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ON QUASI-PROJECTIVE MODULES AND QUASI-INJECTIVE MODULES

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Abstract. In [9, Theorem 3.1] K. R. Fuller characterized indecomposable injective projective modules over artinian rings using i-pairs. In [3] the author generalized this theorem to indecomposable projective quasi-injective modules and indecomposable quasi-projective injective modules over artiniain rings. In [2] the author and K. Oshiro studied the above Fuller's theorem minutely. Further in [12], [13] M. Hoshino and T. Sumioka extended these results to perfect rings. In this paper we studies the results in [3] from the point of view of [2], [12].

Throughout this paper, we let R be a semiperfect ring. By M_R (resp. ${}_RM$) we stress that M is a unitary right (resp. left) R-module. For an R-module M, we denote the injective hull, the Jacobson radical, the socle, the top $M/J(M)$, the Loewy length, and the composition length of M by $E(M)$, $J(M)$, $S(M)$, $T(M)$, $L(M)$, and $|\,M\,|$, respectively. For $x \in R$, $(x)_L$ means the left multiplication map by x.

1 Simple-injectivity and condition $\alpha_r[e, g, f]$.

[2, Theorem 1] is minutely studied and extended to perfect rings by Hoshino and Sumioka in [12]. In this section, we generalized [2, Theorem 1] from the point of view of [3, Theorem 1] and [12].

An R-module M is called *local* (resp. *colocal*) if $J(M)$ is small in M with $M/J(M)$ simple (resp. $S(M)$ is simple and essential in M). And we call a bimodule ${}_RM_S$ colocal if both ${}_RM$ and M_S are colocal.

Let M and N be R-modules. M is called to be N*-injective* if for any submodule X of N and any homomorphism $\varphi : X \to M$ there exists $\tilde{\varphi} \in \mathrm{Hom}_R(N, M)$ such that the restriction map $\tilde{\varphi}|_X$ coincides with φ. In particular, if we only consider homomorphisms with simple images as φ, M is called to be N*-simple-injective*.

The following Proposition gives a relation between M-simple-injective and M-injective. The proof is given by the same way as [3, Lemma 6].

Proposition 1.1 ([3, Lemma 6]). *Let M and N be right R-modules with $S(N_R) \cong T(fR_R)$ for some primitive idempotent f in R. Suppose that N is M-simple-injective and either $L(Nf_{fRf}) < \infty$ or $L(Mf_{fRf}) < \infty$ holds. Then N is M-injective.*

An R-module M is called *quasi-injective* if M is M-injective. And M is called *simple-quasi-injective* if M is M-simple-injective. Dually we define a *quasi-projective* module. We note that quasi-injective modules and quasi-projective modules are characterized as follows by [21]:

Let M be a right R-module and let e be an idempotent in R.

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(1) M is quasi-injective if and only if $\varphi(M) \subseteq M$ for any $\varphi \in \mathrm{End}_R(E(M))$.

(2) Let I be a left eRe-right R-subbimodule of eR. Then eR/I is a quasi-projective right R-module.

Conversely, if M is quasi-projective with a projective cover $\varphi : eR \to M$, then $\mathrm{Ker}\,\varphi$ is a left eRe-right R-subbimodule of eR. In the case, if M is indecomposable, then e is a primitive idempotent.

Now we characterize M-simple-injective modules and simple-quasi-injective modules.

For any primitive idempotents e and f in R and any idempotent g in R, we say that R satisfies $\alpha_r[e, g, f]$ (resp. $\alpha_l[e, g, f]$) if $r_{gRf}l_{eRg}(X) = X$ for any right fRf-module X with $r_{gRf}(eRg) \subseteq X \subseteq gRf$ (resp. $l_{eRg}r_{gRf}(X) = X$ for any left eRe-module X with $l_{eRg}(gRf) \subseteq X \subseteq eRg$).

We easily have the following characterization of $\alpha_r[e, g, f]$ (resp. $\alpha_l[e, g, f]$).

Lemma 1.2 ([2, Lemma 2]). *Let e and f be primitive idempotents in R and let g be an idempotent in R. Then the following two conditions are equivalent.*

(a) *R satisfies $\alpha_r[e, g, f]$ (resp. $\alpha_l[e, g, f]$).*

(b) *There exists $a \in eRg$ such that $a\overline{X} = 0$ but $a\overline{Y} \neq 0$ for any right fRf-modules $\overline{X}$ and $\overline{Y}$ with $\overline{X} \subsetneq \overline{Y} \subseteq gRf/r_{gRf}(eRg)$ (resp. there exists $a \in gRf$ such that $\overline{X}a = 0$ but $\overline{Y}a \neq 0$ for any left eRe-modules $\overline{X}$ and $\overline{Y}$ with $\overline{X} \subsetneq \overline{Y} \subseteq eRg/l_{eRg}(gRf)$).*

Let e and f be primitive idempotents in R. Following Morimoto and Sumioka [15] and Hoshino and Sumioka [13] we call a pair (eR, Rf) a *colocal pair* (abbreviated *c-pair*) if ${}_{eRe}eRf_{fRf}$ is a colocal bimodule.

The following proposition is a generalization of [2, Proposition 3], in which we further characterize $\alpha_r[e, g, f]$ by the simple-injectivity.

Proposition 1.3 ([2, Proposition 3]). *Let (eR, Rf) be a c-pair and let g be an idempotent in R.*

(1) *Consider the following two conditions:*

 (a) *R satisfies $\alpha_r[e, g, f]$.*

 (b) *Quasi-projective module $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective.*

 Then (a) $\Rightarrow$ (b) holds. And if the ring fRf is right or left perfect, the converse also holds.

(2) *The following two conditions are equivalent:*

 (a) *Quasi-projective module $eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective.*

 (b) *The condition (1)(b) and $r_{gRf}(eRg) = 0$ hold.*

Proof. (1). (a) $\Rightarrow$ (b). Let $\overline{I}$ be a right R-submodule of $gR/r_{gR}(eRg)$ and let φ be a homomorphism : $\overline{I}_R \to eR/l_{eR}(Rf)_R$ with $\mathrm{Im}\,\varphi$ simple. Consider a restriction map $\varphi|_{\overline{I}\cdot f}$: $\overline{I} \cdot f \to S(eR/l_{eR}(Rf)_R) \cdot f = S(eRf_{fRf})$. We have $y \in eRg$ such that $y \cdot \mathrm{Ker}(\varphi|_{\overline{I}\cdot f}) = 0$ and $y \cdot \overline{I} \neq 0$ by $\alpha_r[e, g, f]$. There is $y' \in eRe$ such that $\varphi|_{\overline{I}\cdot f} = (y'y)_L$ as right fRf-homomorphisms : $\overline{I} \to S(eRf_{fRf})$ since the left eRe-module $S(eRf_{fRf})$ ($= S({}_{eRe}eRf)$) is simple and essential in ${}_{eRe}eRf$ by [3, Lemma 1 (3)] and its proof. Consider $(y'y)_L \in$

$\mathrm{Hom}_R(gR/r_{gR}(eRg),\, eR/l_{eR}(Rf))$. Then $\varphi = (y'y)_L|_{\overline{I}}$ by [3, Lemma 8] and [13, Corollary 3.3]. Therefore $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective.

$(b) \Rightarrow (a)$. Let X and Y be right fRf-modules with $r_{gRf}(eRg) \subseteq X \subsetneq Y \subseteq gRf$. We have only to show that there is $r \in eRg$ such that $rX = 0$ but $rY \neq 0$ by Lemma 1.2. So we may assume that Y/X_{fRf} is simple since a ring fRf is right or left perfect (see, for instance, [1, 28.4.Theorem]). Then we have a right fRf-epimorphism $\varphi : Y \to S(eR/l_{eR}(Rf)_R) \cdot f$ with $\mathrm{Ker}\,\varphi = X$ since $S(eR/l_{eR}(Rf)_R) \cdot f = S(eRf_{fRf})$ is a simple right fRf-module. And we claim that we can define a right R-epimorphism $\tilde{\varphi} : YR/r_{gRf}(eRg)R \to S(eR/l_{eR}(Rf)_R)$ by $\tilde{\varphi}(\sum_{i=1}^{n} a_i r_i + r_{gRf}(eRg)R) = \sum_{i=1}^{n} \varphi(a_i) r_i$, where $a_i \in Y$ and $r_i \in fR$. Assume that $\sum_{i=1}^{n} \varphi(a_i) r_i \neq \overline{0}$. There exists $s \in Rf$ with $\overline{0} \neq (\sum_{i=1}^{n} \varphi(a_i) r_i) s \in S(eR/l_{eR}(Rf)_R) \cdot f$ by [13, Corollary 3.3]. Then $(0 \neq)\ (\sum_{i=1}^{n} \varphi(a_i) r_i) s = \sum_{i=1}^{n} \varphi(a_i) r_i s = \varphi(\sum_{i=1}^{n} a_i r_i s) = \varphi((\sum_{i=1}^{n} a_i r_i) s)$. So $\sum_{i=1}^{n} a_i r_i \notin r_{gRf}(eRg)R$ because $\mathrm{Ker}\,\varphi = X \supseteq r_{gRf}(eRg)$. Further we have a right R-isomorphism $\eta : (YR + r_{gR}(eRg))/r_{gR}(eRg) \to YR/r_{gRf}(eRg)R$ since $(YR + r_{gR}(eRg))/r_{gR}(eRg) \cong YR/(YR \cap r_{gR}(eRg))$ and $YR \cap r_{gR}(eRg) = r_{gRf}(eRg)R$. Therefore there is $r \in eRg$ with $(r)_L = \tilde{\varphi}\eta$ because $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective. Then $rX = 0$ but $rY \neq 0$.

(2). $(a) \Rightarrow (b)$. Let I be a right R-submodule of gR with $I \supseteq r_{gR}(eRg)$ and let $\varphi \in \mathrm{Hom}_R(I/r_{gR}(eRg),\, S(eR/l_{eR}(Rf)_R))$. A right R-homomorphism $\psi : (I + l_{gR}(Rf))/l_{gR}(Rf) \to S(eR/l_{eR}(Rf)_R)$ is defined by $\psi(x + l_{gR}(Rf)) = \varphi(x + r_{gR}(eRg))$ for any $x \in I$ by [13, Corollary 3.3]. Then because $eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective, there exists $a \in eRg$ with $(a)_L|_{(I+l_{gR}(Rf))/l_{gR}(Rf)} = \psi$, where we consider $(a)_L : gR/l_{gR}(Rf) \to eR/l_{eR}(Rf)$. Define a right R-homomorphism $\tilde{\varphi} : gR/r_{gR}(eRg) \to eR/l_{eR}(Rf)$ by $\tilde{\varphi}(g + r_{gR}(eRg)) = a + l_{eR}(Rf)$. Then $\tilde{\varphi}(x + r_{gR}(eRg)) = ax + l_{eR}(Rf) = (a)_L(x + l_{gR}(Rf)) = \psi(x + l_{gR}(Rf)) = \varphi(x + r_{gR}(eRg))$ for any $x \in I$. Therefore $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective.

Assume that there is a nonzero element $x \in r_{gRf}(eRg)$. Then we have a right R-epimorphism $\xi : (xR + l_{gR}(Rf))/l_{gR}(Rf) \to S(eR/l_{eR}(Rf)_R)$ since $T(xR_R) \cong T(fR_R)$. Therefore because $eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective, there is $a \in eRg$ with $(a)_L = \xi$, where we consider $(a)_L : (xR + l_{gR}(Rf))/l_{gR}(Rf) \to S(eR/l_{eR}(Rf)_R)$. Then $ax \neq 0$. This contradicts with the fact that $x \in r_{gRf}(eRg)$.

$(b) \Rightarrow (a)$. Let I be a right R-submodule of gR with $I \supseteq l_{gR}(Rf)$ and let $\psi \in \mathrm{Hom}_R(I/l_{gR}(Rf),\, S(eR/l_{eR}(Rf)_R))$. Then we can define a right R-homomorphism $\varphi : (I + r_{gR}(eRg))/r_{gR}(eRg) \to S(eR/l_{eR}(Rf)_R)$ by $\varphi(x + r_{gR}(eRg)) = \psi(x + l_{gR}(Rf))$ for any $x \in I$ because the assumption $r_{gRf}(eRg) = 0$ and $S(eR/l_{eR}(Rf)_R) \cong T(fR_R)$ induce $\psi(y + l_{gR}(Rf)) = 0$ for any $y \in I \cap r_{gR}(eRg)$. Since $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective, there exists $a \in eRg$ with $(a)_L|_{(I+r_{gR}(eRg))/r_{gR}(eRg)} = \varphi$, where we consider $(a)_L : gR/r_{gR}(eRg) \to eR/l_{eR}(Rf)$. Then define a right R-homomorphism $\tilde{\psi} : gR/l_{gR}(Rf) \to eR/l_{eR}(Rf)$ by $\tilde{\psi}(g + l_{gR}(Rf)) = a + l_{eR}(Rf)$. For any $x \in I$, $\tilde{\psi}(x + l_{gR}(Rf)) = ax + l_{eR}(Rf) = (a)_L(x + r_{gR}(eRg)) = \varphi(x + r_{gR}(eRg)) = \psi(x + l_{gR}(Rf))$. Therefore $eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective. □

The following is a useful lemma to give simple proofs for the successive results. The proof is given by the same way as [3, Lemma 7].

Lemma 1.4 ([3, Lemma 7])**.** *Let h be a primitive idempotent in R, let g be an idempotent in R and let H be a right R-submodule of gR. Suppose that I is a gR/H-simple-injective right R-module with $S(I_R) \cong T(hR_R)$. Then for each nonzero element $t \in gRh - H$ and for each nonzero element $s \in S(I_R) \cdot h$ we have $x \in I$ such that $xt = s$.*

Now we have a characterization of indecomposable quasi-projective simple-quasi-injective modules. Then $\alpha_r[e, e, f]$ (resp. $\alpha_l[e, f, f]$) plays an important role. By the definition of $\alpha_r[e,g,f]$ (resp. $\alpha_l[e,g,f]$) and Lemma 1.2 we see that R satisfies $\alpha_r[e,e,f]$ (resp. $\alpha_l[e,f,f]$) if and only if $r_{eRf}l_{eRe}(X) = X$ for any right fRf-submodule X of eRf (resp. $l_{eRf}r_{fRf}(Y) = Y$ for any left eRe-submodule Y of eRf), or equivalently, there exists $a \in eRe$ such that $aX = 0$ but $aY \neq 0$ for any right fRf-modules X and Y with $X \subsetneq Y \subseteq eRf$ (resp. there exists $a \in fRf$ such that $Xa = 0$ but $Ya \neq 0$ for any left eRe-submodules X and Y with $X \subsetneq Y \subseteq eRf$).

Now we give an equivalent condition of a quasi-projective module $eR/l_{eR}(Rf)_R$ to be simple-quasi-injective. This proposition will give more important successive results.

Theorem 1.5. *Let R be a left perfect ring and let e and f be primitive idempotents in R with $eRf \neq 0$. Then the following two conditions are equivalent.*

(*a*) *Quasi-projective module $eR/l_{eR}(Rf)_R$ is simple-quasi-injective.*

(*b*) (*i*) *(eR, Rf) is a c-pair, and*

(*ii*) *R satisfies $\alpha_r[e,e,f]$.*

Proof. (a) ⇒ (b). (i). $S(eR/l_{eR}(Rf)_R) \cong T(fR_R)$ by [13, Lemma 3.6] since $eRf \neq 0$. So the statement hols by [13, Lemma 3.5 (1)].

(ii). By Proposition 1.3 and (i) which we already show.

(b) ⇒ (a). By Proposition 1.3. □

Corollary 1.6. *Let R be a semiprimary ring which satisfies ACC on right annihilator ideals and let e and f be primitive idempotents in R with $eRf \neq 0$. Then the following three conditions are equivalent.*

(*a*) *${}_RRf/r_{Rf}(eR)$ is quasi-injective.*

(*b*) *$eR/l_{eR}(Rf)_R$ is quasi-injective.*

(*c*) *(eR, Rf) is a c-pair.*

Proof. (a), (b) ⇒ (c). By Theorem 1.5.

(c) ⇒ (a), (b). Since ACC holds on right annihilator ideals, R satisfies both $\alpha_l[e,f,f]$ and $\alpha_r[e,e,f]$ by [15, Theorem 1.4]. Hence the statement holds by Proposition 1.1 and Theorem 1.5. □

Next we characterize indecomposable projective simple-quasi-injective modules and indecomposable quasi-projective R-simple-injective modules.

Theorem 1.7.

(1) *The following two conditions are equivalent for a right perfect ring R and a primtive idempotent f in R.*

(*a*) *${}_RRf$ is simple-quasi-injective.*

(*b*) *There exists a primitive idempotent e in R such that*

(*i*) *$S({}_RRf)$ is simple and essential in Rf with $S({}_RRf) \cong T({}_RRe)$,*

(*ii*) *$S(eRf_{fRf})$ is simple and essential in eRf, and*

(*iii*) *R satisfies $\alpha_l[e,f,f]$.*

(2) *The following two conditions are equivalent for a left perfect ring R and primitive idempotents e and f in R.*

(a) *Quasi-projective module* $eR/l_{eR}(Rf)_R$ *is* R*-simple-injective.*

(b) (i) $S(_RRf)$ *is simple and essential in* Rf *with* $S(_RRf) \cong T(_RRe)$,

(ii) $S(eRf_{fRf})$ *is simple and essential in* eRf, *and*

(iii) R *satisfies* $\alpha_r[e, e, f]$.

Proof. (1). By Theorem 1.5 and [13, Lemma 3.6].

(2). (a) $\Rightarrow$ (b). $eR/l_{eR}(Rf)_R$ is simple-quasi-injective since it is R-simple-injective. So (ii) and (iii) hold and $S(_{eRe}eRf)$ is also simple and essential in eRf by Theorem 1.5. Therefore $S(_{eRe}eRf) = S(eRf_{fRf})$ by [3, Lemma 1 (3)]. Further $S(eR/l_{eR}(Rf)_R) \cdot f = S(eRf_{fRf})$ by [3, Lemma 1 (1)] because $S(eR/l_{eR}(Rf)_R) \cong T(fR_R)$ by [13, Corollary 3.3]. Take nonzero $s \in S(_{eRe}eRf)$. Then, for any $t \in Rf$, applying Lemma 1.4 (with $I = eR/l_{eR}(Rf)$, $H = 0$, $h = f$ and $g = 1$), we have a nonzero $x \in S(eRf_{fRf})$ such that $xt = s$ since $s \in S(eR/l_{eR}(Rf)_R) \cdot f$. Therefore $R \cdot S(_{eRe}eRf)$ is an essential simple left R-submodule of Rf, i.e., (i) holds.

(b) $\Rightarrow$ (a). Let I be a right ideal of R and let $\varphi : I \to eR/l_{eR}(Rf)$ be a right R-homomorphism with $\operatorname{Im}\varphi$ simple. Consider a right fRf-epimorphism $\varphi|_{If} : If \to S(eR/l_{eR}(Rf)_R) \cdot f = S(eRf_{fRf})$. Now $e \cdot If \neq 0$ since $S(_RRf) \cong T(_RRe)$. Therefore we have $y \in eRe$ such that $y \cdot \operatorname{Ker}(\varphi|_{If}) = 0$ and $y \cdot If \neq 0$ by Lemma 1.2. Then there is $y' \in eRe$ such that $(y'y)_L = \varphi|_{If}$ because $S(eRf_{fRf}) = S(_{eRe}eRf)$ is a simple left eRe-module. We consider $(y'y)_L \in \operatorname{Hom}_R(R_R, eR_R)$ and put $\tilde{\varphi} := \pi(y'y)_L \in \operatorname{Hom}_R(R_R, eR/l_{eR}(Rf)_R)$, where we let $\pi : eR \to eR/l_{eR}(Rf)$ be the natural epimorphism. Then $\varphi|_{If} = \tilde{\varphi}|_{If}$ since $\varphi|_{If} = (y'y)_L$. Therefore $\varphi = \tilde{\varphi}|_I$ by [3, Lemma 8]. Hence $eR/l_{eR}(Rf)_R$ is R-simple-injective. □

Let e and f be primitive idempotents in R. If $S(_RRe)$ and $S(fR_R)$ are essential simple socles with $S(_RRe) \cong T(_RRf)$ and $S(fR_R) \cong T(eR_R)$, then we say that (fR, Re) is an *injective pair* (abbreviated *i-pair*).

The following is [12, Theorem 3.6] which is a generalization of [2, Theorem 1] to left perfect rings.

Corollary 1.8 ([12, Theorem 3.6])**.** *Let* R *be a left perfect ring and let* e *be a primitive idempotent in* R*. Then the following two conditions are equivalent.*

(a) eR_R *is* R*-simple-injective.*

(b) (i) *There exists a primitive idempotent* f *in* R *with* (eR, Rf) *an i-pair, and*

(ii) R *satisfies* $\alpha_r[e, 1, f]$.

2 Injectivity and composition length.

[2, Theorem 2] is minutely studied and extended to perfect rings by Hoshino and Sumioka in [12]. In this section, we generalized [2, Theorem 2] from the point of view of [3, Theorem 1] and [12].

First we give two lemmas.

Lemma 2.1. *Let* (eR, Rf) *be a c-pair and let* g *be an idempotent in* R*. Then for each* $n \in \mathbb{N}$, $r_{gRf}(eJ^ng)/r_{gRf}(eJ^{n-1}g)$ *is either* 0 *or essential socle of a right* fRf*-module* $gRf/r_{gRf}(eJ^{n-1}g)$.

Proof. Assume that $r_{gRf}(eJ^ng) \neq r_{gRf}(eJ^{n-1}g)$. We have $x \in r_{gRf}(eJ^ng) - r_{gRf}(eJ^{n-1}g)$. Then $0 \neq eJ^{n-1}gx \subseteq S(_{eRe}eRf)$ $(= r_{eRf}(eJe))$. So $eJ^{n-1}gx \subseteq S(eRf_{fRf})$ by [3, Lemma 1 (3)]. Therefore $eJ^{n-1}gxfJf = 0$, i.e., $xfJf \subseteq r_{gRf}(eJ^{n-1}g)$, i.e., $r_{gRf}(eJ^ng)/r_{gRf}(eJ^{n-1}g)$

is a semisimple right fRf-module. Further for any $y \in gRf - r_{gRf}(eJ^{n-1}g)$ there is $r \in fRf$ with $0 \neq eJ^{n-1}gyr \in S(eRf_{fRf})$ $(= S({}_{eRe}eRf))$. Therefore $eJe \cdot eJ^{n-1}gyr = 0$, i.e., $yr \in r_{gRf}(eJ^n g) - r_{gRf}(eJ^{n-1}g)$, i.e., $r_{gRf}(eJ^n g)/r_{gRf}(eJ^{n-1}g)$ is the essential socle of a right fRf-module $gRf/r_{gRf}(eJ^{n-1}g)$. □

Lemma 2.2. *Let (eR, Rf) be a c-pair, let g be an idempotent in R, and let X and Y be right fRf-submodules of gRf such that $r_{gRf}(eRg) \subseteq X \subsetneq Y$ and Y/X is the essential socle of a right fRf-module gRf/X. Suppose that $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective and ${}_R Rf/r_{Rf}(eR)$ is $Rg/l_{Rg}(gRf)$-simple-injective. Then $|\,Y/X_{fRf}\,| < \infty$.*

Proof. Assume that $|\,Y/X\,| = \infty$. We have an infinite subset $\{y_\lambda\}_{\lambda\in\Lambda}$ of $Y - X$ such that $\oplus_{\lambda\in\Lambda}(y_\lambda + X)fRf = Y/X$. For each $\lambda \in \Lambda$, put $M_\lambda := y_\lambda J + \sum_{\lambda'\in\Lambda-\{\lambda\}} y_{\lambda'}R + XR$. Each M_λ is a maximal right R-submodule of YR such that $YR/M_\lambda \cong T(fR_R)$ $(\cong S(eR/l_{eR}(Rf)_R))$. Therefore there is $z_\lambda \in eRg$ with $z_\lambda y_\lambda \neq 0$ and $z_\lambda M_\lambda = 0$ for each λ since $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective. Then $z_\lambda \in l_{eRg}(X) - l_{eRg}(Y)$. Moreover we claim that $\{Rz_\lambda\}_{\lambda\in\Lambda}$ is a set of independent elements modulo $l_{Rg}(Y)$. Assume that $\sum_{i=1}^n r_i z_{l_i} \in l_{Rg}(Y)$, where $r_i \in R$ and $l_i \in \Lambda$. For each j, $r_j z_{l_j} y_{l_j} = (\sum_{i=1}^n r_i z_{l_i}) y_{l_j} \in l_{Rg}(Y) \cdot Y = 0$. Hence $r_j z_{l_j} \in l_{Rg}(Y)$ since $z_{l_j} M_{l_j} = 0$.

Now take $l \in \Lambda$. And put $T := \sum_{\lambda\in\Lambda} Rz_\lambda$ and $W := Jz_l + \sum_{\lambda'\in\Lambda-\{l\}} R(z_{\lambda'} - z_l)$. Then ${}_R(T + l_{Rg}(Y))/(W + l_{Rg}(Y)) \cong {}_R T/W \cong T({}_R Re) \cong S({}_R Rf/r_{Rf}(eR))$ since $\{Rz_\lambda\}_{\lambda\in\Lambda}$ is a set of independent elements modulo $l_{Rg}(Y)$. Therefore we have $a \in gRf$ with $Ta \neq 0$ but $Wa = 0$ because ${}_R Rf/r_{Rf}(eR)$ is $Rg/l_{Rg}(gRf)$-simple-injective. Then we claim that $a \in Y$. Assume that $a \notin Y$. Then $afJf \not\subseteq X$ since $Y/X = S(gRf/X_{fRf})$. There is $r \in fJf$ with $ar \notin X$. We may assume that $ar = y_{l'}$ for some $l' \in \Lambda$ because Y/X is the essential socle of gRf/X_{fRf}. Then $z_{l'}ar \neq 0$. On the other hand, $z_l ar = 0$ since $z_l a + r_{Rf}(eR) \in S({}_R Rf/r_{Rf}(eR))$ induces $z_l a \in e \cdot S({}_R Rf/r_{Rf}(eR)) = S({}_{eRe}eRf) = S(eRf_{fRf})$ and $r \in fJf$. Therefore $z_\lambda ar = 0$ for any $\lambda \in \Lambda$ because $Wa = 0$. This is a contradiction. So we can represent $a = \sum_{i=1}^m y_{l'_i} r_i + x$, where $l'_i \in \Lambda$, $r_i \in R$ and $x \in X$. Then $z_l a = z_{\lambda'} a$ for any $\lambda' \in \Lambda - \{l\}$ since $Wa = 0$. And we can take $l'' \in \Lambda - \{l'_i\}_{i=1}^m$ because Λ is an infinite set. Therefore $0 \neq z_l a = z_{l''} a = z_{l''}(\sum_{i=1}^m y_{l'_i} r_i + x) = 0$, a contradiction. □

Using Lemmas 2.1 and 2.2 we easily have the following .

Proposition 2.3. *Let (eR, Rf) be a c-pair and let g be an idempotent in R. Suppose that fRf is a left perfect ring, $eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective and ${}_R Rf/r_{Rf}(eR)$ is $Rg/l_{Rg}(gRf)$-simple-injective. Then $|\,gRf/r_{gRf}(eRg)_{fRf}\,| < \infty$ and $|\,{}_{eRe}eRg/l_{eRg}(gRf)\,| < \infty$.*

Proof. $gRf/r_{gRf}(eRg)_{fRf}$ is artinian by Lemma 2.2 and, for instance, [1, 10.10. Proposition] since fRf is left perfect. Therefore there is $n \in \mathbb{N}$ with $gJ^n f \subseteq r_{gRf}(eRg)$. On the other hand $|\,{}_{eRe}\, l_{eRg}(gJ^i f)/l_{eRg}(gJ^{i-1}f)\,| < \infty$ for any $i = 1, \ldots, n$ by Lemmas 2.1 and 2.2. Therefore $|\,{}_{eRe}eRg/l_{eRg}(gRf)\,| < \infty$. Hence $|\,gRf/r_{gRf}(eRg)_{fRf}\,| < \infty$ by Lemma 2.1. □

Now we give a theorem. The equvalence of (c) and (d) is given by Hoshino and Sumioka in [13, Lemma 2.5].

Theorem 2.4. *Let (eR, Rf) be a c-pair and let g be an idempotent in R. Suppose that fRf is a left perfect ring. Then the following conditions are equivalent.*

(a) (i) *$eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-injective, and* (*ii*) ${}_RRf/r_{Rf}(eR)$ *is* $Rg/l_{Rg}(gRf)$*-injective.*

(*b*) (*i*) $eR/l_{eR}(Rf)_R$ *is* $gR/r_{gR}(eRg)$*-simple-injective, and*

(*ii*) ${}_RRf/r_{Rf}(eR)$ *is* $Rg/l_{Rg}(gRf)$*-simple-injective.*

(*c*) $|\, gRf/r_{gRf}(eRg)_{fRf}\,| < \infty$.

(*d*) $|\, {}_{eRe}eRg/l_{eRg}(gRf)\,| < \infty$.

(*e*) *ACC holds on* $\{r_{gRf}(I) \mid I$ *is a left* eRe*-submodule of* $eRg\}$ (*equivalently, DCC holds on* $\{l_{eRg}(I') \mid I'$ *is a right* fRf*-submodule of* $gRf\}$).

Proof. $(a) \Rightarrow (b)$. Clear.

$(b) \Rightarrow (c), (d)$. By Proposition 2.3.

$(b) \Rightarrow (a)$. We see by Proposition 1.1 since we already show that $(b) \Rightarrow (c), (d)$.

$(c) \Leftrightarrow (d)$. By [13, Lemma 2.5].

$(c) \Rightarrow (b)$. We see that R satisfies $\alpha_r[e, g, f]$ by [15, Lemma 1.1]. Similarly R also satisfies $\alpha_l[e, g, f]$ since we already show $(c) \Leftrightarrow (d)$. Therefore (b) holds by Proposition 1.3 (1).

$(c) \Rightarrow (e)$. Obvious.

$(e) \Rightarrow (c)$. By [15, Theorem 1.4]. □

The following corollaries are easily induced from Theorem 2.4.

Corollary 2.5. *Let* (eR, Rf) *be a c-pair. Suppose that* fRf *is a left perfect ring. Then the following conditions are equivalent.*

(*a*) $eR/l_{eR}(Rf)_R$ *and* ${}_RRf/r_{Rf}(eR)$ *are injective.*

(*b*) $eR/l_{eR}(Rf)_R$ *and* ${}_RRf/r_{Rf}(eR)$ *are R-simple-injective.*

(*c*) $|\, Rf/r_{Rf}(eR)_{fRf}\,| < \infty$.

(*d*) $|\, {}_{eRe}eR/l_{eR}(Rf)\,| < \infty$.

(*e*) *ACC holds on* $\{r_{Rf}(I) \mid I$ *is a left* eRe*-submodule of* $eR\}$.

Proof. Clearly (c), (d), (e) and the following (a′) and (b′) are equivalent by Theorem 2.4 and Proposition 1.3 (2).

(a') $eR/l_{eR}(Rf)_R$ is $R/l_R(Rf)$-injective and ${}_RRf/r_{Rf}(eR)$ is $R/r_R(eR)$-injective.

(b') $eR/l_{eR}(Rf)_R$ is $R/l_R(Rf)$-simple-injective and ${}_RRf/r_{Rf}(eR)$ is $R/r_R(eR)$-simple-injective.

And obviously (a′) (resp. (b′)) is equivalent to (a) (resp. (b)). □

Corollary 2.6 ([2, Theorem 2]). *Let* (eR, Rf) *be an i-pair. Suppose that* fRf *is a left perfect ring. Then the following conditions are equivalent.*

(*a*) eR_R *and* ${}_RRf$ *are injective.*

(*b*) eR_R *and* ${}_RRf$ *are R-simple-injective.*

(*c*) $|\, Rf_{fRf}\,| < \infty$.

(*d*) $|\, {}_{eRe}eR\,| < \infty$.

(e) *ACC holds on* $\{r_{Rf}(I) \mid I$ *is a left eRe-submodule of eR*$\}$.

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