CHARACTERIZATION OF TOTALLY GEODESIC SUBMANIFOLDS IN TERMS OF FRENET CURVES

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Dedicated to Professor Kiyosi Yamaguti

ABSTRACT. In this paper, we give a characterization of totally geodesic submanifolds of a Riemannian manifold in terms of Frenet curves of order 2.

1 Introduction Let $f : M^n \to \widetilde{M}^m$ be an isometric immersion of a Riemannian manifold M^n into an ambient Riemannian manifold $\widetilde{M}^m$. By observing the extrinsic shape of curves of a submanifold M^n, we can study the properties of the immersion f in some cases.

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M^n parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ along γ and positive smooth functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ satisfying the following system of ordinary equations

$$\nabla_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \ldots, d, \tag{1.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M^n. The equation (1.1) is said to be the *Frenet formula* for the Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \ldots, d-1)$ and the orthonormal frame fields $\{V_1, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. A Frenet curve is called a Frenet curve *of order d* if it is a Frenet curve of proper order $r(\leq d)$. If the Riemannian manifold M^n is complete, every Frenet curve γ is defined for $-\infty < s < \infty$ (cf.[1],[5]). A Frenet curve of proper order 2 with constant curvature $k(> 0)$, that is a curve which satisfies $\nabla_{\dot{\gamma}} V_1(s) = kV_2(s)$, $\nabla_{\dot{\gamma}} V_2(s) = -kV_1(s)$ and $V_1(s) = \dot{\gamma}$, is called a *circle* of curvature k. We regard a geodesic as a circle of null curvature.

In their paper [4], Nomizu and Yano proved a well-known theorem which states that a submanifold M^n is an extrinsic sphere of $\widetilde{M}^m$, that is a totally umbilic submanifold with parallel mean curvature vector, if and only if all circles of some positive curvature k in M^n are circles in the ambient space $\widetilde{M}^m$. Recently, Kôzaki and Maeda improved this theorem, namely they showed that M^n is an extrinsic sphere of $\widetilde{M}^m$ if and only if all circles of some positive curvature k in M^n are Frenet curves of order 2 in $\widetilde{M}^m$ ([2]).

In this note, motivated by the results of [2], we study the case that the curve in the submanifold M^n is a Frenet curve of proper order 2. The purpose of this paper is to give a characterization of totally geodesic submanifolds in any Riemannian manifold from this point of view (Theorem 1).

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2 Main Result Let M^n, $\widetilde{M}^m$ be Riemannian manifolds and $f : M^n \to \widetilde{M}^m$ an isometric immersion. Throughout this paper we will identify a vector X of M^n with a vector $f_*(X)$ of $\widetilde{M}^m$. The Riemannian metrics on M^n, $\widetilde{M}^m$ are denoted by the same notation $\langle\ ,\ \rangle$. We here review fundamental equations in submanifold theory. We denote by ∇ and $\widetilde{\nabla}$ the covariant differentiations of M^n and $\widetilde{M}^m$, respectively. Then the second fundamental form σ of the immersion f is defined by

$$\sigma(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y, \tag{2.1}$$

where X and Y are vector fields tangent to M^n. For a vector field ξ normal to M^n, we write

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\widetilde{\nabla}_X\xi$. We define the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\bar{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.3}$$

We shall prove the following:

Theorem 1 *Let M^n be a connected Riemannian submanifold of a Riemannian manifold $(\widetilde{M}^m, \langle\ ,\ \rangle)$ through an isometric immersion f. Assume that there exists a nonconstant positive smooth function $\kappa = \kappa(s)$ satisfying that for every Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^m$. Then M^n is totally geodesic in $\widetilde{M}^m$.*

Proof. Let x be an arbitrary point of M^n and X, Y an orthonormal pair of vectors in $T_x(M^n)$. Without loss of generality we suppose that the function $\kappa = \kappa(s)$ is defined on some open interval $-\epsilon < s < \epsilon$. Let $\gamma = \gamma(s)$ $(|s| < \epsilon)$ be a Frenet curve of proper order 2 with curvature κ in M^n satisfying the equations

$$\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s, \quad \nabla_{\dot\gamma} Y_s = -\kappa\dot\gamma \tag{2.4}$$

and the initial condition

$$\gamma(0) = x, \quad \dot\gamma(0) = X \quad \text{and} \quad Y_0 = Y. \tag{2.5}$$

As a matter of course, the existence and uniqueness theorem on solutions of ordinary differential equations guarantees that there exists locally a unique Frenet curve $\gamma = \gamma(s)$ of proper order 2 satisfying (2.4) and (2.5).

By assumption the curve $f \circ \gamma$ is a Frenet curve of order 2 in $\widetilde{M}^m$. Then there exist a (nonnegative) function $\tilde\kappa = \tilde\kappa(s)$ and a field of unit vectors $\widetilde{Y}_s$ along $f \circ \gamma$ in $\widetilde{M}^m$ satisfying that

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \tilde\kappa \widetilde{Y}_s, \quad \widetilde{\nabla}_{\dot\gamma}\widetilde{Y}_s = -\tilde\kappa\dot\gamma. \tag{2.6}$$

Then by (2.1) we have

$$\tilde\kappa \widetilde{Y}_s = \kappa Y_s + \sigma(\dot\gamma, \dot\gamma), \tag{2.7}$$

so that

$$\tilde\kappa^2 = \kappa^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2. \tag{2.8}$$

We here note that $\tilde{\kappa}(s) > 0$ for each s, because $\kappa(s) > 0$ for for each s.

Now we shall compute the covariant differentiation of (2.7): For the left-hand side, by using (2.6) and (2.7) we see

$$\begin{aligned} \widetilde{\nabla}_{\dot{\gamma}}(\tilde{\kappa}\widetilde{Y}_s) &= \dot{\tilde{\kappa}}\widetilde{Y}_s + \tilde{\kappa}\widetilde{\nabla}_{\dot{\gamma}}\widetilde{Y}_s \\ &= \frac{\dot{\tilde{\kappa}}}{\tilde{\kappa}}(\kappa Y_s + \sigma(\dot{\gamma}, \dot{\gamma})) - \tilde{\kappa}^2\dot{\gamma}. \end{aligned} \tag{2.9}$$

For the right-hand side, by using (2.1), (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} &\widetilde{\nabla}_{\dot{\gamma}}(\kappa Y_s + \sigma(\dot{\gamma}, \dot{\gamma})) \\ &\quad = \dot{\kappa}Y_s + \kappa\widetilde{\nabla}_{\dot{\gamma}}Y_s - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + D_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})) \\ &\quad = \dot{\kappa}Y_s + \kappa(\nabla_{\dot{\gamma}}Y_s + \sigma(\dot{\gamma}, Y_s)) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) \\ &\quad = \dot{\kappa}Y_s - \kappa^2\dot{\gamma} + 3\kappa\sigma(\dot{\gamma}, Y_s) - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}). \end{aligned} \tag{2.10}$$

We compare the tangential components and the normal components for the submanifold M^n in (2.9) and (2.10), respectively. Then we get the following:

$$\dot{\tilde{\kappa}}\kappa Y_s - \tilde{\kappa}^3\dot{\gamma} = \tilde{\kappa}(\dot{\kappa}Y_s - \kappa^2\dot{\gamma} - A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}), \tag{2.11}$$

$$\dot{\tilde{\kappa}}\sigma(\dot{\gamma}, \dot{\gamma}) = \tilde{\kappa}(3\kappa\sigma(\dot{\gamma}, Y_s) + (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma})). \tag{2.12}$$

The equation (2.12) gives at $s = 0$

$$\dot{\tilde{\kappa}}(0)\sigma(X, X) = \tilde{\kappa}(0)(3\kappa(0)\sigma(X, Y) + (\bar{\nabla}_X\sigma)(X, X)),$$

so that

$$\dot{\tilde{\kappa}}(0)\tilde{\kappa}(0)\sigma(X, X) = \tilde{\kappa}(0)^2(3\kappa(0)\sigma(X, Y) + (\bar{\nabla}_X\sigma)(X, X)). \tag{2.13}$$

On the other hand, we see from (2.8)

$$\tilde{\kappa}^2(0) = \kappa^2(0) + \|\sigma(X, X)\|^2, \tag{2.14}$$

and for each s

$$\begin{aligned} 2\dot{\tilde{\kappa}}\tilde{\kappa} &= 2\dot{\kappa}\kappa + \frac{d}{ds}\langle\sigma(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle \\ &= 2\dot{\kappa}\kappa + 2\langle D_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})), \sigma(\dot{\gamma}, \dot{\gamma})\rangle \\ &= 2\dot{\kappa}\kappa + 2\langle(\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle + 4\langle\sigma(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle. \end{aligned}$$

So we have at $s = 0$

$$\begin{aligned} \dot{\tilde{\kappa}}(0)\tilde{\kappa}(0) = \dot{\kappa}(0)\kappa(0) &+ \langle(\bar{\nabla}_X\sigma)(X, X), \sigma(X, X)\rangle \\ &+ 2\kappa(0)\langle\sigma(Y, X), \sigma(X, X)\rangle. \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.13), we obtain

$$\begin{aligned} &(\dot{\kappa}(0)\kappa(0) + \langle(\bar{\nabla}_X\sigma)(X, X), \sigma(X, X)\rangle + 2\kappa(0)\langle\sigma(Y, X), \sigma(X, X)\rangle)\sigma(X, X) \\ &= \tilde{\kappa}(0)^2(3\kappa(0)\sigma(X, Y) + (\bar{\nabla}_X\sigma)(X, X)). \end{aligned} \tag{2.16}$$

Here we consider anther Frenet curve $\tau = \tau(s)$ ($|s| < \epsilon$) of proper order 2 with the same curvature κ in M^n satisfying the equations $\nabla_{\dot{\tau}}\dot{\tau} = \kappa Z_s$ and $\nabla_{\dot{\tau}}Z_s = -\kappa\dot{\tau}$, with initial condition $\tau(0) = x$, $\dot\tau(0) = X$ and $Z_0 = -Y$. By assumption, the curve $f \circ \tau$ is a Frenet curve of order 2 in $\widetilde{M}^m$. So we can apply the above discussion to this curve τ. Then we can see that

$$\begin{aligned}(2.16') \qquad &(\dot\kappa(0)\kappa(0) + \langle(\bar\nabla_X\sigma)(X,X), \sigma(X,X)\rangle - 2\kappa(0)\langle\sigma(Y,X), \sigma(X,X)\rangle)\sigma(X,X) \\ &= \tilde\kappa_1(0)^2(-3\kappa(0)\sigma(X,Y) + (\bar\nabla_X\sigma)(X,X)).\end{aligned}$$

where $\tilde\kappa_1(s) = \|\widetilde\nabla_{\dot\tau}\dot\tau\|$. The equation (2.14) guarantees $\tilde\kappa(0) = \tilde\kappa_1(0)$. Hence from (2.16) and (2.16') we obtain

$$(2.17) \qquad 3\tilde\kappa(0)^2\kappa(0)\sigma(X,Y) - 2\kappa(0)\langle\sigma(X,X), \sigma(X,Y)\rangle\sigma(X,X) = 0.$$

Taking the inner product of both sides of the equation (2.17) with $\sigma(X,X)$, we get

$$3\tilde\kappa(0)^2\kappa(0)\langle\sigma(X,X), \sigma(X,Y)\rangle - 2\kappa(0)\langle\sigma(X,X), \sigma(X,Y)\rangle\|\sigma(X,X)\|^2 = 0.$$

This, together with (2.14), yields

$$(3\kappa(0)^2 + \|\sigma(X,X)\|^2)\langle\sigma(X,X), \sigma(X,Y)\rangle = 0.$$

Hence

$$\langle\sigma(X,X), \sigma(X,Y)\rangle = 0.$$

This, combined with (2.17), shows that $\sigma(X,Y) = 0$ holds for any orthonormal pair of vectors X, Y in $T_x(M^n)$. Since x is an arbitrary point, it follows that our immersion $f : M^n \to \widetilde{M}^m$ is totally umbilic.

By taking the inner product of both sides of the equation (2.11) with Y_s we see that

$$(2.18) \qquad \dot{\tilde\kappa}(s)\kappa(s) = \tilde\kappa(s)\dot\kappa(s)$$

for arbitrary s. Moreover the equation (2.12) reduces to

$$(2.19) \qquad \dot{\tilde\kappa}(s)\mathfrak{h}_{\gamma(s)} = \tilde\kappa(s)(D_{\dot\gamma(s)}\mathfrak{h})_{\gamma(s)}$$

for arbitrary s, where $\mathfrak{h}$ denotes the mean curvature vector of our immersion. Therefore the equations (2.18) and (2.19) imply

$$(2.20) \qquad (D_{\dot\gamma(s)}\mathfrak{h})_{\gamma(s)} = \frac{\dot{\tilde\kappa}(s)}{\tilde\kappa(s)}\mathfrak{h}_{\gamma(s)} = \frac{\dot\kappa(s)}{\kappa(s)}\mathfrak{h}_{\gamma(s)}.$$

In particular, at $s = 0$, we get

$$(D_X\mathfrak{h})_x = \frac{\dot\kappa(0)}{\kappa(0)}\mathfrak{h}_x.$$

This equation shows that $(D_X\mathfrak{h})_x$ is independent of choice of a unit vector $X \in T_x(M^n)$. Changing X into $-X$ we see $(D_X\mathfrak{h})_x = 0$. Since x is an arbitrary point, we have shown that the mean curvature vector $\mathfrak{h}$ of f is parallel.

Finally, by assumption, as the function $\kappa(s)$ is not constant, there exists some $s_0 \in (-\epsilon, \epsilon)$ with $\dot\kappa(s_0) \neq 0$. ¿From (2.20) we have

$$\mathfrak{h}_{\gamma(s_0)} = \frac{\kappa(s_0)}{\dot\kappa(s_0)}(D_{\dot\gamma(s_0)}\mathfrak{h})_{\gamma(s_0)} = 0.$$

Since $\|\mathfrak{h}\|$ is constant on M^n, we have $\mathfrak{h} = 0$ on M^n. Consequently the immersion $f : M^n \to \widetilde{M}^m$ is totally geodesic. $\square$

A curve $\gamma = \gamma(s)$ on a Riemannian manifold M^n is called a *plane curve*, if the curve γ is locally contained in some 2-dimensional totally geodesic submanifold of M^n. Needless to say, every plane curve with positive curvature function is a Frenet curve of proper order 2. But in general, the converse does not hold.

Now, we consider the case that the ambient space $\widetilde{M}^m$ is a real space form $\widetilde{M}^m(c)$ of constant curvature c, that is, $\widetilde{M}^m(c) = \mathbb{R}^m$, $S^m(c)$ or $H^m(c)$ according as the curvature c is zero, positive, or negative. As an immediate consequence of Theorem 1 we obtain the following:

Theorem 2 *Let M^n be a connected Riemannian submanifold of a real space form $\widetilde{M}^m(c)$ through an isometric immersion f. Assume that there exists a nonconstant positive smooth function $\kappa = \kappa(s)$ satisfying that for every Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ in M^n, the curve $f \circ \gamma$ is a plane curve in the ambient space $\widetilde{M}^m(c)$. Then M^n is totally geodesic in $\widetilde{M}^m(c)$.*

Proof. In the ambient real space form $\widetilde{M}^m(c)$, a curve $\gamma = \gamma(s)$ is a Frenet curve of proper order 2 if and only if the curve γ is a plane curve with positive curvature function. In fact, let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 in $\widetilde{M}^m(c)$ satisfying

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa Y_s, \quad \widetilde{\nabla}_{\dot{\gamma}} Y_s = -\kappa\dot{\gamma}, \tag{2.21}$$

with initial condition

$$\gamma(0) = x, \quad \dot{\gamma}(0) = X \quad \text{and} \quad Y_0 = Y, \tag{2.22}$$

for an orthonormal pair of vectors $X, Y \in T_x(\widetilde{M}^m(c))$. Then there exists a 2-dimensional totally geodesic submanifold $\widetilde{M}^2(c)$ passing through the point x of $\widetilde{M}^m(c)$ whose tangent space $T_x(\widetilde{M}^2(c))$ at x is spanned by X and Y. We consider the curve $\gamma_1 = \gamma_1(s)$ in $\widetilde{M}^2(c)$ satisfying the same differential equations (2.21) and the initial condition (2.22). By the uniqueness of solution to ordinary differential equations, we have $\gamma_1(s) = \gamma(s)$ $(|s| < \epsilon)$ for some $\epsilon > 0$. Thus γ is a plane curve. Now the statement of Theorem 2 follows immediately from Theorem 1. □

Moreover, the proof of Theorem 2 yields the following:

Corollary 1 *Let f be an isometric immersion of an n-dimensional real space form $M^n(c)$ of constant curvature c into an m-dimensional real space form $\widetilde{M}^m(\tilde{c})$ of constant curvature $\tilde{c}$. Assume that there exists a nonconstant positive smooth function $\kappa = \kappa(s)$ satisfying that for every plane curve $\gamma = \gamma(s)$ with curvature κ in $M^n(c)$, the curve $f \circ \gamma$ is a plane curve in the ambient space $\widetilde{M}^m(\tilde{c})$. Then $M^n(c)$ is totally geodesic in $\widetilde{M}^m(\tilde{c})$.*

In the assumption of Theorem 1 we here remove the condition that the function $\kappa = \kappa(s)$ is not constant. Then the proof of Theorem 1 gives us the following:

Theorem 3 *Let M^n be a connected Riemannian submanifold of a Riemannian manifold $\widetilde{M}^m$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\widetilde{M}^m$ if and only if there exists a positive smooth function $\kappa = \kappa(s)$ satisfying that for every Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^m$.*

Theorem 3 is a generalization of results in [2] and [4] .

Added in proof. S. Maeda and the author recently studied similar problems in the case of Kähler manifolds to ours in this paper. That is, they characterized totally geodesic Kähler immersions of Kähler manifolds into ambient Kähler manifolds, and parallel isometric immersions of complex space forms into ambient real space forms by using some Frenet curves of order 2 closely related to the complex structure (see [3]). The author also investigated parallel isometric immersions of quaternionic Kähler manifolds and Cayley projective plane into ambient real space forms from the same point of view ([6], [7]).

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