ON CHEBYSHEV TYPE THEORY

Kazuaki KITAHARA

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Abstract. Let $C[a,b]$ be the space of all real-valued continuous functions on a compact interval $[a,b]$ and suppose that $C[a,b]$ is endowed with the supremum norm. If we consider a finite dimensional Chebyshev space G as an approximating space, then a series of important results of best approximation from G are well known as the Chebyshev theory. In this paper, we introduce two other best approximation problems and show that Chebyshev type theory holds in the problems.

1. Introduction

We shall study relations among results of three best approximation problems. Before stating precisely the purpose of this paper, we have to explain some definitions, notations, and best approximation problems.

Let $F[a,b]$ be the space of all real-valued functions on a compact nondegenerate interval $[a,b]$ of $\mathbf{R}$. A finite subset $\{u_1,\dots,u_n\}$ of $F[a,b]$ is called a *system* if $u_1,\dots,u_n$ are linearly independent. We denote by $\mathrm{Span}\{u_1,\dots,u_n\}$ the space spanned by $\{u_1,\dots,u_n\}$. A system $\{u_1,\dots,u_n\}$ of $F[a,b]$ is called a *Chebyshev system* if for any n distinct points $x_1,\dots,x_n\in[a,b]$, the n-th order determinant

$$\begin{vmatrix} u_1(x_1) & u_2(x_1) & \cdots & u_n(x_1) \\ u_1(x_2) & u_2(x_2) & \cdots & u_n(x_2) \\ \vdots & \vdots & & \vdots \\ u_1(x_n) & u_2(x_n) & \cdots & u_n(x_n) \end{vmatrix} \neq 0.$$

In other words, a Chebyshev system $\{u_1,\dots,u_n\}$ is a system such that any nontrivial linear combination possesses at most $n-1$ zeros in $[a,b]$. If a system $\{u_1,\dots,u_n\}$ satisfies that each system $\{u_1,\dots,u_k\}, k=1,\dots,n$ is a Chebyshev system, then $\{u_1,\dots,u_n\}$ is called a *complete Chebyshev system*. Furthermore, we call a space spanned by a Chebyshev system (resp. complete Chebyshev system) a *Chebyshev space* (resp. *complete Chebyshev space*). It is well known that Chebyshev systems are of much use to study best approximation, interpolation, and quadrature formulas in approximation theory. One can see a lot of good properties and important applications of Chebyshev systems in the books [5, 12] and a survey in [11].

Let $C[a,b]$ be the subspace of $F[a,b]$ which consists of continuous functions. $C[a,b]$ is endowed with the following two norms: For $f\in C[a,b]$,

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|,$$

$$\|f\|_{v1} = |f(a)| + \sup_{a\leqq\alpha<\beta\leqq b} |f(\beta)-f(\alpha)|.$$

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For a system $\{u_1,\dots,u_n\}$ of $C[a,b]$, we consider the following problems on best appoximation from $G=\mathrm{Span}\{u_1,\dots,u_n\}$: For any $f\in C[a,b]$,

(P.1) find $p_f\in G$ with $\|f-p_f\|_\infty=\inf_{p\in G}\|f-p\|_\infty$

and

(P.2) find $q_f\in G$ with $\|f-q_f\|_{v1}=\inf_{q\in G}\|f-q\|_{v1}$.

Now we turn to the third best approximation problem. Let $L^1[a,b]$ be the space of all real-valued Lebesgue integrable functions on $[a,b]$ and let $\mathcal{S}$ be the set of all nongenerate closed subintervals of $[a,b]$. For $I,J\in\mathcal{S}$, if $I\cap J$ has no interior points and $x\leqq y$ for all $x\in I$ and $y\in J$, then it is denoted by $I<J$ for this relation. A system $\{v_1,\dots,v_n\}$ of $L^1[a,b]$ is called a *quasi Chebyshev system* if for any n closed subintervals $I_1,\dots,I_n\in\mathcal{S}$ with $I_1<\cdots<I_n$, the n-th order determinant

$$\begin{vmatrix} \int_{I_1} v_1\,dx & \int_{I_1} v_2\,dx & \cdots & \int_{I_1} v_n\,dx \\ \int_{I_2} v_1\,dx & \int_{I_2} v_2\,dx & \cdots & \int_{I_2} v_n\,dx \\ \vdots & \vdots & & \vdots \\ \int_{I_n} v_1\,dx & \int_{I_n} v_2\,dx & \cdots & \int_{I_n} v_n\,dx \end{vmatrix}\neq 0.$$

And, we call a space spanned by a quasi Chebyshev system a *quasi Chebyshev space*. The definition of a quasi Chebyshev system is introduced by Shi[10] (cf.see integral Tchebysheff systems in [6] and $H_{\mathcal{I}}$ systems in [7]). $L^1[a,b]$ is endowed with a norm $\|\cdot\|_I$ such that $\|f\|_I=\sup_{I\in\mathcal{S}}\left|\int_I f(x)\,dx\right|$. For a system $\{u_1,\dots,u_n\}$ of $L^1[a,b]$, we consider the following problem on best appoximation from $G=\mathrm{Span}\{u_1,\dots,u_n\}$: For any $f\in L^1[a,b]$,

(P.3) find $r_f\in G$ with $\|f-r_f\|_I=\inf_{r\in G}\|f-r\|_I$.

Throughout this paper, p_f,q_f, and r_f denote one of best approximations to a function f in the problems (P.1), (P.2), and (P.3), respectively.

The following classical results of best approximation by Chebyshev systems are well known as the Chebyshev theory.

Theorem 1. (see Cheney[2; Chap. 3]) *Let $\{1,u_1,\dots,u_n\}$ be a system of $(C[a,b],\|\cdot\|_\infty)$ and $G=\mathrm{Span}\{1,u_1,\dots,u_n\}$. Then, the following statements hold:*

(1) *If every $f\in C[a,b]$ has a unique best approximation p_f from G, then G is a Chebyshev space.*

Furthermore, suppose that $\{1,u_1,\dots,u_n\}$ is a Chebyshev system.

(2) *For any $f\in C[a,b]$, $p_f\in G$ is a best approximation to f from G if and only if there exist $n+2$ distinct points $x_1,\dots,x_{n+2}$ $(a\leqq x_1<\cdots<x_{n+2}\leqq b)$ such that*

(i) $|f(x_i)-p_f(x_i)|=\|f-p_f\|_\infty,\quad i=1,2,\dots,n+2,$

(ii) $f(x_i)-p_f(x_i)=-(f(x_{i+1})-p_f(x_{i+1})),\quad i=1,2,\dots,n+1.$

(3) *Every $f\in C[a,b]$ has a unique best approximation p_f from G. Furthermore, there exisits a positive number γ depending on f such that*

$$\|f-p\|_\infty\geqq\|f-p_f\|_\infty+\gamma\|p-p_f\|_\infty\quad \textit{for all } p\in G.$$

(4) *For any $f\in C[a,b]$, the second Remez algorithm is a method for obtaining a sequence of G which converges to the unique best approximation to f from G.*

Now we already know several kinds of Chebyshev type theory. Generalized versions of the Chebyshev theory are treated in [1, 3, 4]. Modified versions are given in [9, 10]. Among them, Shi[10] has proven a Chebyshev type theory in $(C[a,b], \|\cdot\|_I)$, that is to say, analogous results to Theorem 1. In this paper, we show that (P.2) and (P.3) are closely related with (P.1) and hence, Chebyshev type theory hold in (P.2) and (P.3). In particular, Chebyshev type theory in (P.3) is an extension of Shi's results[10]. We study (P.2) in section 2 and (P.3) is considered in section 3.

2. Best Approximation in $C[a,b]$

For $f \in C[a,b]$, we use notations $m(f) = \min_{x\in[a,b]} f(x)$, $M(f) = \max_{x\in[a,b]} f(x)$ and $v_1(f) = \sup_{a\leqq\alpha<\beta\leqq b} |f(\beta) - f(\alpha)|$. First we show relations between best approximations in (P.1) and (P.2).

Proposition 2. *Let $\{1, u_1, \dots, u_n\}$ be a system of $C[a,b]$ and put $G = \mathrm{Span}\{1, u_1, \dots, u_n\}$. For any $f \in C[a,b]$, let p_f and q_f be best approximations to f from G in (P.1) and (P.2), respectively. Then, the following hold:*

(1) $p_f + f(a) - p_f(a)$ *is a best approximation to f from G in* $(C[a,b], \|\cdot\|_{v1})$,

(2) $q_f + \dfrac{m(f-q_f) + M(f-q_f)}{2}$ *is a best approximation to f from G in* $(C[a,b], \|\cdot\|_\infty)$,

(3) $2\|f - p_f\|_\infty = \|f - q_f\|_{v1}$.

Proof. (1) First we see that $v_1(f) = M(f) - m(f)$ for $f \in C[a,b]$. Let p_f be a best approximation to f from G in $(C[a,b], \|\cdot\|_\infty)$. Since $1 \in G$, we have $\|f-p_f\|_\infty = M(f-p_f) = -m(f-p_f)$ for $f \in C[a,b]$. Hence, $v_1(f-p_f) = 2\|f-p_f\|_\infty$. Furthermore, $v_1(f-p) \geqq v_1(f-p_f)$ for all $p \in G$, because if $v_1(f-p_0) < v_1(f-p_f)$ for some $p_0 \in G$, then the function $p_0 + \dfrac{m(f-p_0)+M(f-p_0)}{2}$ is a better approximation to f than p_f in $(C[a,b], \|\cdot\|_\infty)$, which contradicts to the assumption of p_f. Noting that $v_1(f - (p_f + f(a) - p_f(a))) = v_1(f - p_f)$ and $(f - (p_f + f(a) - p_f(a)))(a) = 0$, $p_f + f(a) - p_f(a)$ is a best approximation to f from G in $(C[a,b], \|\cdot\|_{v1})$.

(2) Let q_f be a best approximation to f from G in $(C[a,b], \|\cdot\|_{v1})$. If we consider an approximating function $q_f + \dfrac{m(f-q_f)+M(f-q_f)}{2}$, then the function satisfies

$$\left\| f - \left(q_f + \frac{m(f-q_f)+M(f-q_f)}{2} \right) \right\|_\infty = \frac{1}{2} v_1 \left(f - \left(q_f + \frac{m(f-q_f)+M(f-q_f)}{2} \right) \right).$$

Since $v_1 \left(f - \left(q_f + \dfrac{m(f-q_f)+M(f-q_f)}{2} \right) \right) = v_1(f-q_f)$, we have for any $q \in G$,

$$\|f-q\|_\infty \geqq \frac{1}{2} v_1(f-q) \geqq \frac{1}{2} v_1(f-q_f) = \left\| f - \left(q_f + \frac{m(f-q_f)+M(f-q_f)}{2} \right) \right\|_\infty.$$

This means that the function $q_f + \dfrac{m(f-q_f)+M(f-q_f)}{2}$ is a best approximation to f from G in $(C[a,b], \|\cdot\|_\infty)$.

As for (3), one can see that $2\|f-p_f\|_\infty = \|f-q_f\|_{v1}$ without any difficulty. ■

By Theorem 1 and Proposition 2, we immediately have the following statement.

Theorem 3. (Chebyshev Type Theory in $(C[a,b], \|\cdot\|_{v1})$) *Let* $\{1, u_1, \dots, u_n\}$ *be a system of* $(C[a,b], \|\cdot\|_{v1})$ *and* $G = \mathrm{Span}\{1, u_1, \dots, u_n\}$. *Then, the following statements hold:*

(1) *If every* $f \in C[a,b]$ *has a unique best approximation from* G, *then* G *is a Chebyshev space.*

Furthermore, suppose that $\{1, u_1, \dots, u_n\}$ *is a Chebyshev system.*

(2) *For any* $f \in C[a,b]$, $q_f \in G$ *is a best approximation to* f *from* G *if and only if* $|(f-q_f)(a)| = 0$ *and there exist* $n+2$ *distinct points* $x_1, \dots, x_{n+2}$ $(a \leqq x_1 < \cdots < x_{n+2} \leqq b)$ *such that*

(i) $|(f(x_{i+1}) - q_f(x_{i+1})) - (f(x_i) - q_f(x_i))| = \|f - q_f\|_{v1}, \quad i = 1, 2, \dots, n+1,$

(ii) $(f(x_{i+1}) - q_f(x_{i+1})) - (f(x_i) - q_f(x_i)) = -((f(x_{i+2}) - q_f(x_{i+2})) - (f(x_{i+1}) - q_f(x_{i+1}))),$
$i = 1, 2, \dots, n.$

(3) *Every* $f \in C[a,b]$ *has a unique best approximation* q_f *from* G. *Furthermore, there exisits a positive number* η *depending on* f *such that*

$$\|f - q\|_{v1} \geqq \|f - q_f\|_{v1} + \eta\|q - q_f\|_{v1} \quad \textit{for all } q \in G.$$

(4) *For any* $f \in C[a,b]$, *let* $\{p_n\}$ *be a sequence of* G *obtained by the second Remez algorithm in Theorem 1 (4). Then, the sequence* $\{p_n + f(a) - p_n(a)\}$ *converges to the unique best approximation to* f *from* G.

Proof. Since from Theorem 1 and Proposition 2, it is obvious that (1) and (2) hold, we only show the inequality in (3) and convergence of the sequence in (4).

As for (3), it is sufficient to show the existence of positive number $\eta(< 1)$ such that for any $f \in C[a,b]$

$$\|f - q\|_{v1} \geqq \|f - q_f\|_{v1} + \eta\|q - q_f\|_{v1} \quad \textit{for all } q \in G, \tag{2.1}$$

or

$$|(f-q)(a)| + v_1(f-q) \geqq |(f-q_f)(a)| + v_1(f-q_f) + \eta|(q-q_f)(a)| + \eta v_1(q-q_f) \quad \textit{for all } q \in G.$$

Since q_f is a best approximation to f, $f(a) = q_f(a)$. Hence, we have

$$|(f - q)(a)| \geqq |(f - q_f)(a)| + \eta|(q - q_f)(a)| \quad \textit{for all } \eta \ (0 < \eta < 1). \tag{2.2}$$

Put $p_f = q_f + \dfrac{m(f - q_f) + M(f - q_f)}{2}$, which is the unique best approximation to f from G in $(C[a,b], \|\cdot\|_\infty)$. Noting that $v_1(g) = v_1(g + c)$ for any $g \in C[a,b]$ and any $c \in \mathbf{R}$, by Proposition 2, we obtain

$$v_1(f - q) = 2\left\|f - \left(q + \frac{m(f - q) + M(f - q)}{2}\right)\right\|_\infty, \qquad v_1(f - q_f) = 2\|f - p_f\|_\infty,$$

and

$$\left\|\left(q+\frac{m(f-q)+M(f-q)}{2}\right)-p_f\right\|_\infty \geqq \frac{1}{2}v_1\left(\left(q+\frac{m(f-q)+M(f-q)}{2}\right)-p_f\right)=\frac{1}{2}v_1(q-q_f).$$

On the other hand, since by Theorem 1 (3),

$$\left\|f-\left(q+\frac{m(f-q)+M(f-q)}{2}\right)\right\|_\infty \geqq \|f-p_f\|_\infty+\gamma\left\|\left(q+\frac{m(f-q)+M(f-q)}{2}\right)-p_f\right\|_\infty,$$

holds, we have

$$v_1(f-q)\geqq v_1(f-q_f)+\gamma v_1(q-q_f). \tag{2.3}$$

From (2.2) and (2.3), (2.1) follows immdiately.

For any $f\in C[a,b]$, let $\{p_n\}$ be a sequence of G obtained by the second Remez algorithm in Theorem 1 (4). Since $\{p_n\}$ converges to the unique best approximation p_f to f from G, $\{p_n+f(a)-p_n(a)\}$ converges to $p_f+f(a)-p_f(a)$, which is the unique best approximation to f from G in $(C[a,b],\|\cdot\|_{v1})$. ■

3. Chebyshev Type Theory in $(L^1[a,b],\|\cdot\|_I)$

First we consider a property of complete Chebyshev systems which is necessary to show a Chebyshev type theory in $(L^1[a,b],\|\cdot\|_I)$.

Definition 1. Let $\{1,u_1,\dots,u_n\}$ be a system of $C[a,b]$. Then, $\{1,u_1,\dots,u_n\}$ is said to have (*) *property* if for any n subintervals $[a_1,b_1],\dots,[a_n,b_n]\in\mathcal{S}$ with $[a_1,b_1]<\cdots<[a_n,b_n]$, the n-th order determinant

$$\begin{vmatrix} u_1(b_1)-u_1(a_1) & u_2(b_1)-u_2(a_1) & \cdots & u_n(b_1)-u_n(a_1)\\ u_1(b_2)-u_1(a_2) & u_2(b_2)-u_2(a_2) & \cdots & u_n(b_2)-u_n(a_2)\\ \vdots & \vdots & & \vdots\\ u_1(b_n)-u_1(a_n) & u_2(b_n)-u_2(a_n) & \cdots & u_n(b_n)-u_n(a_n)\end{vmatrix}\neq 0.$$

For convenience, we write $u(I),I=[\alpha,\beta]$ for $u(\beta)-u(\alpha)$.

Now we obtain

Lemma 4. *If a system $\{1,u_1,\dots,u_n\}$ of $C[a,b]$ has (*) property, then $\{1,u_1,\dots,u_n\}$ is a Chebyshev system.*

Proof. On the contrary, suppose that $\{1,u_1,\dots,u_n\}$ is not a Chebyshev system. Then, there exists a nontrivial linear combination $c_0+c_1u_1(x)+\cdots+c_nu_n(x)$ which possesses $n+1$ zeros $x_0,x_1,\dots,x_n$ $(x_0<x_1<\cdots<x_n)$ in $[a,b]$. And $c_1^2+c_2^2+\cdots+c_n^2\neq 0$ because $c_0+c_1u_1(x)+\cdots+c_nu_n(x)$ can't possess $n+1$ zeros for $c_0\neq 0,c_1=\cdots=c_n=0$. Since it is seen that

$$(c_0+c_1u_1+\cdots+c_nu_n)(x_i)-(c_0+c_1u_1+\cdots+c_nu_n)(x_{i-1})=0,\quad i=1,2,\dots,n,$$

if we put $I_i=[x_{i-1},x_i],i=1,2,\dots,n$, we obtain

$$\begin{pmatrix} u_1(I_1) & u_2(I_1) & \cdots & u_n(I_1)\\ u_1(I_2) & u_2(I_2) & \cdots & u_n(I_2)\\ \vdots & \vdots & & \vdots\\ u_1(I_n) & u_2(I_n) & \cdots & u_n(I_n)\end{pmatrix}\begin{pmatrix} c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\ 0\end{pmatrix}.$$

But this contradicts the conditon that $\{1, u_1, \dots, u_n\}$ of $C[a,b]$ has (*) property. ■

Remark 1. As is easily seen, Lemma 4 holds without continuity of $u_1, u_2, \dots, u_n$.

To prove the converse of Lemma 4, we prepare some definitions and propositions.

Definition 2. (Zielke[12; p.33, 34]) (1) Let f be a real-valued function on $[a,b]$. Points $x_1, x_2, \dots, x_k$ $(a \leqq x_1 < x_2 < \cdots < x_k \leqq b)$ are called a *strong* (*resp. weak*) *oscillation of f of length k* if

$$(-1)^i(f(x_{i+1}) - f(x_i)) > 0 \ \ (\textit{resp.} \ \leqq 0) \ , \qquad i = 1, 2, \dots, k-1$$

or

$$(-1)^i(f(x_{i+1}) - f(x_i)) < 0 \ \ (\textit{resp.} \ \leqq 0) \ , \qquad i = 1, 2, \dots, k-1$$

hold.
(2) Let U be an n dimensional space which consists of real-valued functions on $[a,b]$. Then U is called a *strong* (*resp.weak*) *oscillation space* if no nonconstant $f \in U$ has a weak (resp. strong) oscillation of length $n+1$.

We show a slightly special version of Theorem 8.8 in Zielke[12] for this paper.

Proposition 5. (Zielke[12; Theorem 8.8 in p.39]) *Let U be an $(n+1)$-dimensional Chebyshev space of $F[a,b]$ which contains contant functions. Then the following statements are equivalent:*
(1) *U is a strong oscillation space.*
(2) *U is a weak oscillation space.*
(3) *U has a basis $\{1, u_1, \dots, u_n\}$ which is a complete Chebyshev system.*

From Lemma 4 and Proposition 5, we have a property of a complete Chebyshev system.

Theorem 6. *Let U be an $(n+1)$-dimensional space of $C[a,b]$ which contains contant functions. Then the following statements are equivalent:*
(1) *U has a basis $\{1, u_1, \dots, u_n\}$ which has (*) property.*
(2) *U has a basis $\{1, v_1, \dots, v_n\}$ which is a complete Chebyshev system.*

Proof. (1) $\Rightarrow$ (2). U is a Chebyshev space by Lemma 4. Hence, by Proposition 5, it is sufficient to show that U is a weak oscillation space. On the contrary, suppose that U isn't a weak oscillation space. There exists a nonconstant $u \in U$ and $n+2$ ponits $x_1, x_2, \dots, x_{n+2}$ $(x_1 < x_2 < \cdots < x_{n+2})$ in $[a,b]$ such that

$$(-1)^i(u(x_{i+1}) - u(x_i)) > 0 \ \ , \qquad i = 1, 2, \dots, n+1.$$

Put $d = \min\{(-1)^i(u(x_{i+1}) - u(x_i)) \mid i = 1, 2, \dots, n+1\}$. Since $u = c_0 + \sum_{i=1}^n c_i u_i$ is continuous, there exist n subintervals $I_j = [a_j, b_j], j = 1, 2, \dots, n$ of $[a,b]$ satisfying that

$$x_j < a_j < x_{j+1} < b_j < x_{j+2} \ \ , \qquad j = 1, 2, \dots, n,$$

$$I_1 < I_2 < \cdots < I_n,$$

$$u(a_j) = u(b_j) = u(x_{j+1}) + \frac{(-1)^{j+1}d}{3} \ \ , \qquad j = 1, 2, \dots, n.$$

Noting that $u = c_0 + \sum_{i=1}^n c_i u_i$ isn't a nonconstant function, $c_1^2 + \cdots + c_n^2 \neq 0$. Thus, we have

$$\begin{pmatrix} u_1(I_1) & u_2(I_1) & \cdots & u_n(I_1) \\ u_1(I_2) & u_2(I_2) & \cdots & u_n(I_2) \\ \vdots & \vdots & & \vdots \\ u_1(I_n) & u_2(I_n) & \cdots & u_n(I_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But this contradicts the conditon that $\{1, u_1, \ldots, u_n\}$ of $C[a, b]$ has (*) property.
(2) $\Rightarrow$ (1). Let $\{1, v_1, \ldots, v_n\}$ be a basis which is a complete Chebyshev system. We show that the system $1, v_1, \ldots, v_n$ has (*) property. On the contrary, suppose that $\{1, v_1, \ldots, v_n\}$ doesn't have (*) property. Then, there exist n subintervals $I_1, \ldots, I_n (I_1 < \cdots < I_n)$ such that

$$\begin{vmatrix} v_1(I_1) & v_2(I_1) & \cdots & v_n(I_1) \\ v_1(I_2) & v_2(I_2) & \cdots & v_n(I_2) \\ \vdots & \vdots & & \vdots \\ v_1(I_n) & v_2(I_n) & \cdots & v_n(I_n) \end{vmatrix} = 0.$$

Hence, there is a $(c_1, c_2, \ldots, c_n) \in \mathbf{R}^n - \{\mathbf{0}\}$ satisfying that $\sum_{i=1}^n c_i v_i(I_j) = 0, j = 1, 2, \ldots, n$. If we write $I_j = [\alpha_j, \beta_j]$ for $j = 1, 2, \ldots, n$, then

$$\sum_{i=1}^n c_i v_i(\alpha_j) = \sum_{i=1}^n c_i v_i(\beta_j) \quad , \quad j = 1, 2, \ldots, n.$$

Since $\{1, v_1, \ldots, v_n\}$ is a complete Chebyshev system, there exist $\gamma_j \in (\alpha_j, \beta_j), j = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^n c_i v_i(\alpha_j) < \sum_{i=1}^n c_i v_i(\gamma_j) \quad \text{or} \quad \sum_{i=1}^n c_i v_i(\alpha_j) > \sum_{i=1}^n c_i v_i(\gamma_j).$$

Clearly $\alpha_1, \gamma_1, \beta_1$ are a strong oscillation of $\sum_{i=1}^n c_i v_i$ of length 3 and some 4 points among points $\alpha_i, \gamma_i, \beta_i, i = 1, 2$ are a strong oscillation of $\sum_{i=1}^n c_i v_i$ of length 4. Analogously, we can find $n+2$ points among points $\alpha_i, \gamma_i, \beta_i, i = 1, 2, \ldots, n$ which are a strong oscillation of $\sum_{i=1}^n c_i v_i$ of length $n+2$. Since $\sum_{i=1}^n c_i v_i$ is nonconstant, this contradicts Proposition 5. Thus, $\{1, v_1, \ldots, v_n\}$ has (*) property. ∎

Now we proceed to the Chebyshev theory in $(L^1[a, b], \|\cdot\|_I)$. Let $AC[a, b]$ be the subspace of $F[a, b]$ which consists of absolutely continuous functions. Then, we begin by the following lemma.

Lemma 7. *Let $\{u_1, \ldots, u_n\}$ be a system in $L^1[a, b]$. Put $w_0(x) = 1$ and $w_i(x) = \int_a^x u_i(t)\, dt \in AC[a, b], i = 1, 2, \ldots, n$. If for any $f \in AC[a, b]$, $c_0^* w_0 + \sum_{i=1}^n c_i^* w_i$ is a best approximation to f from* Span$\{w_0, w_1, \ldots, w_n\}$ *in $(AC[a, b], \|\cdot\|_{v1})$, then $\sum_{i=1}^n c_i^* u_i$ is a best approximation to f' from* Span$\{u_1, \ldots, u_n\}$ *in $(L^1[a, b], \|\cdot\|_I)$.*

Proof. Suppose that $c_0^* w_0 + \sum_{i=1}^n c_i^* w_i$ is a best approximation to f from Span$\{w_0, w_1, \ldots, w_n\}$ in $(AC[a, b], \|\cdot\|_{v1})$. For any $c_0 w_0 + \sum_{i=1}^n c_i w_i \in$ Span$\{w_0, w_1, \ldots, w_n\}$, we observe that

$$v_1(f - (c_0^* w_0 + \sum_{i=1}^n c_i^* w_i)) \leqq v_1(f - (c_0 w_0 + \sum_{i=1}^n c_i w_i)),$$

because if $v_1(f - (c_0^* w_0 + \sum_{i=1}^n c_i^* w_i)) > v_1(f - (c_0 w_0 + \sum_{i=1}^n c_i w_i))$ for some $c_0 w_0 + \sum_{i=1}^n c_i w_i \in$ Span$\{w_0, w_1, \ldots, w_n\}$, then $f(a) w_0 + \sum_{i=1}^n c_i w_i$ is a better approximation to f than $c_0^* w_0 + \sum_{i=1}^n c_i^* w_i$, which contradicts the assumption of $c_0^* w_0 + \sum_{i=1}^n c_i^* w_i$. Hence, we have for any $\sum_{i=1}^n c_i u_i \in$ Span$\{u_1, \ldots, u_n\}$,

$$\left\| f' - \sum_{i=1}^{n} c_i u_i \right\|_I = v_1 \left(f - \sum_{i=1}^{n} c_i w_i \right)$$
$$\geqq v_1 \left(f - (c_0^* w_0 + \sum_{i=1}^{n} c_i^* w_i) \right) = v_1 \left(f - \sum_{i=1}^{n} c_i^* w_i \right) = \left\| f' - \sum_{i=1}^{n} c_i^* u_i \right\|_I .$$

Thus, $\sum_{i=1}^{n} c_i^* u_i$ is a best approximation to f' from $\mathrm{Span}\{u_1, \dots, u_n\}$ in $(L^1[a,b], \|\cdot\|_I)$. ■

We are in position to state

Theorem 8. (Chebyshev Type Theory in $(L^1[a,b], \|\cdot\|_I)$) *Let $\{u_1, \dots, u_n\}$ be an approximating system $(L^1[a,b], \|\cdot\|_I)$ and $G_k = \mathrm{Span}\{u_1, \dots, u_k\}$, $k = 1, 2, \dots, n$. Then, the following statements hold:*

(1) *If every $f \in L^1[a,b]$ has a unique best approximation from $G_k, k = 1, 2, \dots, n$, then G_n is a quasi Chebyshev space.*

Furthermore, suppose that G_n is a quasi Chebyshev space.

(2) *For any $f \in L^1[a,b]$, $r_f \in G_n$ is a best approximation to f from G_n if and only if there exist $n+1$ subintervals $I_1, I_2, \dots, I_{n+1} \in \mathcal{S}$ with $I_1 < \cdots < I_{n+1}$ of $[a,b]$ such that*

(i) $|\int_{I_j} (f - r_f)\, dx| = \|f - r_f\|_I, \quad j = 1, 2, \dots, n+1,$

(ii) $\int_{I_j} (f - r_f)\, dx = -\int_{I_{j+1}} (f - r_f)\, dx \quad j = 1, 2, \dots, n.$

(3) *Every $f \in L^1[a,b]$ has a unique best approximation r_f from G_n. Furthermore, there exisits a positive number η depending on f such that*

$$\|f - r\|_I \geqq \|f - r_f\|_I + \eta \|q - r_f\|_I \quad \textit{for all } r \in G_n. \tag{2.4}$$

(4) *Let f be any function in $L^1[a,b]$ and put $\varphi_f(x) = \int_a^x f(t)\, dt$. Let $\{w_0 = 1, w_1 = \int_a^x u_1(t)\, dt, \dots, w_n = \int_a^x u_n(t)\, dt\}$ be an approximating system of $(C[a,b], \|\cdot\|_{v1})$. For $\varphi_f \in AC[a,b]$, let $\{q_k\}$ be a sequence of $\mathrm{Span}\{w_0, \dots, w_n\}$ obtained by the algorithm in Theorem 3 (4). Then, the sequence $\{q_n'\}$ converges to the unique best approximation to f from G_n.*

Proof. (1) We show that the system $\{w_0 = 1, w_1 = \int_a^x u_1(t)\, dt, \dots, w_n = \int_a^x u_n(t)\, dt\}$ has (*) property. By Theorem 6, it is sufficient to prove that each system $\{1, w_1, \dots, w_k\}, k = 1, 2, \dots, n$ is a Chebyshev system.

Let f be any function in $AC[a,b]$. From the assumption, $f' \in L^1[a,b]$ has a unique best approximation $\sum_{i=1}^{k} c_i u_i$ from G_k in $(L^1[a,b], \|\cdot\|_I)$. Hence, by Proposition 7, $f \in AC[a,b]$ has a unique best approximation $q_f = f(a)w_0 + \sum_{i=1}^{k} c_i w_i$ from $\mathrm{Span}\{1, w_1, \dots, w_k\}$ in $(AC[a,b], \|\cdot\|_{v1})$. Moreover, by Proposition 2, f has a unique best approximation $q_f + \dfrac{m(f - q_f) + M(f - q_f)}{2}$ from $\mathrm{Span}\{w_0, w_1, \dots, w_k\}$ in $(AC[a,b], \|\cdot\|_\infty)$. Using the same proof as that of Theorem 1 (1), we can verify that $\{1, w_1, \dots, w_k\}$ is a Chebyshev system. But, to make sure, we show a proof for this setting.

On the contrary, suppose that $\{w_0, w_1, \dots, w_k\}$ isn't a Chebyshev system. Then, there exist $k+1$ distinct points $x_0, x_1, \dots, x_k$ in $[a,b]$ such that

$$\det A := \begin{vmatrix} w_0(x_0) & w_1(x_0) & \cdots & w_k(x_0) \\ w_0(x_1) & w_1(x_1) & \cdots & w_k(x_1) \\ \vdots & \vdots & & \vdots \\ w_0(x_k) & w_1(x_k) & \cdots & w_k(x_k) \end{vmatrix} = 0.$$

Since A is singular, there are $\mathbf{c} = (c_i), \mathbf{d} = (d_i) \in \mathbf{R}^{k+1} - \{\mathbf{0}\}$ satisfying that

$$\sum_{i=0}^{k} c_i w_j(x_i) = 0 \ , \quad j = 0, 1, \dots, k \quad , \qquad \sum_{j=0}^{k} d_j w_j(x_i) = 0 \ , \quad i = 0, 1, \dots, k. \tag{2.5}$$

Without loss of generality, we assume that $h(x) = \sum_{j=0}^{k} d_j w_j(x)$ satisfies $\|h\|_\infty < 1$. Let us consider $f_1 \in AC[a,b]$ with $\|f_1\|_\infty = 1$ and $f_1(x_i) = \operatorname{sign}(c_i), i = 0, 1, \dots, n$ (Polygonal lines are suitable for f_1.). If we put $f(x) = f_1(x)(1 - |h(x)|)$, then f is a function in $AC[a,b]$ such that

$$\|f\|_\infty = 1 \qquad \text{and} \qquad f(x_i) = f_1(x_i)(1 - |h(x_i)|) = f_1(x_i) = \operatorname{sign}(c_i) \quad , \quad i = 0, 1, \dots, k.$$

Now we show that $\left\| f - \sum_{i=0}^{k} a_i w_i \right\|_\infty \geqq 1$ for all $\mathbf{a} = (a_i) \in \mathbf{R}^{k+1}$. On the contrary, suppose that $\left\| f - \sum_{i=0}^{k} a_i w_i \right\|_\infty < 1$ for some $\mathbf{a} = (a_i) \in \mathbf{R}^{k+1}$. For any x_j with $c_j \neq 0$, we have

$$\operatorname{sign}\left(\sum_{i=0}^{k} a_i w_i(x_j) \right) = \operatorname{sign}(f(x_j)) = \operatorname{sign}(c_j) \neq 0.$$

Hence, $\sum_{i=0}^{k} c_j \left(\sum_{i=0}^{k} a_i w_i(x_j) \right) > 0$ holds. But we see that by (2.5),

$$\sum_{j=0}^{k} c_j \left(\sum_{i=0}^{k} a_i w_i(x_j) \right) = \sum_{i=0}^{k} a_i \left(\sum_{j=0}^{k} c_j w_i(x_j) \right) = 0,$$

which leads to a contradiction. Hence, $\left\| f - \sum_{i=0}^{k} a_i w_i \right\|_\infty \geqq 1$ for all $\mathbf{a} = (a_i) \in \mathbf{R}^{k+1}$.

We consider $\gamma h(x), 0 \leqq \gamma \leqq 1$ as approximations to f from $\operatorname{Span}\{w_0, w_1, \dots, w_k\}$. Then, we obtain for any $x \in [a,b]$,

$$|f(x) - \gamma h(x)| \leqq |f(x)| + \gamma |h(x)| = |f_1(x)|(1 - |h(x)|) + \gamma |h(x)| \leqq 1 + (\gamma - 1)|h(x)| \leqq 1.$$

This implies that $\|f - \sum_{i=0}^{k} \gamma d_i w_i\|_\infty = 1$ for $0 \leqq \gamma \leqq 1$ and $\sum_{i=0}^{k} \gamma d_i w_i$ for $0 \leqq \gamma \leqq 1$ are best approximations to f from $\operatorname{Span}\{w_0, w_1, \dots, w_k\}$ in $(AC[a,b], \|\cdot\|_\infty)$. But this contradicts to the uniqueness of best approximation to f from $\operatorname{Span}\{w_0, w_1, \dots, w_k\}$.
(2) From Theorem 3 (2) and Lemma 7, (2) follows immdediately.
(3) Let f be any function in $L^1[a,b]$ and put $\varphi_f(x) = \int_a^x f(t)\,dt$, $w_0(x) = 1$, $w_i(x) = \int_a^x u_i(t)\,dt, i = 1, 2, \dots, n$. Since $\{u_1, \dots, u_n\}$ is a quasi Chebyshev system, $\{w_0, w_1, \dots, w_n\}$ has (*) property. By Theorem 3 (3), there exist a unique best approximation $\sum_{i=0}^{n} c_i^* w_i$ to φ_f from $\operatorname{Span}\{w_0, w_1, \dots, w_n\}$ and a positive number η depending on f such that for any $\sum_{i=0}^{n} c_i w_i \in \operatorname{Span}\{w_0, w_1, \dots, w_n\}$,

$$\left\| \varphi_f - \sum_{i=0}^{n} c_i w_i \right\|_{v1} \geqq \left\| \varphi_f - \sum_{i=0}^{n} c_i^* w_i \right\|_{v1} + \eta \left\| \sum_{i=0}^{n} c_i w_i - \sum_{i=0}^{n} c_i^* w_i \right\|_{v1}.$$

Since $\varphi_f(a) = 0$ and $w_i(a) = 0, i = 0, 1, \dots, n$, we have

$$\left\| \varphi_f - \sum_{i=0}^{n} c_i w_i \right\|_{v1} = \left\| f - \sum_{i=1}^{n} c_i u_i \right\|_I , \quad \left\| \varphi_f - \sum_{i=0}^{n} c_i^* w_i \right\|_{v1} = \left\| f - \sum_{i=1}^{n} c_i^* u_i \right\|_I ,$$

and

$$\left\| \sum_{i=0}^{n} c_i w_i - \sum_{i=0}^{n} c_i^* w_i \right\|_{v1} = \left\| \sum_{i=1}^{n} c_i u_i - \sum_{i=1}^{n} c_i^* u_i \right\|_I .$$

Thus we see that the inequality (2.4) holds.
(4) Put $q_k = \sum_{i=0}^{n} a_i^k w_i, k = 1, 2, \dots$. Since φ_f has a unique best approximation $q = \sum_{i=0}^{n} a_i w_i$ from Span$\{w_0, w_1, \dots, w_n\}$, each sequence $\{a_i^k\}, i = 0, \dots, n$ converges to a_i. Hence, $\{q_k'\}$ converges to $q' = \sum_{i=0}^{n} a_i w_i'$ in $(L^1[a,b], \|\cdot\|_I)$ which is a unique best approximation to f from Span$\{u_1, \dots, u_n\}$. ■

Remark 2. In $(C[a,b], \|\cdot\|_I)$, Shi[10] gave constructive proofs of Theorem 8 (2), (3), and Kitahara and Sakamori[8] showed an algorithm to obtain a best approximation from a quasi Chebyshev space.

Finally, we give a problem in order to find best approximation problems in which Chebyshev type theory holds.

Problem. For a positive integer k, let $\|\cdot\|_{vk}$ be a norm on $C[a,b]$ such that

$$\|f\|_{vk} = |f(a)| + \sup_{a \leqq x_0 < \cdots < x_k \leqq b} \sum_{i=0}^{k-1} |f(x_{i+1}) - f(x_i)| , \qquad f \in C[a,b].$$

Let G be a finite dimensional complete Chebyshev space of $C[a,b]$ which contains constant functions. If G is an approximating space, then is it true that Chebyshev type theory holds?

Remark 3. In case $k = 1$, the problem stated above is (P.2) itself. Let us consider the function space $C^1[a,b]$ of all real-valued continuously differentiable functions on $[a,b]$. As a norm on $C^1[a,b]$, we consider $\|f\|_{v\infty} = \lim_{k\to\infty} \|f\|_{vk} = |f(a)| + v(f), f \in C^1[a,b]$, where $v(f)$ denotes the total variation of f. It is stated in [7] that every finite dimensional complete Chebyshev space which contains constant functions isn't a unicity space in $(C^1[a,b], \|\cdot\|_{v\infty})$. Hence, we have a negative answer of Problem for $k = \infty$.

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School of Science and Technology, Kwansei Gakuin University
Sanda 669-1337, Japan
E-mail : z95040@ksc.kwansei.ac.jp