i}{K}) + 1/2(\zeta_t^2 + is_n^2)}{\sqrt{\zeta_t^2 + is_n^2}}, \\ d_2' &= d_1 - \sqrt{\zeta_t^2 + is_n^2}, \\ \zeta_t &= \frac{1+\delta \mathbb{P}(\tau > T_{n+1}|\tau > T) F^j(t,T_n)}{\delta \mathbb{P}(\tau > T_{n+1}|\tau > T) F^j(t,T_n)} \int_t^{T_n} \sqrt{v^f(t,s)^2 + \sigma^2} ds \end{aligned}$$

Because of the same reason in section 3.2, the difference between $\mathbb{E}[C(t,T_n)]$ and $C(t,T_n)^e$ is bounded but the sign of the difference can not be determined.

5 Numerical example In this section, we calculate the implied volatility numerically by the approximate caplet formula given in the previous section and argue the effect of the hazard rate through this implied volatility.

We use a set of parameters for the conditional hazard rate in Corollary 3.1: $h(0,0) = 0.01$, $t = 0$, $T_n = 0.5, \cdots, 1.5$, $c = 0.2$, $m = 0$, $\sigma = 0.0025$. The parameters for the caplet formula (60) are $\delta = 0.5$, $\lambda = 1$, $m_n = \{-0.3, 0, 0.3\}$, $s_n = 0.45$, $\zeta_t^2 = 0.05^2(T_n - t)$, $F(t,T_n) = 0.06$, $K = 0.03, \cdots, 0.09$. Figure 1 shows the survival probability and the conditional survival probability with this parameter set.

Under these parameters, we can obtain the implied volatility surface with the default risk shown in Figure 2: The right side of upper row shows the case $m_n = -0.3$, the left

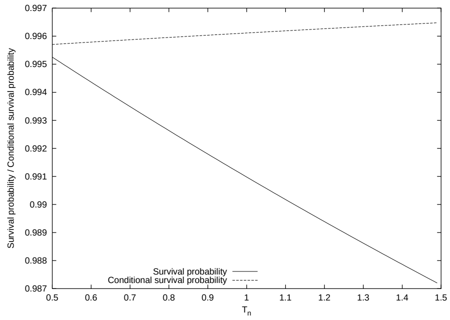

Figure 1: Survival probability and Conditional survival probability.

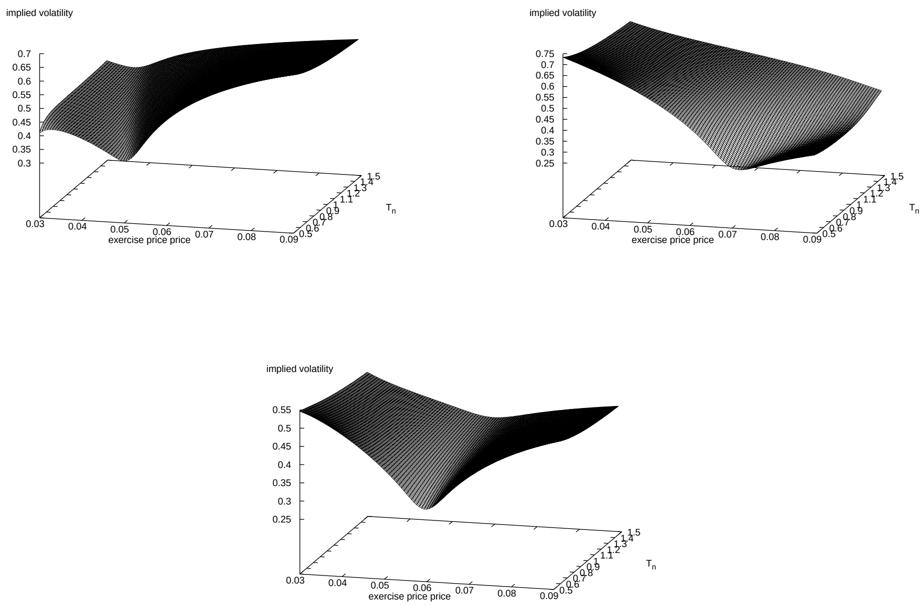

Figure 2: Implied volatility with hazard rate.

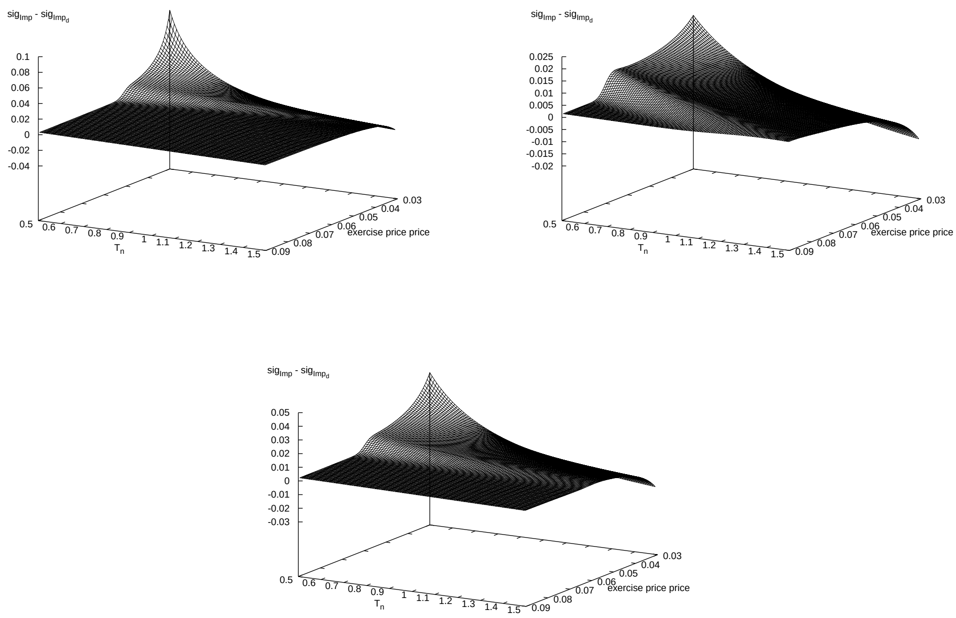

Figure 3: Difference between volatilities.

side $m_n = 0.3$ and the lower $m_n = 0$. The implied volatility surface with and without the default risk respectively are given by

$$\begin{aligned} \hat{\sigma}_{Imp_d} &:= \arg\min_\sigma (C(t, T_n)^e - BS(\sigma)) \\ \hat{\sigma}_{Imp} &:= \arg\min_\sigma (C(t, T_n) - BS(\sigma)), \end{aligned}$$

where $C(t, T_n)^e$ is represented by (60), $C(t, T_n)$ is the caplet price with jump risk given by [12] and $BS(\sigma)$ is Black-Scholes formula of call option with same parameters except for the volatility σ. The figure of the lower in Figure 2 gives the result with $m_n = 0$, which can illustrate 'smile'. When $m_n = 0.3$, at each maturity date, the implied volatility has the up slope shape or the exercise price in which the implied volatility that takes minimum value is taken lowers(The left side of upper row in Figure 2). Oppositely, when $m_n = -0.3$, the down slope shape or the exercise price where the minimum value of implied volatility is high (The right side of upper row in Figure 2). This is the same behavior as no default case, that is, jump-diffusion model with default risk has the flexibility to illustrate skew or smile.

To see the influence of the survival probability above, it is necessary to obtain the difference between implied volatilities: with default risk and without one. The difference between the implied volatility surfaces is shown in Figure 3: The right side of upper row is the case $m_n = -0.3$, the left side is $m_n = 0.3$ and the lower is $m_n = 0$.

Figure 3 provides several implications. Firstly, when T_n is short, it occurs that as the exercise price rises, the difference between the implied volatilities: $\hat{\sigma}_{Imp_d}$ and $\hat{\sigma}_{Imp_d}$ is decreasing. That is, the implied volatility $\hat{\sigma}_{Imp_d}$ is comparatively lower than $\hat{\sigma}_{Imp}$ where the exercise price is low (see Left: Figure 4). Therefore, including the default risk, the

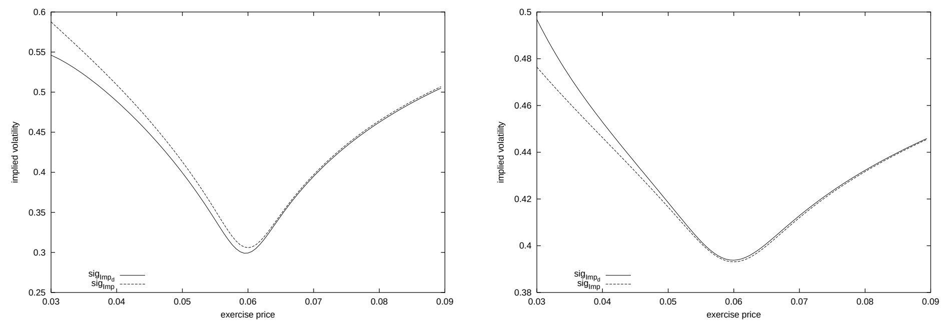

Figure 4: $\hat{\sigma}_{Imp_d}$ and $\hat{\sigma}_{Imp}$ for $m = 0, T_n = 0.5$ (left), 1.5 (right)

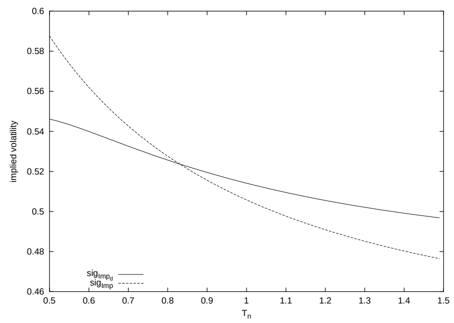

Figure 5: $\hat{\sigma}_{Imp_d}$ and $\hat{\sigma}_{Imp}$ for $m = 0, K = 0.03$

implied volatility smile (skew) becomes gentle.

However when T_n is enough large, oppositely as the exercise price rises, the implied volatility $\hat{\sigma}_{Imp_d}$ is close to $\hat{\sigma}_{Imp}$ with decreasing (see Right: Figure 4). Thus, the effect of the default risk to $\hat{\sigma}_{Imp_d}$ makes the sharp of the implied volatility whether steep or gentle with depending on the T_n.

Secondly, when m_n is low, the default risk has not good effect. It means that when $m_n = -0.3$, the influence of the default risk tends not to be received: in this case, the range of the difference between implied volatilities is in $[-0.01796, 0.020771]$ (for $m = 0.3$, $[-0.2195, 0.093425]$ and for $m = 0$, $[-0.0204, 0.041512]$). Lastly, for the low exercise price as T_n grows, $\hat{\sigma}_{Imp_d}$ becomes larger than $\hat{\sigma}_{Imp}$ (see Figure 5).

6 Conclusion We considered the forward rate model with both default and jump risks. The model with the default was showed in [20] and one with the jump risk in [12], respectively. Our constructed model was based on their models. Moreover so as to derive the survival probability of the forward rate, it is the Vasicek-type hazard rate model of [17]. That is, it is assumed that the hazard rate follows the Vasicek model.

The smile or skew of the implied volatility is well known in the market. Therefore the jump model is used to express the smile or skew. We investigated the effect of the default risk of the underling asset through the implied volatility. The results by the sensitivity analysis show that the default risk reduces or amplifies the implied volatility depending on the time interval. Moreover, as the exercise price increases, the implied volatility with the default risk becomes the one without the default risk.

The model in this paper is based on the independence of the random variables of forward rates and hazard rate. By dint of the independence, the approximate closed form of the Caplet was obtained simply. As for the model with jump risks, moreover, in [13] they simulated the LIBOR market model with the Jump risk for the case of path dependence. Their technique of discretization scheme is based on [19]. For a somewhat more general case, their simulation technique can be applied to our model.

A Appendix

A.1 Proofs of Proposition 2.2 and 2.3 We prove proposition 2.3. Assuming that there are not jump and default risks under assumption 2.3 i.e. $q^{f,j}, q^{d,f} = 0$, it yields proposition 2.1 from proposition 2.3. Moreover under the existence of only the jump risk, it is proved in [4].

Let the dynamics of the bond price and the forward rate be given by assumption 2.3. Then the relation between the forward rate and the instantaneous forward rate can be expressed as the equation (11). And as (11) is re-written with substituting (6),

$$\begin{aligned}
&F^{j,d}(t,T)\\
&\quad= \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta} f^{j,d}(t,s)ds\right\} - \exp\left\{\int_T^{T+\delta} \lambda_s^f ds\right\}\right)\\
&\quad= \frac{1}{\delta}\left(\exp\left\{\int_T^{T+\delta}\left[f^{j,d}(0,s) + \int_0^t b^f(u,s)du + \int_0^t v^f(u,s)dW_u\right.\right.\right.\\
&\quad\left.\left. + \int_0^t\int_E q^{f,j}(u,x,s)\mu(du,dx) + \int_0^t\int_{E'} q^{f,d}(u,x,s)\mu'(du,dx)\right] ds\right\}\\
&\quad\left. - \exp\left\{\int_t^{T+\delta} \lambda_s^f ds\right\}\right)
\end{aligned} \tag{61}$$

In the parenthesis $\{\cdot\}$ let us consider $\int_t^{T+\delta} f^{j,d}(t,s)ds$ [9]. It can be manipulated by Fubini's theorem as follows:

$$\begin{aligned}
&\int_t^{T+\delta} f^{j,d}(t,s)ds\\
&\quad= \int_t^{T+\delta} f^{j,d}(0,s)ds + \int_0^t\int_t^{T+\delta} b^f(u,s)dsdu\\
&\quad+ \int_0^t\int_t^{T+\delta} v^f(u,s)dsdW_u + \int_0^t\int_t^{T+\delta}\int_E q^{f,j}(u,x,s)ds\mu(du,dx)\\
&\quad+ \int_0^t\int_t^{T+\delta}\int_{E'} q^{f,d}(u,x,s)ds\mu'(du,dx).
\end{aligned}$$

Furthermore, extending the interval $[s,T]$ to two parts, i.e. $[u,s]$ and $[u,T]$, changing

[9]It has the relationship: $B^{j,d}(t,T+\delta) = \exp\left\{-\int_t^{T+\delta} f^{j,d}(t,s)ds\right\}$.

the order of integration and by $f^{j,d}(s,s) = r_s$,

$$\begin{aligned}
&\int_t^T f^{j,d}(t,s)ds \\
&= \int_0^T f^{j,d}(0,s)ds - \int_0^t f^{j,d}(0,s)ds + \int_0^t \int_u^T b^f(u,s)dsdu \\
&- \int_0^t \int_u^s b^f(u,s)dsdu + \int_0^t \int_u^T v^f(u,s)dsdW_u \\
&- \int_0^t \int_u^s v^f(u,s)dsdW_u + \int_0^t \int_u^T \int_E q^{f,j}(u,x,s)ds\mu(du,dx) \\
&- \int_0^t \int_u^s \int_E q^{f,j}(u,x,s)ds\mu(du,dx) + \int_0^t \int_u^T \int_{E'} q^{f,d}(u,x,s)ds\mu'(du,dx) \\
&- \int_0^t \int_u^s \int_{E'} q^{f,d}(u,x,s)ds\mu'(du,dx) \\
&= \int_0^T f^{j,d}(0,s)ds - \int_0^t r_s ds + \int_0^t \int_u^T b^f(u,s)dsdu \\
&+ \int_0^t \int_u^T v^f(u,s)dsdW_u + \int_0^t \int_u^T \int_E q^{f,j}(u,x,s)ds\mu(du,dx) \\
&+ \int_0^t \int_u^T \int_{E'} q^{f,d}(u,x,s)ds\mu'(du,dx).
\end{aligned}$$

Therefore $d\int_t^T f^{j,d}(t,s)ds$ with $u = t$ is yielded as follows:

$$\begin{aligned}
&d\int_t^T f^{j,d}(t,s)ds \\
&= \left\{ -r_t + \int_t^T b^f(t,s)ds \right\} dt + \int_t^T v^f(t,s)dsdW_t \\
&+ \int_t^T \int_E q^{f,j}(t,x,s)ds\mu(dt,dx) + \int_t^T \int_{E'} q^{f,d}(t,x,s)ds\mu'(dt,dx)
\end{aligned} \tag{62}$$

Thus, applying itô formula to $B^{j,d}(t,T) = \exp\left\{ -\int_t^T f^{j,d}(t,s)ds \right\}$ [10],

$$\begin{aligned}
\frac{dB^{j,d}(t,T)}{B^{j,d}(t-,T)} &= -d\left\{ \int_t^T f^{j,d}(t,s)ds \right\} + \frac{1}{2}d\left\{ \int_t^T f^{j,d}(t,s)ds \right\}^2 \\
&+ \int_E \left[\exp\left\{ -\int_t^T q^{f,j}(t,x,s)ds \right\} - 1 \right] \mu(dt,dx) \\
&+ \int_{E'} \left[\exp\left\{ -\int_t^T q^{f,d}(t,x,s)ds \right\} - 1 \right] \mu'(dt,dx)
\end{aligned}$$

[10]cf. Elliott [11], to apply itô formula with the jump process.

Assume assumption 2.4 and using (41), then

$$\begin{aligned}
&\frac{dB^{j,d}(t,T)}{B^{j,d}(t-,T)} \\
&= \left\{ r_t - \int_t^T b^f(t,s)ds + \frac{1}{2}\left(\int_t^T v^f(t,s)ds\right)^2 \right\} dt - \int_t^T v^f(t,s)dsdW_t \\
&+ \int_E \left[\exp\left\{ - \int_t^T q^{f,j}(t,x,s)ds \right\} - 1 \right] \mu(du,dx) \\
&+ \int_{E'} \left[\exp\left\{ - \int_t^T q^{f,d}(t,x,s)ds \right\} - 1 \right] \mu'(du,dx).
\end{aligned}$$

The proof of proposition 2.3 is complete. Moreover if there are not the jump and default risks, then proposition 2.2 is obtained. □

Remark A.1
This proof is based on the procedure in Björk, Kabanov and Runggaldier [4]. When there is no default risk, proposition 2.2 in [4] is established.

A.2 Proof of Theorem 2.2 Following to theorem 3.13 in [4], to prove theorem 2.2 we construct the equivalence of drift terms between the stochastic process of the bond price under Q-martingale measure and one of forward rate with same measure.

First, let the bond price and forward rate dynamics be as assumption 2.3. Then the bond price dynamics with Q-measure is

$$dB(t,T) = r_t B(t,T)dt + dM^Q. \tag{63}$$

Substituting $dW_t = \theta_t dt + dW'_t$, $\lambda_Q(t,dx) = \Phi(t,x)\lambda(t,dx)$ and $\lambda'_Q(t,dx) = \Phi'(t,x)\lambda'(t,dx)$ into (21) of proposition 2.3, we have

$$\begin{aligned}
&\frac{dB(t,T)}{B(t-,T)} \\
&= \left\{ r_t - \int_t^T b^f(t,s)ds + \frac{1}{2}\left(\int_t^T v^f(t,s)ds\right)^2 - \theta_t \int_t^T v^f(t,s)ds \right. \\
&+ \int_E \left[\exp\left\{ - \int_t^T q^{f,j}(t,x,s)ds \right\} - 1 \right] \Phi(t,x)\lambda(t,dx) \\
&\left. + \int_{E'} \left[\exp\left\{ - \int_t^T q^{f,d}(t,x,s)ds \right\} - 1 \right] \Phi'(t,x)\lambda'(t,dx) \right\} dt \\
&-\theta_t \int_t^T v^f(t,s)dsdW'_t + \int_E \left[\exp\left\{ - \int_t^T q^{f,j}(t,x,s)ds \right\} - 1 \right] \\
&(\mu(dt,dx) - v(dt,dx)) + \int_{E'} \left[\exp\left\{ - \int_t^T q^{f,d}(t,x,s)ds \right\} - 1 \right] \\
&(\mu'(dt,dx) - v'(dt,dx)).
\end{aligned} \tag{64}$$

If (63) and (64) are equivalent, it must be established:

$$
\begin{aligned}
0 &= -\int_t^T b^f(t,s)ds + \frac{1}{2}\left(\int_t^T v^f(t,s)ds\right)^2 - \theta_t \int_t^T v^f(t,s)ds \\
&\quad + \int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\Phi(t,x)\lambda(t,dx) \\
&\quad + \int_{E'} \left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\} - 1\right]\Phi'(t,x)\lambda'(t,dx), \qquad (65)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
0 &= -b^f(t,s) + v^f(t,s)\left(\int_t^T v^f(t,s)ds\right) - \theta_t v^f(t,s) \\
&\quad + \frac{\partial}{\partial s}\int_E \left[\exp\left\{-\int_t^T q^{f,j}(t,x,s)ds\right\} - 1\right]\Phi(t,x)\lambda(t,dx) \\
&\quad + \frac{\partial}{\partial s}\int_{E'} \left[\exp\left\{-\int_t^T q^{f,d}(t,x,s)ds\right\} - 1\right]\Phi'(t,x)\lambda'(t,dx).
\end{aligned}
$$

□

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