UPPER BOUNDS FOR THE DIFFERENCE BETWEEN SYMMETRIC OPERATOR MEANS

MASATOSHI FUJII*, SAICHI IZUMINO** AND RITSUO NAKAMOTO***

Received October 25, 2005

Dedicated to Professor Sin-Ei Takahasi on his sixtieth birthday

Abstract. Upper bounds for the difference between symmetric operator means, particularly, related to the arithmetic and the harmonic means, are discussed. The differenc for commuting operators is also studied.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is said to be positive, and denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. For two positive operators A and B the arithmetic mean $A\nabla B$ and the harmonic mean $A!B$ are defined by

$$A\nabla B = \frac{A+B}{2} \quad \text{and} \quad A!B = 2(A:B)$$

respectively. Here $A : B = (A^{-1} + B^{-1})^{-1}$ is the parallel sum of A and B if both A and B are invertible. (Without invertibility assumption $A : B$ is defined by the limit of $(A+\epsilon I) : (B+\epsilon I)$ as $\epsilon(> 0) \downarrow 0$, where I expresses the identity operator on H.) Concerning the above two operator means the following inequality

$$A\nabla B \geq A!B \tag{1.1}$$

is well-known as the arithmetic-harmonic mean inequality. Related to its reverse type inequality, recently, J. I. Fujii et al. [2], by using Mond-Pečarić method [7], have established the following theorem, which yields *a noncommutative Kantorovich type inequality* (cf. [3]), or both the ratio and the difference type reverse inequalities of (1.1), simultaneously:

Theorem 1.1. ([2, Theorem 6].) *Let A and B be positive operators such that $mI \leq A, B \leq MI$ for some constants $0 < m < M$. Then for any $\alpha > 0$*

$$A + B \leq \alpha(A : B) + \beta(m, M, \alpha), \tag{1.2}$$

where

$$\beta(m, M, \alpha) = \begin{cases} 2(m+M) - 2\sqrt{\alpha m M} & \text{if } m \leq \frac{\sqrt{\alpha m M}}{2}, \\ (2 - \frac{\alpha}{2})M & \text{if } \frac{\sqrt{\alpha m M}}{2} < m, \\ (2 - \frac{\alpha}{2})m & \text{if } M > \frac{\sqrt{\alpha m M}}{2}. \end{cases}$$

Putting $\alpha = \dfrac{(M+m)^2}{Mm}$ in (1.2), we obtain a ratio type reverse inequality

$$A\nabla B \leq \frac{(M+m)^2}{4Mm} A!B, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 47A63, 47A64.

Key words and phrases. operator mean, arithmetic-harmonic mean inequality and the reverse inquality, Kantorovich inequality.

and similarly, putting $\alpha = 4$, *we obtain a difference type reverse inequality*

$$A\nabla B - A!B \le (\sqrt{M} - \sqrt{m})^2 I \tag{1.4}$$

Now from the stand-point of commutativity and noncommutativity, for a simple observation, if we replace, in the above (1.3) and (1.4), A and B by mI and MI, respectively, or equivalently, both of them by corresponding positive numbers, then

$$m\nabla M = \frac{(m+M)^2}{4mM} m!M \text{ and } m\nabla M - m!M = \frac{(M-m)^2}{2(m+M)} \le (\sqrt{M} - \sqrt{m})^2$$

hold, so that the equality holds in (1.3) but *not in* (1.4) *generally.* The upper bound $(\sqrt{M} - \sqrt{m})^2$ in (1.4) seems too large. This suggests to study more about the reverse inequality of difference type in details, which is the aim of this paper.

2. Diffrence between the arithmetic and the harmonic means

For the difference between the arithmetic and the harmonic means of positive operators, we have the following fact, which is essentially obtaind in [1].

Theorem 2.1. (cf. [1].) *For positive invertible operators* A *and* B

$$A\nabla B - A!B = \frac{1}{2}(A-B)(A+B)^{-1}(A-B). \tag{2.1}$$

Proof . By a direct computation, or using the facts $A + B = (A + B)C(A + B)$ and $A : B = ACB = BCA$ for $C = (A + B)^{-1}$, we easily obtain (2.1).

¿From the above theorem we at once obtain the following

Corollary 2.2. *If* $mI \le A, B \le MI$ *for some constants* $0 < m \le M$, *then* $A\nabla B \ge A!B$ *and*

$$0 \le A\nabla B - A!B \le \frac{(M-m)^2}{4m} I. \tag{2.2}$$

Later in Section 3, we state Theorem 3.1 cited from [2], which extends the inequality (2.2).

Remark 2.3. Concerning the differece between the arithmetic and the harmonic means, T. Furuta [4], very recently, has presented a very simple identity: Let $A_1, ..., A_n$ be positive invertible operators and let $\lambda_1, ..., \lambda_n$ be positive numbers with $\sum_{k=1}^n \lambda_k = 1$. Then

$$A - H = \sum_{k=1}^{n} \lambda_k (I - HA_k^{-1}) A_k (I - HA_k^{-1})^* (\ge 0),$$

where $A = \sum_{k=1}^n \lambda_k A_k$ and $H = (\sum_{k=1}^n \lambda_k A_k^{-1})^{-1}$.

Employing the above identity T. Furuta gave an extremely short proof of the extended arithmetic-harmonic mean inequality and its ratio type reverse inequality.

Comparing the upper bounds in (1.4) with that in (2.2), we see that the former is better than the latter, because $(\sqrt{M} - \sqrt{m})^2 \le \dfrac{(M-m)^2}{4m}$. Concerning this fact we notice the following

Example 2.4. The upper bound $(\sqrt{M}-\sqrt{m})^2$ in (1.4) is really the best possible. For a concrete example, let A and B be 2×2 matrices as follows:

$$A=\begin{bmatrix}4 & 0\\ 0 & 1\end{bmatrix} \quad \text{and} \quad B=\frac{1}{3}\begin{bmatrix}4 & 2\sqrt{2}\\ 2\sqrt{2} & 11\end{bmatrix}. \tag{2.3}$$

Then the equality in (1.4) holds.

In fact, both A and B have eigenvalues 1 and 4, so that we can put $m=1$ and $M=4$. By an elementary computation we have

$$A\nabla B=\frac{1}{3}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix}, \quad A!B=\frac{2}{9}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix} \tag{2.4}$$

and

$$D:=A\nabla B-A!B=\frac{1}{9}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix}.$$

The eigenvalues of D are then 1 and 2/3, so that the least upper bound of D, that is, the infimum of $\lambda>0$ such that $D\le \lambda I$ is $1=(\sqrt{4}-\sqrt{1})^2=(\sqrt{M}-\sqrt{m})^2$. This implies that the constant $(\sqrt{M}-\sqrt{m})^2$ is the best possible.

Now for a more minute observation, let

$$A=\begin{bmatrix}M & 0\\ 0 & m\end{bmatrix}, \quad U=\begin{bmatrix}c & -s\\ s & c\end{bmatrix} \quad \text{and} \quad B=UAU^*.$$

Here c and s are positive numbers such that $c^2+s^2=1$. *We have to determine* c *(and* s*) such that the equality holds in* (1.4). Since U is unitary, we see that $mI\le A,B\le MI$. By an elementary computation we have

$$D:=A\nabla B-A!B=\frac{1}{2}(A-B)(A+B)^{-1}(A-B)=kE,$$

where

$$k=\frac{(1-c^2)(M-m)^2}{2\{(M+m)^2-c^2(M-m)^2\}}$$

and

$$E=\begin{bmatrix}(1+c^2)M+s^2m & sc(M-m)\\ sc(M-m) & s^2M+(1+c^2)m\end{bmatrix}.$$

Calculating the eigenvalues λ of E, we have $\lambda=M+m\pm c(M-m)$, so that the eigenvalues of D are $k\{M+m\pm c(M-m)\}$. Hence in order to *realize the equality sign in* (1.4) we have to find the value c (>0) such that the identity

$$k\{M+m+c(M-m)\}=(\sqrt{M}-\sqrt{m})^2$$

holds. Then we obtain $c=\dfrac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}\left(\text{and } s=\dfrac{2\sqrt[4]{Mm}}{\sqrt{M}+\sqrt{m}}\right)$, which is the desired.

As stated in the above example, the constant $(\sqrt{M}-\sqrt{m})^2$ is the best possible for noncommutative operators, however, it seems too large stated before, whenever A and B are real numbers. Under the circumstances we have the following

Theorem 2.5. *Let* A *and* B *be positive operators such that* $mI\le A,B\le MI$ *for some constants* $0<m\le M$. *If we assume that* A *and* B *commute, then*

$$A\nabla B-A!B\le d_0(m,M)I, \tag{2.5}$$

where $d_0(m,M) = \dfrac{(M-m)^2}{2(M+m)}$. *The constant* $d_0(m,M)$ *is the best possible.*

Proof . Since A and B commute, we have to cosider the function

$$\phi(x,y) = x\nabla y - x!y = \frac{(x-y)^2}{2(x+y)}, \tag{2.6}$$

replaced A and B with real variables x and y respectively, in the left-hand side of (2.5). Then it suffices to see that the maximum of $\phi(x,y)$ $(m \le x,\ y \le M)$ is $d_0(m,M)$. First fix $y = a,\ m \le a \le M$. Then $\phi_a(x) = \dfrac{(x-a)^2}{2(x+a)}$ is decreasing on $[m,a]$ and increasing on $[a,M]$ because $\phi_a'(x) = \dfrac{(x-a)(x+3a)}{2(x+a)^2}$. Hence we have

$$\max_{m\le x\le M} \phi_a(x) = \max\left\{\frac{(m-a)^2}{2(m+a)}, \frac{(M-a)^2}{2(M+a)}\right\}.$$

By the same reason as above, $\psi(a) = \dfrac{(m-a)^2}{2(m+a)}$ (resp. $\dfrac{(M-a)^2}{2(M+a)}$) takes the maximum at $a = M$ (resp. $a = m$). This implies that the maximum of $\phi(x,y)$ is $\dfrac{(M-m)^2}{2(M+m)} = d_0(m,M)$.

Later we shall show a theorem (Theorem 3.2) as an extension of Theorem 2.5.

Considering the upper bound of the ratio type reverse inequality (1.3), instead of that of the difference type reverse inequality, with the same assumption that A and B commute, we still obtain the same upper bound $\frac{(M+m)^2}{4Mm}$ (by putting $A = mI$ and $B = MI$) as before in the case without commutativity assumption.

3. Diffrence between the arithmetic (harmonic) mean and a symmetric mean

By the Kubo-Ando theory [6] a map $(A,B) \mapsto A\sigma B$ in the cone of positive operators on H is called an operator mean if the following conditions are satisfied:

monotonicity: $A \le C$ and $B \le D$ imply $A\sigma B \le C\sigma D$,
upper semicontinuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$,
transformer inequality: $T^*(A\sigma B)T \le T^*AT\sigma T^*BT$ for every operator T.
normalized condition: $A\sigma A = A$.

A key fact in the theory is that there is a one-to-one correspondance between an operator mean σ and a nonnegtive monotone function $f = f_\sigma$ defined on $[0,\infty)$ through the formula

$$A\sigma B = A^{1/2}(1\sigma A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

for all positive invertible operators A and B. Replaced A and B by I and tI respectively, it follows that $f_\sigma(tI) = I\sigma(tI)$ or $f_\sigma(t) = 1\sigma t$. The function f_σ is called the representing function for σ, and it is known as a concave function. For an operator mean σ, its transpose σ° is defined by $A\sigma^\circ B = B\sigma A$ for all positive (invertible) operators A and B. This is an operator mean again, and if $\sigma = \sigma^\circ$ then σ is said to be symmetric. The adjoint σ^* of σ is similarly defined by $A\sigma^*B = A\sigma^*B = (A^{-1}\sigma B^{-1})^{-1}$. The representing function f_σ for a symmetric operator mean σ is characterized as one satisfying $f_\sigma(t) = tf_\sigma(t^{-1})$. The representing function corresponding to the adjoint σ^* of σ is given as $f_{\sigma^*}(t) = f_\sigma(t^{-1})^{-1}$.

It is not difficult to see that both the arithmetic mean ∇ and the harmonic mean ! are symmetric, and that $\nabla^* = !$ and $!^* = \nabla$.

We here have to state the geometric mean $A\sharp B$, which is defined by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for all positive invertible operators A and B. The geometric mean is a symmetric and $\sharp^* = \sharp$. With respect to the arithmetic, the geometric and the harmonic means the following inequalities

$$A\nabla B \geq A\sharp B \geq A!B \tag{3.1}$$

are well-known as well as the inequality (1.1). From the general theory of operator means, as an extension of (3.1), the following inequalities are known:

$$A\nabla B \geq A\sigma B \geq A!B \tag{3.2}$$

for any symmetric operator meam σ. Correspondingly, among their representing functions the following inequalities hold:

$$\frac{1+t}{2} = f_\nabla(t) \geq f_\sigma(t) \geq f_!(t) = \frac{2t}{1+t}. \tag{3.3}$$

As an extension of the noncommutative Kantorovich type inequality, or estimates of the ratio and the difference between the arithmetic (resp. the harmonic) mean and a symmetric mean (resp. its adjoint), the following result has been shown in [2].

Theorem 3.1. ([2, Theorem 11].) *Let σ be a symmetric operator mean with the representing function $f = f_\sigma$. If A and B are positive operators such that $mI \leq A, B \leq MI$ for some constants $0 < m \leq M$, then*

$$A\nabla B \leq \frac{m\nabla M}{m\sigma M} A\sigma B. \tag{3.4}$$

$$A\sigma^* B \leq \frac{m\sigma^* M}{m!M} A!B. \tag{3.5}$$

$$A\nabla B - A\sigma B \leq \frac{M}{m\sigma M} \cdot (m\nabla M - m\sigma M)I. \tag{3.6}$$

and

$$A\sigma^* B - A!B \leq \frac{M}{m\sigma^* M} \cdot (m\sigma^* M - m!M)I \tag{3.7}$$

Putting $\sigma = !$ in (3.6) (or $\sigma^* = \nabla$ in (3.7)), we obtain (2.2) of Corollary 2.2 again.

Now for the ratio type reverse inequalities (3.4) and (3.5), it is easy to see that the upper bound of each inequality is the best possible because the equality holds if we put $A = mI$ and $B = MI$. Clearly each of the constants is the best possible even if the operators are restricted to those that are commuting.

Unlike to the ratio type reverse inequalities, for the difference type reverse inequalities (3.6) and (3.7) the best upper bounds seem different from those which are obtained under the restriction that the operators are commuting. First corresponding to (3.6) we have the following fact, which is a generalization of Theorem 2.5.

Theorem 3.2. *Let A and B be positive operators satisfying $mI \leq A, B \leq MI$ for some constants $0 < m < M$. Assume that A and B commute. Then for any symmetric operator mean σ*

$$A\nabla B - A\sigma B \leq (m\nabla M - m\sigma M)I \left(\leq \frac{M}{m\sigma M}(m\nabla M - m\sigma M)I \right) \tag{3.8}$$

holds and the upper bound $m\nabla M - m\sigma M$ *is the best possible.*

In particular,

$$A\nabla B - A\sharp B \le (m\nabla M - m\sharp M)I \left(= \frac{(\sqrt{M}-\sqrt{m})^2}{2} I\right). \tag{3.9}$$

and

$$A\nabla B - A!B \le (m\nabla M - m!M)I \left(= \frac{(M-m)^2}{2(m+M)} I\right).$$

Proof. Since A and B commute, we can replace them by real variables x and y, respectively, with $m \le x, y \le M$, or $(x,y) \in [m,M]^2 = [m,M]\times[m,M]$. Put

$$\phi = \phi(x,y) = x\nabla y - x\sigma y = \frac{x+y}{2} - x f_\sigma\left(\frac{y}{x}\right). \tag{3.10}$$

Calculating the Hessian matrix H_ϕ of ϕ, we have

$$H_\phi = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{bmatrix} = -\frac{1}{x^3} f_\sigma''\left(\frac{y}{x}\right) \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}. \tag{3.11}$$

Since f_σ is concave, or $f_\sigma'' \le 0$, we then have $H_\phi \ge 0$, that is, ϕ is convex. Hence the maximum of ϕ is attained at a vertex of the square $[m,M]^2$. Since $\phi(m,m) = \phi(M,M) = 0$, we obtain $\phi(m,M) (= \phi(M,m))$ as the maximum of ϕ, which implies the desired inequality (3.8). The best possibility of $\phi(m,M)$ is clear.

Now if we put $\sigma = \sharp$ specially, then from Theorem 3.1 (3.6), we have

$$A\nabla B - A\sharp B \le \frac{M}{m\sharp M}\cdot(m\nabla M - m\sigma M)I = \frac{1}{2}\sqrt{\frac{M}{m}}(\sqrt{M}-\sqrt{m})^2 I, \tag{3.12}$$

so that we obtain $\frac{1}{2}\sqrt{\frac{M}{m}}(\sqrt{M}-\sqrt{m})^2$ as an upper bound of the difference $A\nabla B - A\sharp B$. On the other hand, for commuting operators, from Theorem 3.2, (3.9), we have a smaller constant $\frac{(\sqrt{M}-\sqrt{m})^2}{2}$ as the best possible upper bound. Related to those upper bounds we want to state an example by using the same matrices A and B as in Example 2.4.

Example 3.3. Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \frac{1}{3}\begin{bmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 11 \end{bmatrix}$. Then $m = 1 \le A, B \le 4 = M$ and from (2.4)

$$A\nabla B = \frac{1}{3}\begin{bmatrix} 8 & \sqrt{2} \\ \sqrt{2} & 7 \end{bmatrix},$$

By an elementary computation we have

$$A\sharp B = \frac{\sqrt{6}}{9}\begin{bmatrix} 8 & \sqrt{2} \\ \sqrt{2} & 7 \end{bmatrix}. \tag{3.13}$$

Hence

$$D_1 := A\nabla B - A\sharp B = \frac{3-\sqrt{6}}{9}\begin{bmatrix} 8 & \sqrt{2} \\ \sqrt{2} & 7 \end{bmatrix}.$$

Since the eigenvalues of the matrix D_1 are $3-\sqrt{6}$ and $\frac{2}{3}(3-\sqrt{6})$. we see that the least upper bound of D_1 is $3-\sqrt{6} = 0.5505...$ On the other hand, the best upper bound of the difference of the same type under the restriction of commuting operators is, from (3.9),

$$m\nabla M - m\sharp M = (1+4)/2 - \sqrt{1\cdot 4} = 0.5,$$

which is too small for an upper bound of D_1.

Now corresponding to (3.7), consider the difference:

$$A\sigma^* B - A!B \tag{3.14}$$

with the assumption that A and B commute. Then we have the corresponding function

$$\psi = \psi(x,y) = x\sigma^* y - x!y = x f_{\sigma^*}\left(\frac{y}{x}\right) - \frac{2xy}{x+y},$$

similarly as ϕ. It is desirable to obtain the maximum of ψ on $[m,M]^2$. But it seems not so easy as in case of ϕ to compute the value by means of the Hessian matrix of ψ.

However, in particular, for $\sigma(=\sigma^*) = \sharp$, the best upper bound of (3.14) is determined as follows:

Theorem 3.4. *Let A and B be commuting positive operators satisfying $mI \le A, B \le MI$ for some constants $0 < m < M$. Then*

$$A\sharp B - A!B \le d(m,M)I, \tag{3.15}$$

where $d(m,M)$ is defined as follows (and is the best possible upper bound):

$$d(m,M) = \begin{cases} (\sqrt{\tau} - \frac{2\tau}{1+\tau})MI, & \text{if } \sqrt{\frac{m}{M}} \le 1-\tau, \\ (\sqrt{\frac{m}{M}} - \frac{2m/M}{1+m/M})MI, & \text{if } \sqrt{\frac{m}{M}} \ge 1-\tau, \end{cases} \tag{3.16}$$

Here τ $(= 0.7044...)$ is the (unique) positive solution $\in (0,1)$ of the cubic equation

$$t^3 - 4t^2 + 8t - 4 = 0. \tag{3.17}$$

Proof. Let

$$\psi = x\sharp y - x!y = \sqrt{xy} - \frac{2xy}{x+y}.$$

Then we have to compute the maximum of ψ for $m \le x, y \le M$. Without loss of generality we may assume that $M = 1$. The function ψ is rewritten as follows:

$$\psi = \frac{\sqrt{xy}(\sqrt{x} - \sqrt{y})^2}{(\sqrt{x} - \sqrt{y})^2 + 2\sqrt{xy}}.$$

Putting $u = \sqrt{x} - \sqrt{y}$ and $v = \sqrt{xy}$, we have

$$\psi = \psi(u,v) = \frac{u^2 v}{u^2 + 2v}. \tag{3.18}$$

Note that $\sqrt{x} + \sqrt{y} = \sqrt{u^2 + 4v}$. Hence we see that

$$\sqrt{x} = \frac{u + \sqrt{u^2+4v}}{2} \quad \text{and} \quad \sqrt{y} = \frac{-u + \sqrt{u^2+4v}}{2}.$$

Since $m \le x, y \le M = 1$, we then have

$$2\sqrt{m} \le u + \sqrt{u^2+4v} \le 2$$

and

$$2\sqrt{m} \le -u + \sqrt{u^2+4v} \le 2.$$

Write Ω the domain in the $uv-$plane satisfying the above inequalities. Then Ω is a quadrilateral surrounded by the four straight lines

$$v = \pm\sqrt{m}u + m, \quad \text{and} \quad v = \pm u + 1.$$

(The verteces of Ω are the four points $(0,m),(1-\sqrt{m},\sqrt{m}),(0,1)$ and $(-1+\sqrt{m},\sqrt{m})$ in the uv-plane.) We have to find the maximum of $\psi=\psi(u,v)$ on Ω. Since ψ is an even function in u, so that we may consider $(u,v)\in\Omega$ for $0\le u\le 1-\sqrt{m}$. It is easy to see that for a fixed $u=a$ $(0\le a\le 1-\sqrt{m})$, the function $\psi_a(v)=\psi(a,v)$ is increasing on $[0,b],\ b=-a+1$. Hence

$$\max_{0\le v\le b}\psi_a(v)=\psi_a(b)=\psi(a,-a+1). \tag{3.19}$$

Now define

$$\tilde{\psi}(t)=\psi(t,-t+1)=\frac{t^2-t^3}{t^2-2t+2} \tag{3.20}$$

by replacing a with $t\in[0,\infty)$ in (3.19). Then by an elementary calculation we see that $\tilde{\psi}(t)$ is increasing on $(0,\tau)$ and decreasing on (τ,∞), where $t=\tau$ is the unique positive solution of $\tilde{\psi}'(t)=0$, or the equation (3.17). Hence the maximum $\psi_{\max}$ of ψ on Ω is $\tilde{\psi}(\tau)$ if $\tau\le 1-\sqrt{m}$, and $\psi_{\max}=\tilde{\psi}(1-\sqrt{m})$ if $\tau>1-\sqrt{m}$. We then have the desired inequality (3.15) with the constant $d(m,M)$ defined by (3.16). It is clear that $d(m,M)$ is the best possible upper bound.

The upper bound $d(m,M)$ defined in Theorem 3.4 is, as stated, the best possible for operators with the assumtion of commutativity, but is, in general, too small for operators without the assumption. The following example shows this fact.

Example 3.5. Let A and B be the same matrices as used in Examples 2.4 and 3.3. Then $m=1\le A,B\le 4=M$, (from (3.13) and (2.4))

$$A\sharp B=\frac{\sqrt{6}}{9}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix}\quad\text{and}\quad A!B=\frac{2}{9}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix}.$$

Hence

$$D_2:=A\sharp B-A!B=\frac{\sqrt{6}-2}{9}\begin{bmatrix}8 & \sqrt{2}\\ \sqrt{2} & 7\end{bmatrix}.$$

Since the eigenvalues of D_2 are $\sqrt{6}-2$ and $\dfrac{2}{3}(\sqrt{6}-2)$. we see that the least upper bound of D_2 is $\sqrt{6}-2=0.4497...$, which is larger than $d(m,M)=\left(\sqrt{\frac{1}{4}}-\frac{2\cdot(1/4)}{1+(1/4)}\right)\cdot 4=0.4$, the best possible upper bound obtained from (3.16).

Acknowledgement The authors would like to thank Professor M. Nakamura for valuable advice. The authors would like to thank Professors J. I. Fujii et al for giving an opportunity to read [2], and Professor T. Furuta for informing of the result in [4].

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* Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address: mfujii@@cc.osaka-kyoiku.ac.jp

** Toyama University, Gofuku, Toyama 930-8555, Japan.
E-mail address: s-izumino@@h5.dion.ne.jp

*** Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-0033,Japan.
E-mail address: nakamoto@@mx.ibaraki.ac.jp