COMMON FIXED POINT STRUCTURES FOR MULTIVALUED OPERATORS

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ABSTRACT. In this paper we introduce the notions of common fixed point structure and common strict fixed point structure for multivalued operators and we prove some common fixed point theorems and common strict fixed point theorems in the terms of these structures.

1 Introduction   The notion of fixed point structure for singlevalued operators was introduced by I. A. Rus in 1986 ([5], [6]) and it generalizes notions such as "partially ordered set with the fixed point property" or "topological space with the fixed point property". In [7] he presents in a unified form the results which he obtained regarding the fixed point structure and in [9] he formulates some open problems in the fixed point theory, in terms of the fixed point structures (problems about invariant subsets under an operator, coincidence theory, common fixed points, involution operators or retractible mappings). The fixed point structures with the common fixed point property are also studied by I. A. Rus in [10].

Using the fixed point structures, F. Aldea studies in [1] the surjectivity. In [15] F. Ştefănescu introduces the notion of Goebel's type coincidence structure and she proves some coincidence theorems. M.-A. Şerban presents in [14] the technique of the fixed point structure for mappings on product spaces.

I. A. Rus extended in 1993 the technique of the fixed point structure to the multivalued operators by introducing the notions of fixed point structure for multivalued operators and strict fixed point structure ([8]).

Also, I. A. Rus presents in [11] some open problems in the fixed point theory, in terms of the fixed point structures both for singlevalued and for multivalued operators.

Having a preliminary character, the second section of the paper contains notions and results which will appear in the next sections and specifies the terminology and the notations used in the paper. In this second section we shall prove a general lemma of invariant subset under two multivalued operators and we shall introduce two notions: (θ, φ)-contraction pair of multivalued operators and θ-condensing pair of multivalued operators.

Section 3 will be dedicated to the common fixed point structures for multivalued operators. We shall introduce this notion, we shall give examples of such a structure and we shall prove some general common fixed point theorems for multivalued operators.

In the last section we shall introduce the notion of common strict fixed point structure and we shall prove some general common strict fixed point theorems.

2 Some notions and preliminary results   Let X and Y be two nonempty sets. We denote by $\mathcal{P}(X)$ the set of all subsets of X, i. e. $\mathcal{P}(X) := \{ A \mid A \subseteq X \}$ and by $P(X)$ the set of all nonempty subsets of X, i. e. $P(X) := \{ A \mid \emptyset \neq A \subseteq X \}$.

Let $f : X \to X$ be a singlevalued operator. We denote by F_f the fixed points set of f, i. e. $F_f := \{\ x \in X \mid f(x) = x\ \}$.

For a multivalued mapping $T : X \to \mathcal{P}(Y)$ we use the notation $T : X \multimap Y$. We denote by $\mathbb{M}^\circ(X, Y)$ the set of all multivalued mappings $T : X \multimap Y$. We also write $\mathbb{M}^\circ(X) := \mathbb{M}^\circ(X, X)$.

Let $T : X \to P(X)$ be a multivalued operator. We denote by $I(T)$ the set of all nonempty invariant subsets of T, i. e. $I(T) := \{\ A \mid A \in P(X),\ T(A) = \cup_{a \in A} T(a) \subseteq A\ \}$. We denote by F_T the fixed points set of T, i. e. $F_T := \{\ x \in X \mid x \in T(x)\ \}$ and by $(SF)_T$ the strict fixed points set of T, i. e. $(SF)_T := \{\ x \in X \mid T(x) = \{x\}\ \}$.

Let $T_1, T_2 : X \to P(X)$ be two multivalued operators. We denote by $(CF)_{T_1,T_2}$ the common fixed points set of T_1 and T_2, i. e. $(CF)_{T_1,T_2} := F_{T_1} \cap F_{T_2}$ and by $(CSF)_{T_1,T_2}$ the common strict fixed points set of T_1 and T_2, i. e. $(CSF)_{T_1,T_2} := (SF)_{T_1} \cap (SF)_{T_2}$.

Let (X, d) be a metric space.

We denote by $P_{cl}(X)$ the set of all nonempty and closed subsets of X, i. e. $P_{cl}(X) := \{\ A \mid A \in P(X),\ A \text{ is a closed set}\ \}$ and by $P_b(X)$ the set of all nonempty and bounded subsets of X, i. e. $P_b(X) := \{\ A \mid A \in P(X),\ A \text{ is a bounded set}\ \}$.

We also remind the functional $D : P(X) \times P(X) \to \mathbb{R}_+$, defined by $D(A, B) = \inf \{\ d(a, b) \mid a \in A,\ b \in B\ \}$, for each $A, B \in P(X)$, and the generalized functionals $\delta : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by $\delta(A, B) = \sup \{\ d(a, b) \mid a \in A,\ b \in B\ \}$, for each $A, B \in P(X)$ (we notice that $\delta(A) := \delta(A, A)$, for each $A \in P(X)$), and $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, defined by $H(A, B) = \max \left\{ \sup_{a \in A} D(a, B),\ \sup_{b \in B} D(b, A) \right\}$, for each $A, B \in P(X)$.

Definition 2.1. *Let X be a nonempty set. An operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a* closure operator *if*

   (i) $A \subseteq \eta(A)$, *for each $A \in \mathcal{P}(X)$;*

   (ii) $A \subseteq B$ *implies that $\eta(A) \subseteq \eta(B)$, for each $A, B \in \mathcal{P}(X)$;*

   (iii) $\eta(\eta(A)) = \eta(A)$, *for each $A \in \mathcal{P}(X)$.*

Lemma 2.1. *Let X be a nonempty set, $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator and $A_i \in F_\eta$, $i \in I$.*

   Then $\cap_{i \in I} A_i \in F_\eta$.

Lemma 2.2. *Let X be a nonempty set and $T : X \to P(X)$ a multivalued operator. If $T(X) \subseteq Y \subseteq X$, then $Y \in I(T)$.*

   The next result extends the general lemma of invariant subset for a multivalued operator, given by I. A. Rus in [8], to a general lemma of invariant subset for two multivalued operators.

Lemma 2.3. *Let X be a nonempty set, $\eta : P(X) \to P(X)$ a closure operator, $Y \in F_\eta$, $T_1, T_2 : Y \to P(Y)$ two multivalued operators and $A \in P(Y)$.*

   Then there exists $A_0 \subseteq Y$ such that:

   (a) $A_0 \supseteq A$;

   (b) $A_0 \in I(T_1) \cap I(T_2) \cap F_\eta$;

   (c) $\eta(T_1(A_0) \cup T_2(A_0) \cup A) = A_0$.

Proof. Let $\mathcal{B} := \{ B \mid A \subseteq B \subseteq Y$ and $B \in I(T_1) \cap I(T_2) \cap F_\eta \}$. We observe that $\mathcal{B} \neq \emptyset$, because $Y \in \mathcal{B}$. We have $\cap_{B \in \mathcal{B}} B \in \mathcal{B}$ and we put $A_0 := \cap_{B \in \mathcal{B}} B$. With this choice of A_0 we have that (a) and (b) are satisfied.

We shall prove that $\eta(T_1(A_0) \cup T_2(A_0) \cup A) \in \mathcal{B}$.

Obviously $A \subseteq \eta(T_1(A_0) \cup T_2(A_0) \cup A)$ and $\eta(T_1(A_0) \cup T_2(A_0) \cup A) \in F_\eta$. For each $i \in \{1, 2\}$ we have

$$T_i(A_0) \subseteq T_1(A_0) \cup T_2(A_0) \cup A \subseteq \eta(T_1(A_0) \cup T_2(A_0) \cup A) \subseteq \eta(A_0) = A_0.$$

So $\eta(T_1(A_0) \cup T_2(A_0) \cup A) \in I(T_1) \cap I(T_2)$. Now we are able to write that $\eta(T_1(A_0) \cup T_2(A_0) \cup A) \in \mathcal{B}$.

It follows that $\eta(T_1(A_0) \cup T_2(A_0) \cup A) = A_0$, because $A_0 = \cap_{B \in \mathcal{B}} B \in \mathcal{B}$ and $\eta(T_1(A_0) \cup T_2(A_0) \cup A) \subseteq A_0$. $\square$

Definition 2.2 (I. A. Rus [5], [6], [7]). *Let X be a nonempty set and $Z \in P(P(X))$. A functional $\theta : Z \to \mathbb{R}_+$ has the* intersection property *if $A_n \in Z$, $A_{n+1} \subseteq A_n$, for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} \theta(A_n) = 0$ imply that $A_\infty := \cap_{n \in \mathbb{N}} A_n \in Z$ and $\theta(A_\infty) = 0$.*

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. We consider the following conditions:

(i_φ) φ is nondecreasing, i. e. $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$, for each $t_1, t_2 \in \mathbb{R}_+$;

(ii_φ) $\varphi(t) < t$, for each $t > 0$;

(iii_φ) $\varphi(0) = 0$;

(iv_φ) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0, as $n \to \infty$, for each $t \geq 0$.

Lemma 2.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function which satisfies the conditions (i_φ) and (iv_φ). Then (ii_φ) holds.*

Lemma 2.5. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function which satisfies the conditions (i_φ) and (ii_φ). Then (iii_φ) holds.*

Definition 2.3. *A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called* comparison function *if satisfies the conditions (i_φ) and (iv_φ).*

Further on we give two notions.

Definition 2.4. *Let X be a nonempty set, $Y \in P(X)$, $Z \in P(P(X))$ and $\theta : Z \to \mathbb{R}_+$. The multivalued operators $T_1, T_2 : Y \to P(Y)$ form a (θ, φ)-contraction pair if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

(i) $A \in P(Y) \cap Z$ *implies* $T_1(A) \cup T_2(A) \in Z$;

(ii) $\theta(T_1(A) \cup T_2(A)) \leq \varphi(\theta(A))$, *for each* $A \in I(T_1) \cap I(T_2) \cap Z$.

Definition 2.5. *Let X be a nonempty set, $Y \in P(X)$, $Z \in P(P(X))$ and $\theta : Z \to \mathbb{R}_+$. The multivalued operators $T_1, T_2 : Y \to P(Y)$ form a θ-condensing pair if*

(i) $A_i \in Z$, $i \in I$ *and* $\cap_{i \in I} A_i \neq \emptyset$ *imply* $\cap_{i \in I} A_i \in Z$;

(ii) $A \in P(Y) \cap Z$ *implies* $T_1(A) \cup T_2(A) \in Z$;

(iii) $\theta(T_1(A) \cup T_2(A)) < \theta(A)$, *for each* $A \in I(T_1) \cap I(T_2) \cap Z$, *with the property that* $\theta(A) \neq 0$.

These two notions extend the notions of (θ, φ)-contraction and θ-condensing given for a multivalued operator by I. A. Rus in [8] and generalize the corresponding notions for a pair of singlevalued operators, given by I. A. Rus in [10].

3   Common fixed point structures for multivalued operators   In this section
we shall introduce the notion of common fixed point structure for multivalued operators,
which extends the notion of fixed point structure for multivalued operators, given by I. A.
Rus in [8]. The results which we shall give here extend those presented by I. A. Rus in [8].

Definition 3.1. *Let X be a nonempty set. A triplet (X, S, M_C°) is a* common fixed point
structure *for multivalued operators* (*briefly* c. f. p. s.) *if*

(i) $S \in P(P(X))$;

(ii) $M_C^\circ : P(X) \multimap \cup_{Y \in P(X)} \mathbb{M}^\circ(Y) \times \mathbb{M}^\circ(Y),\ Y | \multimap M_C^\circ(Y) \subseteq \mathbb{M}^\circ(Y) \times \mathbb{M}^\circ(Y)$ *is a*
multivalued mapping such that if $Z \in P(Y)$, then

$$\{\ (T_1|_Z, T_2|_Z)\ |\ (T_1, T_2) \in M_C^\circ(Y)\ and\ Z \in I(T_1) \cap I(T_2)\ \} \subseteq M_C^\circ(Z);$$

(iii) *each $Y \in S$ has the common fixed point property relative to $M_C^\circ(Y)$, i. e.*

$$Y \in S\ and\ (T_1, T_2) \in M_C^\circ(Y)\ imply\ (CF)_{T_1, T_2} \neq \emptyset.$$

We illustrate this definition through some examples. An important source of such e-
xamples can be given by using the common fixed point theorems for multivalued operators.

Example 3.1. *Let X be a nonempty set, $S = \{\ \{x\}\ |\ x \in X\ \}$ and for each $Y \in P(X)$ we
take $M_C^\circ(Y)$ the set of all pairs of multivalued operators (T_1, T_2), where $T_1, T_2 : Y \to P(Y)$.
In this case the triplet (X, S, M_C°) is a c. f. p. s. and it is called the trivial common fixed
point structure.*

Example 3.2. *Let (X, d) be a complete metric space, $S = P_{cl}(X)$ and for each $Y \in
P(X)$ we take $M_C^\circ(Y)$ the set of all pairs of multivalued operators (T_1, T_2), where $T_1, T_2 :
Y \to P_{cl}(Y)$, with the property that there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 +
2 \max \{a_4, a_5\} < 1$, such that*

$$H(T_1(x), T_2(y)) \le a_1\ d(x, y) + a_2\ D(x, T_1(x)) + a_3\ D(y, T_2(y)) +$$

$$+a_4\ D(x, T_2(y)) + a_5\ D(y, T_1(x)),$$

for each $x, y \in Y$.
    *Then the triplet (X, S, M_C°) is a c. f. p. s., taking into account a result given by V.
Popa in [3].*

Example 3.3. *Let (X, d) be a complete metric space, $S = P_{cl}(X)$ and for each $Y \in P(X)$
we put $M_C^\circ(Y)$ the set of all pairs of multivalued operators (T_1, T_2), where $T_1, T_2 : Y \to
P_{cl}(Y)$, with the property that there exists $a \in [0, 1[$ such that*

$$H(T_1(x), T_2(y)) \le a\ \max \{d(x, y), D(x, T_1(x)), D(y, T_2(y)),$$

$$1/2\ [D(x, T_2(y)) + D(y, T_1(x))]\},$$

for each $x, y \in Y$.
    *Then the triplet (X, S, M_C°) is a c. f. p. s., taking into account a result given by V.
Popa in [4].*

Example 3.4. *Let (X, d) be a complete metric space, $S = P_{cl}(X)$ and for each $Y \in P(X)$ we take $M_C^\circ(Y)$ the set of all pairs of multivalued operators (T_1, T_2), where $T_1, T_2 : Y \to P_{cl}(Y)$, with the property that for each $i, j \in \{1, 2\}$, with $i \neq j$, there exist $a_{i1}, \dots, a_{i5} \in \mathbb{R}_+$, with $a_{i1} + a_{i2} + a_{i3} + 2\ a_{i4} < 1$, such that for each $x \in Y$, any $u_x \in T_i(x)$ and for all $y \in Y$, there exists $u_y \in T_j(y)$ so that*

$$d(u_x, u_y) \le a_{i1}\ d(x, y) + a_{i2}\ d(x, u_x) + a_{i3}\ d(y, u_y) + a_{i4}\ d(x, u_y) + a_{i5}\ d(y, u_x).$$

Then (X, S, M_C°) is a c. f. p. s., according to the Theorem 2.2 from [13].

Example 3.5. *Let (X, d) be a complete metric space, $S = P_{cl}(X)$ and for each $Y \in P(X)$ we take $M_C^\circ(Y)$ the set of all pairs of multivalued operators (T_1, T_2), where $T_1, T_2 : Y \to P_{cl}(Y)$, with the property that for each $i, j \in \{1, 2\}$, with $i \neq j$, there exists $a_i \in [0, 1[$ such that for each $x \in Y$, any $u_x \in T_i(x)$ and for all $y \in Y$, there exists $u_y \in T_j(y)$ so that*

$$d(u_x, u_y) \le a_i\ \max\{d(x, y), d(x, u_x), d(y, u_y), 1/2\ [d(x, u_y) + d(y, u_x)]\}.$$

Then (X, S, M_C°) is a c. f. p. s., according to the Theorem 2.2 from [13].

Definition 3.2. *Let (X, S, M_C°) be a c. f. p. s., $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. The pair (θ, η) is a* compatible pair *with the c. f. p. s. (X, S, M_C°) if*

(i) η is a closure operator, $S \subseteq \eta(Z) \subseteq Z \subseteq P(X)$ and $\theta(\eta(A)) = \theta(A)$, for each $A \in Z$;

(ii) $F_\eta \cap Z_\theta \subseteq S$, where $Z_\theta = \{\ A \mid A \in Z,\ \theta(A) = 0\ \}$.

Theorem 3.1. *Let (X, S, M_C°) be a c. f. p. s. and (θ, η) a compatible pair with the c. f. p. s. (X, S, M_C°), where $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. Let $Y \in \eta(Z)$ and $(T_1, T_2) \in M_C^\circ(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) (T_1, T_2) is a (θ, φ)-contraction pair.

Then

(a) $(CF)_{T_1, T_2} \neq \emptyset$;

(b) if $(CF)_{T_1, T_2} \in I(T_1) \cap I(T_2) \cap Z$, then $\theta((CF)_{T_1, T_2}) = 0$.

Proof. (a) We put $Y_0 = Y$. We observe that $Y_0 \in I(T_1) \cap I(T_2) \cap F_\eta \cap Z$.

We have that $Y_{n+1} := \eta(T_1(Y_n) \cup T_2(Y_n)) \in I(T_1) \cap I(T_2) \cap F_\eta \cap Z$, for each $n \in \mathbb{N}$.

We also have that

$$\theta(Y_{n+1}) = \theta(\eta(T_1(Y_n) \cup T_2(Y_n))) = \theta(T_1(Y_n) \cup T_2(Y_n)) \le$$

$$\le \varphi(\theta(Y_n)) \le \dots \le \varphi^{n+1}(\theta(Y)),$$

for each $n \in \mathbb{N}$.

It follows that $\theta(Y_{n+1}) \to 0$, as $n \to \infty$, taking into account that $\varphi^{n+1}(\theta(Y)) \to 0$, as $n \to \infty$.

Because $Y_{n+1} = \eta(T_1(Y_n) \cup T_2(Y_n)) \subseteq \eta(Y_n) = Y_n$, for each $n \in \mathbb{N}$, and using the fact that $\theta|_{\eta(Z)}$ has the intersection property, we are able to write that

$$Y_\infty := \cap_{n \in \mathbb{N}} Y_n \in Z \text{ and } \theta(Y_\infty) = 0.$$

Also, from the Lemma 2.1 we have that $Y_\infty \in F_\eta$. So $Y_\infty \in S$.

We remark that $Y_\infty \in I(T_1) \cap I(T_2)$ and hence $(T_1|_{Y_\infty}, T_2|_{Y_\infty}) \in M_C^\circ(Y_\infty)$.

It follows that $(CF)_{T_1|_{Y_\infty}, T_2|_{Y_\infty}} \neq \emptyset$ and therefore $(CF)_{T_1, T_2} \neq \emptyset$.

(b) We have

$$\theta((CF)_{T_1, T_2}) = \theta(T_1((CF)_{T_1, T_2}) \cup T_2((CF)_{T_1, T_2})) \leq$$

$$\leq \varphi(\theta((CF)_{T_1, T_2})) \leq \ldots \leq \varphi^n(\theta((CF)_{T_1, T_2})),$$

for each $n \in \mathbb{N}$.

¿From here we get that $\theta((CF)_{T_1, T_2}) = 0$, because $\varphi^n(\theta((CF)_{T_1, T_2})) \to 0$, as $n \to \infty$. $\square$

Theorem 3.2. *Let (X, S, M_C°) be a c. f. p. s. and (θ, η) a compatible pair with the c. f. p. s. (X, S, M_C°), where $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. Let $Y \in F_\eta$ and $(T_1, T_2) \in M_C^\circ(Y)$ such that $T_1(Y) \cup T_2(Y) \in Z$. We suppose that:*

(i) *$\theta|_{\eta(Z)}$ has the intersection property;*

(ii) *(T_1, T_2) is a (θ, φ)-contraction pair.*

Then

(a) *$(CF)_{T_1, T_2} \neq \emptyset$;*

(b) *if $(CF)_{T_1, T_2} \in I(T_1) \cap I(T_2) \cap Z$, then $\theta((CF)_{T_1, T_2}) = 0$.*

Proof. We notice that the multivalued operators
$T_1|_{\eta(T_1(Y) \cup T_2(Y))}, T_2|_{\eta(T_1(Y) \cup T_2(Y))} : \eta(T_1(Y) \cup T_2(Y)) \to P(\eta(T_1(Y) \cup T_2(Y)))$ satisfy the conditions from the Theorem 3.1. $\square$

Theorem 3.3. *Let (X, S, M_C°) be a c. f. p. s. and (θ, η) a compatible pair with the c. f. p. s. (X, S, M_C°), where $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. Let $Y \in \eta(Z)$ and $(T_1, T_2) \in M_C^\circ(Y)$. We suppose that:*

(i) *$x \in Y$, $A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;*

(ii) *(T_1, T_2) is a θ-condensing pair.*

Then

(a) *$(CF)_{T_1, T_2} \neq \emptyset$;*

(b) *if $(CF)_{T_1, T_2} \in I(T_1) \cap I(T_2) \cap Z$, then $\theta((CF)_{T_1, T_2}) = 0$.*

Proof. (a) Let $y_0 \in Y$. ¿From the Lemma 2.3 and the conditions (i) and (ii) of the Definition 2.5, as well as from the condition (i) of the hypothesis, we get that there exists $A_0 \subseteq Y$ such that $y_0 \in A_0$, $A_0 \in I(T_1) \cap I(T_2) \cap F_\eta \cap Z$ and $\eta(T_1(A_0) \cup T_2(A_0) \cup \{y_0\}) = A_0$.

If we suppose that $\theta(A_0) \neq 0$, then we reach the contradiction

$$\theta(A_0) = \theta(\eta(T_1(A_0) \cup T_2(A_0) \cup \{y_0\})) = \theta(T_1(A_0) \cup T_2(A_0) \cup \{y_0\}) =$$

$$= \theta(T_1(A_0) \cup T_2(A_0)) < \theta(A_0).$$

So $\theta(A_0) = 0$. We got that $A_0 \in F_\eta \cap Z_\theta$ and hence $A_0 \in S$. Also, we have that $(T_1|_{A_0}, T_2|_{A_0}) \in M_C^\circ(A_0)$. Therefore $(CF)_{T_1|_{A_0}, T_2|_{A_0}} \neq \emptyset$ and this implies that $(CF)_{T_1, T_2} \neq \emptyset$.

(b) If we suppose that $\theta((CF)_{T_1, T_2}) \neq 0$, then we have

$$\theta((CF)_{T_1, T_2}) = \theta(T_1((CF)_{T_1, T_2}) \cup T_2((CF)_{T_1, T_2})) < \theta((CF)_{T_1, T_2}),$$

which is a contradiction. So $\theta((CF)_{T_1, T_2}) = 0$. $\square$

Theorem 3.4. *Let (X, S, M_C°) be a c. f. p. s. and (θ, η) a compatible pair with the c. f. p. s. (X, S, M_C°), where $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$. Let $Y \in F_\eta$ and $(T_1, T_2) \in M_C^\circ(Y)$ such that $T_1(Y) \cup T_2(Y) \in Z$. We suppose that:*

(i) $x \in Y$, $A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(iii) (T_1, T_2) is a θ-condensing pair.

Then

(a) $(CF)_{T_1, T_2} \neq \emptyset$;

(b) if $(CF)_{T_1, T_2} \in I(T_1) \cap I(T_2) \cap Z$, then $\theta((CF)_{T_1, T_2}) = 0$.

Proof. We observe that the multivalued operators $T_1|_{\eta(T_1(Y) \cup T_2(Y))}, T_2|_{\eta(T_1(Y) \cup T_2(Y))}$: $\eta(T_1(Y) \cup T_2(Y)) \to P(\eta(T_1(Y) \cup T_2(Y)))$ satisfy the conditions from the Theorem 3.3. $\square$

4   Common strict fixed point structures   The notion of common strict fixed point structure, which we shall introduce further on, extends the notion of strict fixed point structure, given by I. A. Rus in [8]. We shall also give abstract results relative to the common strict fixed point structure, results which extend those presented by I. A. Rus in [8]. The proofs of these abstract results are made as the proofs of the corresponding results from the section 3.

Definition 4.1. *Let X be a nonempty set. A triplet (X, S, M_C°) is a common strict fixed point structure (briefly c. s. f. p. s.) if*

(i) $S \in P(P(X))$;

(ii) $M_C^\circ : P(X) \multimap \cup_{Y \in P(X)} \mathbb{M}^\circ(Y) \times \mathbb{M}^\circ(Y)$, $Y \longmapsto M_C^\circ(Y) \subseteq \mathbb{M}^\circ(Y) \times \mathbb{M}^\circ(Y)$ is a mapping such that if $Z \in P(Y)$, then

$$\{ (T_1|_Z, T_2|_Z) \mid (T_1, T_2) \in M_C^\circ(Y) \text{ and } Z \in I(T_1) \cap I(T_2) \} \subseteq M_C^\circ(Z);$$

(iii) each $Y \in S$ has the common strict fixed point property relative to $M_C^\circ(Y)$, i. e.

$$Y \in S \text{ and } (T_1, T_2) \in M_C^\circ(Y) \text{ imply } (CSF)_{T_1, T_2} \neq \emptyset.$$

The triplet (X, S, M_C°) from the Example 3.1 is in fact a c. s. f. p. s. .

Using the common strict fixed point theorems can be presented examples of c. s. f. p. s., as the next example shows.

Example 4.1. *Let (X, d) be a complete metric space, $S = P_{cl}(X)$ and for each $Y \in P(X)$ we take $M_C^\circ(Y)$ to be the set of all pairs of multivalued operators (T_1, T_2), with $T_1, T_2 : Y \to P_b(Y)$ and for which there exist $a, b, c \in \mathbb{R}_+$, with $a + 2b + 4c < 1$, such that*

$$\delta(T_1(x), T_2(y)) \leq a \ d(x, y) + b \ [\delta(x, T_1(x)) + \delta(y, T_2(y))] + c \ [\delta(x, T_2(y)) + \delta(y, T_1(x))],$$

for each $x, y \in Y$.

Then (X, S, M_C°) is a c. s. f. p. s., according to the Theorem 2.1 *given by M. Avram in [2].*

Having a c. s. f. p. s. (X, S, M_C°), $\theta : Z \to \mathbb{R}_+$ and $\eta : P(X) \to P(X)$, we say that the pair (θ, η) is compatible with the c. s. f. p. s. (X, S, M_C°) if the conditions (i) and (ii) from the Definition 3.2 are satisfied.

Theorem 4.1. *Let* (X, S, M_C°) *be a c. s. f. p. s. and* (θ, η) *a compatible pair with the c. s. f. p. s.* (X, S, M_C°), *where* $\theta : Z \to \mathbb{R}_+$ *and* $\eta : P(X) \to P(X)$. *Let* $Y \in \eta(Z)$ *and* $(T_1, T_2) \in M_C^\circ(Y)$. *We suppose that:*

(i) $\theta|_{\eta(Z)}$ *has the intersection property;*

(ii) (T_1, T_2) *is a* (θ, φ)*-contraction pair.*

Then

(a) $(CSF)_{T_1, T_2} \neq \emptyset$;

(b) *if* $(CSF)_{T_1, T_2} \in Z$, *then* $\theta((CSF)_{T_1, T_2}) = 0$.

Theorem 4.2. *Let* (X, S, M_C°) *be a c. s. f. p. s. and* (θ, η) *a compatible pair with the c. s. f. p. s.* (X, S, M_C°), *where* $\theta : Z \to \mathbb{R}_+$ *and* $\eta : P(X) \to P(X)$. *Let* $Y \in F_\eta$ *and* $(T_1, T_2) \in M_C^\circ(Y)$ *such that* $T_1(Y) \cup T_2(Y) \in Z$. *We suppose that:*

(i) $\theta|_{\eta(Z)}$ *has the intersection property;*

(ii) (T_1, T_2) *is a* (θ, φ)*-contraction pair.*

Then

(a) $(CSF)_{T_1, T_2} \neq \emptyset$;

(b) *if* $(CSF)_{T_1, T_2} \in Z$, *then* $\theta((CSF)_{T_1, T_2}) = 0$.

The next result is a consequence of the Theorem 4.1 (or of the Theorem 4.2).

Theorem 4.3. *Let* (X, d) *be a complete and bounded metric space and* $T_1, T_2 : X \to P(X)$ *two multivalued operators which form a* (δ, φ)*-contraction pair* $(\delta : P(X) \to \mathbb{R}_+$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+)$.
Then $(CF)_{T_1, T_2} = (CSF)_{T_1, T_2} = \{x^*\}$.

Proof. We take $S = \{ \{x\} \mid x \in X \}$ and $M_C^\circ(Y)$ the set of all pairs of multivalued operators $(\widetilde{T_1}, \widetilde{T_2})$, where $\widetilde{T_1}, \widetilde{T_2} : Y \to P(Y)$, for each $Y \in P(X)$.

Also, we choose $Z = P(X)$, $\theta = \delta$ and $\eta : P(X) \to P(X)$ defined by $\eta(A) = \overline{A}$, for each $A \in P(X)$.

¿From the Theorem 4.1 (or the Theorem 4.2) it follows that $\delta((CSF)_{T_1, T_2}) = 0$, taking into account that (X, S, M_C°) is a c. s. f. p. s.. So $(CSF)_{T_1, T_2} = \{x^*\}$.

We put $X_0 = X$ and let $X_{n+1} = \overline{T_1(X_n) \cup T_2(X_n)}$, for each $n \in \mathbb{N}$. We observe that $(CF)_{T_1, T_2} \subseteq X_\infty := \cap_{n \in \mathbb{N}} X_n$. Because $\{x^*\} = (CSF)_{T_1, T_2} \subseteq (CF)_{T_1, T_2}$ and $\delta(X_\infty) = 0$, we get that $(CF)_{T_1, T_2} = (CSF)_{T_1, T_2} = \{x^*\}$. $\square$

Theorem 4.4. *Let* (X, S, M_C°) *be a c. s. f. p. s. and* (θ, η) *a compatible pair with the c. s. f. p. s.* (X, S, M_C°), *where* $\theta : Z \to \mathbb{R}_+$ *and* $\eta : P(X) \to P(X)$. *Let* $Y \in \eta(Z)$ *and* $(T_1, T_2) \in M_C^\circ(Y)$. *We suppose that:*

(i) $x \in Y$, $A \in Z$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$;

(ii) (T_1, T_2) *is a θ-condensing pair.*

Then

(a) $(CSF)_{T_1,T_2} \neq \emptyset$;

(b) *if* $(CSF)_{T_1,T_2} \in Z$, *then* $\theta((CSF)_{T_1,T_2}) = 0$.

Theorem 4.5. *Let* (X, S, M_C°) *be a c. s. f. p. s. and* (θ, η) *a compatible pair with the c. s. f. p. s.* (X, S, M_C°), *where* $\theta : Z \to \mathbb{R}_+$ *and* $\eta : P(X) \to P(X)$. *Let* $Y \in F_\eta$ *and* $(T_1, T_2) \in M_C^\circ(Y)$ *such that* $T_1(Y) \cup T_2(Y) \in Z$. *We suppose that:*

(i) $x \in Y$, $A \in Z$ *imply* $A \cup \{x\} \in Z$ *and* $\theta(A \cup \{x\}) = \theta(A)$;

(iii) (T_1, T_2) *is a θ-condensing pair.*

Then

(a) $(CSF)_{T_1,T_2} \neq \emptyset$;

(b) *if* $(CSF)_{T_1,T_2} \in Z$, *then* $\theta((CSF)_{T_1,T_2}) = 0$.

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