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THE DISCRETE WALSH-TYPE TRANSFORM FOR INFINITE SEQUENCES

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Abstract. We introduce a set of Walsh-type functions on positive integers and give the inversion formula by using these functions for an infinite sequence of complex numbers whose series converges absolutely.

Discrete orthogonal transforms apply to finite sequences. In this paper we introduce a special transform for infinite sequences and give the inversion formula. For this purpose we modify the generalized Walsh functions constructed by Chrestenson [1].

Let $q \geq 2$ be a fixed integer. We define a set of functions, for $j, k \geq 1$, by

$$\psi_j(k) = \exp\left(\frac{2\pi i}{q}\sum_{t=1}^{\infty}\left[\frac{j}{q^{t-1}}\right]\left[\frac{k}{q^{t-1}}\right]\right),$$

where $[x]$ is the greatest integer $\leq x$.

Lemma 1 *Let $\lfloor x \rfloor$ denote the least integer $\geq x$. Then*

$$\sum_{k=1}^{\infty}\left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}\psi_m(k)\overline{\psi_n(k)} = \begin{cases} \frac{q^{2n}}{q^{2n}-1} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. If $m = n$, then

$$\begin{aligned} \sum_{k=1}^{\infty}\left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}\psi_m(k)\overline{\psi_n(k)} &= \sum_{k=1}^{\infty}\left(\frac{1}{q^{2n}}\right)^{\left\lfloor \frac{k}{q^{2n}} \right\rfloor} = q^{2n}\sum_{k=1}^{\infty}\left(\frac{1}{q^{2n}}\right)^{k} \\ &= \frac{q^{2n}}{q^{2n}-1}. \end{aligned}$$

Next, suppose $m \neq n$. Then

$$\begin{aligned} (1) \qquad & \sum_{k=1}^{\infty}\left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}\psi_m(k)\overline{\psi_n(k)} \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}\exp\left(\frac{2\pi i}{q}\sum_{t=1}^{\infty}\left(\left[\frac{m}{q^{t-1}}\right]-\left[\frac{n}{q^{t-1}}\right]\right)\left[\frac{k}{q^{t-1}}\right]\right). \end{aligned}$$

Here we write

$$A(m,n) = \left\{t-1 : \left[\frac{m}{q^{t-1}}\right]-\left[\frac{n}{q^{t-1}}\right] \not\equiv 0 \pmod{q}\right\}$$

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and express m, n as q-adic expansions

$$m = \sum_{k=0}^{t_1} a_k q^k, \quad n = \sum_{k=0}^{t_2} b_k q^k, \quad 0 \le a_k, b_k < q.$$

Since $A(m,n) = \{k : a_k - b_k \neq 0\} \subseteq \{k : 0 \le k \le \max\{t_1, t_2\}\}$ and $\max\{t_1, t_2\} < m + n$, for $t - 1 \in A(m,n)$,

$$\begin{aligned}\left\{ \left[\frac{k}{q^{t-1}} \right] : (j-1)q^{m+n} + 1 \le k \le jq^{m+n} \right\} &= \{ \underbrace{(j-1)q^{m+n-t+1}, \ldots, (j-1)q^{m+n-t+1}}_{q^{t-1}-1} \} \\ &\cup \{ \underbrace{(j-1)q^{m+n-t+1} + 1, \ldots, (j-1)q^{m+n-t+1} + 1}_{q^{t-1}} \} \\ &\cup \cdots \cup \{ \underbrace{jq^{m+n-t+1} - 1, \ldots, jq^{m+n-t+1} - 1}_{q^{t-1}} \} \cup \{ jq^{m+n-t+1} \}.\end{aligned}$$

Hence

$$\sum_{k=(j-1)q^{m+n}+1}^{jq^{m+n}} \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} \psi_m(k) \overline{\psi_n(k)} = 0, \quad j = 1, 2, \ldots.$$

Since (1) converges absolutely,

$$\sum_{k=1}^{N} \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} \psi_m(k) \overline{\psi_n(k)} \to 0 \quad \text{as } N \to \infty. \quad \square$$

Next we define the Fourier coefficients of a sequence of complex numbers $\{f_1, f_2, \ldots\}$ by

$$c_n(k) = \sum_{m=1}^{\infty} f_m \psi_k(m) \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}.$$

The Fourier coefficients $c_n(k)$ have the following properties.

Lemma 2 *If $\sum_{m=1}^{\infty} f_m$ converges absolutely, then, for any $n \ge 1$,*

(i) *$\sum_{k=1}^{\infty} c_n(k)$ converges absolutely,*

(ii) $\lim_{k \to \infty} c_n(k) = 0$.

Proof. It follows that

$$\sum_{k=1}^{\infty} |c_n(k)| \le \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |f_m| \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}.$$

Here we consider the partial sum

$$s_n(N_1, N_2) = \sum_{k=1}^{N_1} \sum_{m=1}^{N_2} |f_m| \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor}.$$

For any $N_1, N_2 \ge 1$,

$$\begin{aligned} s_n(N_1, N_2) &< \sum_{m=1}^{N_2} |f_m| \sum_{k=1}^{\infty} \left(\frac{1}{q^{m+n}} \right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} = \sum_{m=1}^{N_2} |f_m| \frac{1}{1 - \frac{1}{q^{m+n}}} \\ &< \sum_{m=1}^{N_2} 2|f_m| < \sum_{m=1}^{\infty} 2|f_m| < \infty. \end{aligned}$$

This proves (i), and (ii) follows from (i). □

¿From Lemmas 1 and 2 we deduce the following inversion formula.

Theorem 3 *If* $\sum_{m=1}^{\infty} f_m$ *converges absolutely, then*

$$f_n = \frac{q^{2n}-1}{q^{2n}} \sum_{k=1}^{\infty} c_n(k)\overline{\psi_n(k)}.$$

Proof. We note that $\psi_j(k) = \psi_k(j)$ and that

$$\begin{aligned} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \left| f_m \psi_k(m) \left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} \overline{\psi_n(k)} \right| &= \lim_{N_1\to\infty} \lim_{N_2\to\infty} s_n(N_1, N_2) \\ &< \infty. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{k=1}^{\infty} c_n(k)\overline{\psi_n(k)} &= \sum_{k=1}^{\infty} \left(\sum_{m=1}^{\infty} f_m \psi_k(m) \left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} \right) \overline{\psi_n(k)} \\ &= \sum_{m=1}^{\infty} f_m \sum_{k=1}^{\infty} \left(\frac{1}{q^{m+n}}\right)^{\left\lfloor \frac{k}{q^{m+n}} \right\rfloor} \psi_m(k)\overline{\psi_n(k)} \\ &= \frac{q^{2n}}{q^{2n}-1} f_n. \quad \square \end{aligned}$$

References

[1] H. E. Chrestenson, *A class of generalized Walsh functions*, Pacific J. Math. 5 (1955), 17–31.

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