CONTROLLED CONVERGENCE THEOREM
FOR NUCLEAR HILBERTIAN (UCs-N) SPACES
VALUED HENSTOCK-KURZWEIL INTEGRALS

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ABSTRACT. In [9], S. Nakanishi generalized the definition of Henstock-Kurzweil integral to functions with values in (UCs-N) spaces, and pointed out that the Saks-Henstock lemma holds for the case when the (UCs-N) spaces are nuclear Hilbertian (UCs-N) spaces, which include the spaces $S, S', D$ and $D'$ occurring in distribution theory of L. Schwartz as typical spaces. In [12], L. I. Paredes and T. S. Chew studied a controlled convergence theorem for Banach space valued $HL$ integrals. The purpose of this paper is to study a controlled convergence theorem for Henstock-Kurzweil integrals of functions taking values in nuclear Hilbertian (UCs-N) spaces.

In [1], S. S. Cao studied the Henstock-Kurzweil integral for Banach space valued functions, and pointed out that the Saks-Henstock lemma holds for finite dimensional Banach space valued functions, but it does not always hold for the case of infinite dimension, and introduced a definition of $HL$ integrability. In [9], S. Nakanishi generalized the definition of Henstock-Kurzweil integral to functions taking values in (UCs-N) spaces, and pointed out that the Saks-Henstock lemma holds for the case when the (UCs-N) spaces are nuclear Hilbertian (UCs-N) spaces, which include the spaces $S, S', D$ and $D'$ occurring in distribution theory of L. Schwartz as typical spaces (see [5-11]). In [12], L. I. Paredes and T. S. Chew studied a controlled convergence theorem for Banach space valued $HL$ integrals. The purpose of this paper is to study a controlled convergence theorem for Henstock-Kurzweil integrals of functions taking values in nuclear Hilbertian (UCs-N) spaces.

1. Preliminaries.
Throughout this paper, “vector space” means a vector space over the field of real numbers, and we denote the set of all non-negative integers by $N = \{0, 1, 2, \cdots \}$.

First, according to Nakanishi, we recall the definitions of (UCs-N) spaces ([11, pp.1-3]) and $H-K$ integrals ([9, p.320 and p.327]):

(1.1) (UCs-N) spaces. Let $X$ be a vector space, and let $(X_\alpha, \{p^\alpha_n\}_{n=0}^{\infty}) (\alpha \in \Xi)$ be a family of vector subspaces $X_\alpha$ of $X$ such that a sequence of semi-norms $\{p^\alpha_n\}_{n=0}^{\infty}$ is defined on $X_\alpha$ for each $\alpha \in \Xi$. Suppose that they satisfy the following conditions (I)-(V):

(I) $\bigcup_{\alpha \in \Xi} X_\alpha = X$.

(II) $\Xi$ is an upward directed set with the ordering $\leq$.

(III) $\alpha \leq \beta$ if and only if $X_\alpha \subset X_\beta$.

(IV) For each $\alpha \in \Xi$, $p^\alpha_0(x) \leq p^\alpha_1(x) \leq \cdots$ for every $x \in X_\alpha$.

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(V) If \( \alpha \leq \beta \), then \( p_n^\alpha(x) \geq p_n^\beta(x) \) for every \( x \in X_\alpha \) and every \( n \in N \).

In the space \( X \) mentioned in the above, the notion concerned with "convergence" is defined only for the countable sequence of points as follows.

(C1) A sequence \( \{x_i\} \) is said to be convergent to \( x \) in \( X \) if and only if there exists an \( \alpha \in \Xi \) such that \( x_i(i = 1, 2, \ldots) \) and \( x \) are contained in \( X_\alpha \) and the sequence is convergent to \( x \) in the space \( X_\alpha \) topologized by \( \{p_n^\alpha\}_{n=0}^\infty \).

(C2) A sequence \( \{x_i\} \) is said to be a Cauchy sequence in \( X \) if and only if there exists an \( \alpha \in \Xi \) such that \( x_i(i = 1, 2, \ldots) \) are contained in \( X_\alpha \) and the sequence is a Cauchy sequence in the space \( X_\alpha \) topologized by \( \{p_n^\alpha\}_{n=0}^\infty \).

(C3) The space \( X \) is said to be separated if \( x = y \) whenever \( \lim x_i = x \) and \( \lim x_i = y \).

By (C1) and (C2), we see that the space \( X \) is separated if and only if for every \( \alpha \in \Xi \), the space \( X_\alpha \) topologized by \( \{p_n^\alpha\}_{n=0}^\infty \) is separated.

If \( X \) is a vector space endowed with \( (X_\alpha, \{p_n^\alpha\}_{n=0}^\infty)(\alpha \in \Xi) \) satisfying (I)-(V) and if, on \( X \), convergence, Cauchy sequence and separation axiom are defined by (C1), (C2) and (C3), respectively, then the space \( X \) is called a (UCs-N) space with component spaces \( (X_\alpha, \{p_n^\alpha\}) \) \( (\alpha \in \Xi) \).

In particular, when \( \Xi \) is a set consisting of a single element, say \( \alpha \), and \( p_0^\alpha(x) \leq p_n^\alpha(x) \leq \cdots \) for every \( x \in X \), the space \( X \) is called a (Cs-N) space and denoted by \( (X, \{p_n^\alpha\}) \).

(1.2) \( H-K \) integrals.

Two intervals are called non-overlapping if there are no common inner points. Let \( \delta \) be a positive function defined on \([a, b]\), and let \( P = \{(c_i, d_i) : i = 1, 2, \ldots, h\} \) be a finite collection of interval-point pairs, where \([c_1, d_1], \ldots, [c_h, d_h]\) are non-overlapping intervals and \( \xi_1, \ldots, \xi_h \) are real numbers. We say that \( P \) is a \( \delta \)-fine Perron partition (abbr. \( P \)-partition) in \([a, b]\) if \( \cup_{i=1}^h [c_i, d_i] \subset [a, b] \) and \( \xi_i \in [c_i, d_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \) for \( i = 1, 2, \ldots, h \); if, in addition, \( \cup_{i=1}^h [c_i, d_i] = [a, b] \), we say that \( P \) is a \( \delta \)-fine \( P \)-partition of \([a, b]\).

Definition 1.1. Let \((X, p)\) be a normed space endowed with a norm \( p \) and let \( f \) be an \( X \)-valued function defined on \([a, b]\). The function \( f \) is said to be Henstock-Kurzweil (abbr. \( H-K \)) integrable to a vector \( z \in X \) on \([a, b]\) if for given \( \varepsilon > 0 \) there is a positive function \( \delta_\varepsilon \) on \([a, b]\) such that for any \( \delta \)-fine \( P \)-partition \( P = \{[u_i, v_i], \xi_i \} : i = 1, 2, \ldots, h \} \) of \([a, b]\), we have

\[
p \left( \sum_{i=1}^h f(\xi_i)(v_i - u_i) - z \right) < \varepsilon,
\]
or alternatively,

\[
p \left( \sum_{P} f(\xi)(v - u) - z \right) < \varepsilon,
\]

where \(([u, v], \xi)\) denotes a typical interval-point pair in \( P \) with \( \xi \in [u, v] \subset (\xi - \delta_\varepsilon(\xi), \xi + \delta_\varepsilon(\xi)) \).

It is easy to see that the vector \( z \) is uniquely determined. The integral of \( f \) on \([a, b]\) is given by the vector \( z \), and it is written \( \int_a^b f(t) dt \). The function \( f \) is said to be \( H-K \) integrable on a set \( A \subset [a, b] \) if \( A \) is a Lebesgue measurable subset of \([a, b]\) and the function \( \chi_A f \) is \( H-K \) integrable on \([a, b]\), where \( \chi_A \) is the characteristic function of \( A \).

Let \( f \) be an \( X \)-valued \( H-K \) integrable function defined on \([a, b]\). Then, \( f \) is also \( H-K \) integrable on any subinterval \([c, d]\) of \([a, b]\). The primitive of \( f \) is the function \( F \) such that \( F(x) = \int_a^x f(t) dt \) for each \( x \in [a, b] \) and \( F(a) = 0 \). We say that the Saks-Henstock Lemma holds for \( f \), if, given \( \varepsilon > 0 \), there is a positive function \( \delta_\varepsilon \) on \([a, b]\) such that for any \( \delta_\varepsilon \)-fine
P-partition \( \{(c_i, d_i], \xi_i) : i = 1, 2, \ldots, h \} \) in \([a, b]\) we have

\[
\sum_{i=1}^{h} p(f(\xi_i))(d_i - c_i) - (F(d_i) - F(c_i)) < \varepsilon.
\]

**Definition 1.2.** Let \((X, \{p_n\})\) be a separated \((Cs-N)\) space. An \(X\)-valued function \(f\) defined on \([a, b]\) is said to be \(H-K\) integrable to a vector \(z \in X\) on \([a, b]\) if for every \(n \in N\) there is a positive function \(\delta_n(\xi)\) on \([a, b]\) such that for any \(\delta_n\)-fine P-partition \(P = \{(u, v)\}\) of \([a, b]\), we have

\[
p_n \left( \sum_{p} f(\xi)(v - u) - z \right) < 1/2^n,
\]

It is easy to see that the vector \(z\) is uniquely determined. The definitions of the integral \(\int_{a}^{b} f(t)dt\) and the primitive of \(f\) are similar to the normed space valued case.

Let \(X\) be a \((Cs-N)\) space \((X, \{p_n\})\). Put \(N(n) = \{x \in X : p_n(x) = 0\}\). Then, the quotient space \(X/N(n)\) is a normed space. We denote the element of the quotient space with \(x \in X\) as a representative by \(\{x\}_n\). We denote the completion of the normed space \(X/N(n)\) by \((\hat{X}_n, \hat{p}_n)\), where \(\hat{p}_n\) denotes the norm on \(\hat{X}_n\). In particular, we denote the element of \(\hat{X}_n\) with a Cauchy sequence \(\{(x_n, [x]_n, \ldots) : x \in X\}\) as a representative by \(\{[x]_n\}^\wedge\). For an \(X\)-valued function \(f\), we define \(\hat{X}_n\)-valued function \(\hat{f}_n\) by \(\hat{f}_n(t) = \{[f(t)]_n\}^\wedge\). (see [11, p.8]).

Then, the following proposition holds from [11, Proposition 3].

**Proposition 1.3.** Let \((X, \{p_n\})\) be a separated complete \((Cs-N)\) space, and \(f\) an \(X\)-valued function. Then, the function \(f\) is \(H-K\) integrable on \([a, b]\) as an \((X, \{p_n\})\)-valued function if and only if for every \(n \in N\), the function \(\hat{f}_n\) is \(H-K\) integrable on \([a, b]\) as an \((\hat{X}_n, \hat{p}_n)\)-valued function. In this case, \(\int_{a}^{b} \hat{f}_n(t)dt = \{\int_{a}^{b} [f(t)]_n dt\}^\wedge\) for every \(n \in N\).

**Definition 1.4.** Let \(X\) be a separated \((UCs-N)\) space with component spaces \((X_\alpha, \{p_n^\alpha\})\) \((\alpha \in \Xi)\). An \(X\)-valued function \(f\) defined on \([a, b]\) is said to be \(H-K\) integrable to a vector \(z \in X\) on \([a, b]\) if there is a component space \(X_\alpha\) such that:

- (1) The image of \([a, b]\) by \(f\) is contained in \(X_\alpha\) and \(z \in X_\alpha\);
- (2) \(f\) is \(H-K\) integrable to \(z\) on \([a, b]\) as an \((X_\alpha, \{p_n^\alpha\})\)-valued function.

If it is necessary to indicate such an \(X_\alpha\) explicitly, for convenience we will say that \(f\) is \(H-K\) integrable\((X_\alpha)\) to \(z\) on \([a, b]\). By [10, (0.13)] the vector \(z\) is determined uniquely independently of the choice of \(X_\alpha\). The definitions of the integral and the primitive are similar to the normed space valued case.

Next, according to Paredes and Chew([12]), we recall the controlled convergence theorem.

**1.3** HL integrals and the controlled convergence theorem.

An interval function in \([a, b]\) means a function defined on the family of all subintervals of \([a, b]\). An interval function \(F\) in \([a, b]\) is called finitely additive if \(F(I_1 \cup I_2) = F(I_1) + F(I_2)\) for any pair of non-overlapping intervals \(I_1\) and \(I_2\) in \([a, b]\) whose union is an interval(see [14, p.61]). Let \(F\) be a function defined on \([a, b]\). Then \(F\) can be treated as a function of intervals by defining \(F([u, v]) = F(v) - F(u)\).
**Definition 1.5.** (cf. [1]) Let \((X, p)\) be a Banach space with a norm \(p\). An \(X\)-valued function \(f\) defined on \([a, b]\) is said to be HL integrable on \([a, b]\) if there is an \(X\)-valued interval function \(F\) in \([a, b]\) which is finitely additive and having the following property: for given \(\varepsilon > 0\) there is a positive function \(\delta_\varepsilon\) on \([a, b]\) such that for any \(\delta_\varepsilon\)-fine \(P\)-partition \(P = \{(\xi, [u, v])\}\) of \([a, b]\) we have

\[
\sum_P p(f(\xi)(v - u) - F([u, v])) < \varepsilon.
\]

It is easy to see that the vector \(F([a, b])\) is uniquely determined. The HL integral of \(f\) on \([a, b]\) is given by the vector \(F([a, b])\), and it is denoted by \((HL) \int_a^b f(t)dt\). Setting \(F(t) = F([a, t])\) when \(t \in (a, b]\), and \(F(a) = 0\), the function \(F\) is called the HL-primitive of \(f\) on \([a, b]\), or simply the primitive.

**Definition 1.6.** (cf. [4]) Let \((X, p)\) be a normed space and let \(F\) be an \(X\)-valued function defined on \([a, b]\). Let \(E\) be a subset of \([a, b]\).

1. \(F\) is said to be absolutely continuous (abbr. AC) on \(E\) if for every \(\varepsilon > 0\) there exists an \(\eta > 0\) such that for every finite collection of non-overlapping intervals \(\{[u_i, v_i] : i = 1, 2, \cdots, h\}\) with the endpoints belonging to \(E\) and with \(\sum_{i=1}^h (v_i - u_i) < \eta\), we have

\[
\sum_{i=1}^h p(F([u_i, v_i])) < \varepsilon.
\]

2. \(F\) is said to be absolutely continuous in the restricted sense (abbr. \(AC_\varepsilon\)) on \(E\) if for every \(\varepsilon > 0\) there exists an \(\eta > 0\) such that for every finite collection of non-overlapping intervals \(\{[u_i, v_i] : i = 1, 2, \cdots, h\}\) with one of the endpoints belonging to \(E\) and with \(\sum_{i=1}^h (v_i - u_i) < \eta\), we have

\[
\sum_{i=1}^h p(F([u_i, v_i])) < \varepsilon.
\]

3. \(F\) is said to be generalized absolutely continuous (abbr. \(AC\Gamma\)) on \(E\) if \(E\) can be written as a countable union of sets on each of which \(F\) is \(AC\). \(F\) is said to be generalized absolutely continuous in the restricted sense (abbr. \(AC\Gamma_\varepsilon\)) on \(E\) if \(E\) can be written as a countable union of sets on each of which \(F\) is \(AC\Gamma_\varepsilon\).

The following statement holds from the Theorem 3.1 in [12].

**Theorem 1.7 (Controlled convergence theorem).** Let \((X, p)\) be a Banach space, let \(\{f_j\}\) be a sequence of \(X\)-valued functions which are HL integrable on \([a, b]\), and let \(F_j\) be the primitive of \(f_j\) for every \(j\). Suppose that:

1. \(\lim_{j \to \infty} f_j(t) = f(t)\) almost everywhere on \([a, b]\).
2. \(\{F_j\}\) is \(AC\Gamma\) on \([a, b]\) uniformly in \(j\), i.e., \([a, b]\) is the union of a sequence \(\{E_s\}\) of closed sets such that \(\{F_j\}\) is \(AC\Gamma_\varepsilon\) on each \(E_s\) uniformly in \(j\).
3. \(\{F_j\}\) converges uniformly on \([a, b]\).

Then, \(f\) is also HL integrable on \([a, b]\) and

\[
\lim_{j \to \infty} (HL) \int_a^b f_j(t)dt = (HL) \int_a^b f(t)dt.
\]
2. Controlled convergence theorem for H-K integrals of functions with values in Hilbert spaces.

Throughout this section, $H_1$ and $H_2$ are Hilbert spaces and $T$ is a nuclear operator of $H_1$ into $H_2$.

The following lemma holds from [10, (0.7), and Lemmas 1, 2 and 9].

**Lemma 2.1.** Let $f$ be an $H_1$-valued function defined on $[a, b]$. If $f$ is H-K integrable on $[a, b]$ and $F$ is the primitive of $f$, then $Tf$ has the following properties as an $H_2$-valued function.

1. $Tf$ is measurable on $[a, b]$.
2. $Tf$ is H-K integrable on $[a, b]$, and $\int_a^b Tf \, dt = T\int_a^b f \, dt$.
3. $TF$ is the primitive of $Tf$.
5. $TF$ is continuous on $[a, b]$.

Let $\{f_j\}$ be a sequence of $H_1$-valued functions which are H-K integrable on $[a, b]$, and $F_j$ the primitive of $f_j$ for every $j$. By Lemma 2.1, for every $j$, $Tf_j$ is H-K integrable on $[a, b]$, $TF_j$ is the primitive of $Tf_j$, and Saks-Henstock Lemma holds for $Tf_j$. Hence $\{Tf_j\}$ is a sequence of $H_2$-valued functions which are HL integrable on $[a, b]$. Therefore, the following statement holds from Theorem 1.7.

**Theorem 2.2 (Controlled convergence theorem).** Let $\{f_j\}$ be a sequence of $H_1$-valued functions which are H-K integrable on $[a, b]$

1. $\lim_{j \to \infty} Tf_j(t) = f(t)$ in $H_2$ almost everywhere on $[a, b]$.
2. $\{TF_j\}$ is ACG$_*$ on $[a, b]$ uniformly in $j$.
3. $\{TF_j\}$ converges uniformly on $[a, b]$.

Then, $f$ is also H-K integrable on $[a, b]$ and

$$\lim_{j \to \infty} \int_a^b Tf_j(t) \, dt = \int_a^b f(t) \, dt \quad \text{in } H_2.$$

3. Generalized AC$_*$ functions with values in (UCs-N) spaces.

**Definition 3.1.** Let $(X, \{p_n\})$ be a (Cs-N) space and let $F$ be an $X$-valued function defined on $[a, b]$ and let $E$ be a subset of $[a, b]$.

1. $F$ is said to be AC on $E$ if for every $n \in N$ there exists an $\eta_n > 0$ such that for every finite collection of non-overlapping intervals $\{[u_i, v_i] : i = 1, 2, \ldots, h\}$ with the endpoints belonging to $E$ and with $\sum_{i=1}^h (v_i - u_i) < \eta_n$, we have

$$\sum_{i=1}^h p_n(F([u_i, v_i])) < 1/2^n.$$

2. $F$ is said to be AC$_*$ on $E$ if for every $n \in N$ there exists an $\eta_n > 0$ such that for every finite collection of non-overlapping intervals $\{[u_i, v_i] : i = 1, 2, \ldots, h\}$ with one of the endpoints belonging to $E$ and with $\sum_{i=1}^h (v_i - u_i) < \eta_n$, we have

$$\sum_{i=1}^h p_n(F([u_i, v_i])) < 1/2^n.$$

3. $F$ is said to be ACG(resp. ACG$_*$) on $E$ if $E$ can be written as a countable union of sets on each of which $F$ is AC(resp. AC$_*$).
The proofs of the next two propositions are essentially similar to the real-valued case (see [4] or [3]).

**Proposition 3.2.** Let \( X \) be a separated complete (Cs-N) space. Let \( E \) be a closed subset of \([a, b]\) and let \((a, b) \setminus E\) be the union of \((a_k, b_k)\) for \(k = 1, 2, \ldots\). Suppose that an \( X \)-valued function \( F \) is continuous on \([a, b]\). Then the following statements are equivalent:

1. \( F \) is \( AC_\ast \) on \( E \).
2. \( F \) is \( AC \) on \( E \) and \( \sum_{k=1}^{\infty} \omega_k(F; [a_k, b_k]) < \infty \) for every \( n \in \mathbb{N} \).
3. For every \( n \in \mathbb{N} \) there exists an \( \eta_n > 0 \) such that for every finite collection \([u_i, v_i] : i = 1, 2, \ldots, h\) of non-overlapping intervals in \([a, b]\) with the endpoints belonging to \( E \) and with \( \sum_{i=1}^{h} (v_i - u_i) < \eta_n \), we have

\[
\sum_{i=1}^{h} \omega_n(F; [u_i, v_i]) < 1/2^n
\]

where \( \omega_n(F; [u, v]) = \sup \{ p_n (F(x) - F(y)) : x, y \in [u, v] \} \).

**Proposition 3.3.** Let \( X \) be a separated complete (Cs-N) space. Let \( E \) be a subset of \([a, b]\). If an \( X \)-valued function \( F \) is \( AC_\ast \) on \( E \) and continuous on \([a, b]\), then \( F \) is \( AC_\ast \) on \( \overline{E} \), where \( \overline{E} \) is the closure of \( E \).

**Definition 3.4.** Let \( X \) be a (UCs-N) space with component spaces \((X_\alpha, \{p_\alpha^n\}) (\alpha \in \Xi)\). Let \( F \) be an \( X \)-valued function defined on \([a, b]\) and let \( E \) be a subset of \([a, b]\).

\( F \) is said to be \( AC \) (resp. \( AC_\ast \), \( ACG \), \( ACG_\ast \) ) on \( E \) if there is a component space \((X_\alpha, \{p^n_\alpha\})\) such that the image of \([a, b]\) by \( F \) is contained in \( X_\alpha \) and \( F \) is \( AC \) (resp. \( AC_\ast \), \( ACG \), \( ACG_\ast \) ) on \( E \) as an \((X_\alpha, \{p^n_\alpha\})\)-valued function.

**Proposition 3.5.** Let \( X \) be a separated (UCs-N) space with complete component spaces \((X_\alpha, \{p^n_\alpha\}) (\alpha \in \Xi)\). Let \( E \) be a closed subset of \([a, b]\) and let \((a, b) \setminus E\) be the union of \((a_k, b_k)\) for \(k = 1, 2, \ldots\). Suppose that an \( X \)-valued function \( F \) defined on \([a, b]\) is continuous on \([a, b]\). Then the following statements are equivalent:

1. \( F \) is \( AC_\ast \) on \( E \).
2. \( F \) is \( AC \) on \( E \) and there exists a \( \beta \in \Xi \) such that \( \sum_{k=1}^{\infty} \omega_k(F; [a_k, b_k]) < \infty \) for every \( n \in \mathbb{N} \), where \( \omega_k(F; [u, v]) = \sup \{ p_k (F(x) - F(y)) : x, y \in [u, v] \} \).
3. There is a component space \( X_\beta \) such that the image of \([a, b]\) by \( F \) is contained in \( X_\beta \) and for every \( n \in \mathbb{N} \) there exists an \( \eta_n > 0 \) such that for every finite collection \([u_i, v_i] : i = 1, 2, \ldots, h\) of non-overlapping intervals in \([a, b]\) with the endpoints belonging to \( E \) and with \( \sum_{i=1}^{h} (v_i - u_i) < \eta_n \), we have

\[
\sum_{i=1}^{h} \omega_n(F; [u_i, v_i]) < 1/2^n
\]

**Proof.** (1) \( \Rightarrow \) (2) : Since \( F \) is \( AC_\ast \) on \( E \), there is a component space \((X_\beta, \{p_\beta^n\})\) such that the image of \([a, b]\) by \( F \) is contained in \( X_\beta \) and \( F \) is \( AC_\ast \) on \( E \) as an \((X_\beta, \{p^n_\beta\})\)-valued function. Hence, by Proposition 3.2, \( F \) is \( AC \) on \( E \) as an \((X_\beta, \{p^n_\beta\})\)-valued function and \( \sum_{k=1}^{\infty} \omega_k(F; [a_k, b_k]) < \infty \) for every \( n \in \mathbb{N} \).

(2) \( \Rightarrow \) (3) : Let \( F \) be \( AC \) on \( E \) and there exists a \( \beta \in \Xi \) such that \( \sum_{k=1}^{\infty} \omega_k(F; [a_k, b_k]) < \infty \) for every \( n \in \mathbb{N} \). Since \( F \) is \( AC \) on \( E \), there is a component space \((X_\gamma, \{p^n_\gamma\})\) such
that the image of $[a,b]$ by $F$ is contained in $X_\gamma$ and $F$ is $AC$ on $E$ as an $(X_\gamma, \{p_\alpha^n\})$-valued function. By (1.1) (I), choose an $\alpha \in \Xi$ such that $\beta \leq \alpha$ and $\gamma \leq \alpha$. Then, by (1.1) (III) and (V), $X_\gamma \subset X_\alpha$ and $F$ is $AC$ on $E$ as an $(X_\alpha, \{p_\alpha^n\})$-valued function, and 
\[ \sum_{k=1}^{\infty} \omega_n^\alpha(F; [a_k, b_k]) \leq \sum_{k=1}^{\infty} \omega_n^\beta(F; [a_k, b_k]) < \infty \]
for every $n \in \mathbb{N}$. Hence, (3) holds by Proposition 3.2.

(3) $\Rightarrow$ (1) : By Proposition 3.2, it is clear.

**Proposition 3.6.** Let $X$ be a separated (UCs-N) space with complete component spaces $(X_\alpha, \{p_\alpha^n\}) (\alpha \in \Xi)$ and let $E$ be a subset of $[a,b]$. If an $X$-valued function $F$ is $AC_\alpha$ on $E$ and continuous on $[a,b]$, then $F$ is $AC_\alpha$ on $\overline{E}$.

**Proof.** Since an $X$-valued function $F$ is $AC_\alpha$ on $E$, by definition, there is a component space $(X_\alpha, \{p_\alpha^n\})$ such that the image of $[a,b]$ by $F$ is contained in $X_\alpha$ and $F$ is $AC_\alpha$ on $E$ as an $(X_\alpha, \{p_\alpha^n\})$-valued function. Hence, by Proposition 3.3, $F$ is $AC_\alpha$ on $\overline{E}$ as an $(X_\alpha, \{p_\alpha^n\})$-valued function. Thus, $F$ is $AC_\alpha$ on $\overline{E}$ as an $X$-valued function.

### 4. Controlled convergence theorem for $H$-$K$ integrals of functions with values in nuclear Hilbertian (UCs-N) spaces.

According to Nakanishi [11, pp.5-6], we recall the definition of nuclear Hilbertian (UCs-N) spaces:

Let $X$ be a separated (UCs-N) space with complete component spaces $(X_\alpha, \{p_\alpha^n\}) (\alpha \in \Xi)$ such that, on each component space $(X_\alpha, \{p_\alpha^n\})$, for every $n \in \mathbb{N}$ there is defined a positive hermitian form $(\ , )_n^\alpha$ and $p_\alpha^n$ is the semi-norm associated with $(\ , )_n^\alpha$.

Put $N(\alpha, n) = \{ x \in X_\alpha : p_\alpha^n(x) = 0 \}$ and consider the quotient space $X_\alpha/N(\alpha, n)$. Then, we can regard $(\ , )_n^\alpha$ as a nondegenerate positive hermitian form on $X_\alpha/N(\alpha, n)$, and therefore the quotient space $X_\alpha/N(\alpha, n)$, denoted by $X^\alpha_n$, can be considered to be a prehilbert space with the scalar product $(\ , )_n^\alpha$. We denote the element of $X^\alpha_n$ having $x \in X_\alpha$ as a representative by $[x]_n^\alpha$.

Let $\alpha \leq \beta$ and $m \geq n$. Since $X$ is a (UCs-N) space, we have $X_\alpha \subset X_\beta$ and $p_\alpha^m(x) \geq p_\beta^m(x)$ for $x \in X_\alpha$. We denote the completion of prehilbert spaces $X^\alpha_m$ and $X^\beta_m$ with respect to $p_\alpha^m$ and $p_\beta^m$ by $\hat{X}^\alpha_m$ and $\hat{X}^\beta_m$, respectively. If $\{[x]_m^\alpha\}_{i=1}^{\infty}$ is a Cauchy sequence in $X^\alpha_m$, then $\{[x]_m^\beta\}_{i=1}^{\infty}$ is a Cauchy sequence in $X^\beta_m$. Hence, the element of $\hat{X}^\beta_m$ having the Cauchy sequence $\{[x]_m^\beta\}_{i=1}^{\infty}$ as a representative is uniquely determined by the element of $\hat{X}^\alpha_m$ having the Cauchy sequence $\{[x]_m^\alpha\}_{i=1}^{\infty}$ as a representative. We denote the correspondence by $\hat{T}^\alpha_m$. Then, $\hat{T}^\alpha_m$ is a continuous linear mapping of $\hat{X}^\alpha_m$ into $\hat{X}^\beta_m$ such that

\[ \hat{p}_m^\alpha(\hat{x}_m^\alpha) \geq \hat{p}_m^\beta(\hat{T}^\alpha_m(\hat{x}_m^\alpha)) \]

for $\hat{x}_m^\alpha \in \hat{X}^\alpha_m$,

where $\hat{p}_m^\alpha$ and $\hat{p}_m^\beta$ are the norms associated with the scalar products on $\hat{X}^\alpha_m$ and $\hat{X}^\beta_m$, respectively.

Now, suppose that, for every $\alpha \in \Xi$, corresponding to $\alpha$ we can find

(i) a $\beta$ and two increasing sequences of non-negative integers $\{m(0) < m(1) < \cdots \}$ and

{\{n(0) < n(1) < \cdots \}} such that:

(4.1) $\beta \geq \alpha$,

(4.2) $m(i) \geq n(i)$ for every $i \in \mathbb{N}$, and

(4.3) $\hat{T}^\alpha_{\beta, n(i)}$ is nuclear for every $i \in \mathbb{N}$, where $\hat{T}^\alpha_{\beta, n(i)}$ is the continuous linear mapping of $X^\alpha_{m(i)}$ into $\hat{X}^\beta_{n(i)}$ defined in the above.
Then we call such a space $X$ a nuclear Hilbertian (UCs-N) space with component spaces $(X_\alpha, \{p^\alpha_n\})(\alpha \in \Xi)$.

Let $X$ be a nuclear Hilbertian (UCs-N) space with component spaces $(X_\alpha, \{p^\alpha_n\})(\alpha \in \Xi)$. We denote the element of $X^\alpha_n$ with a Cauchy sequence $\{x^\alpha_n, [x^\alpha_n]_\alpha, \cdots\} (x \in X_\alpha)$ as a representative by $\{[x^\alpha_n]\}^\wedge$. For an $X^\alpha_n$-valued function $f$ defined on $[a, b]$, we define an $X^\alpha_n$-valued function $f^\alpha_n$ by $f^\alpha_n(t) = \{[f(t)]^\alpha_n\}^\wedge$.

Now, we obtain the following convergence theorem.

**Theorem 4.1 (Controlled convergence theorem).** Let $X$ be a nuclear Hilbertian (UCs-N) space with component spaces $(X_\alpha, \{p^\alpha_n\})(\alpha \in \Xi)$. Let $\{f_j\}$ be a sequence of $X^\alpha_n$-valued functions which are H-K integrable on $(X_\alpha, \{p^\alpha_n\})$ almost everywhere on $[a, b]$. Suppose that there is a $\beta$ such that:

1) The image of $[a, b]$ by $f_j$ is contained in $X_\beta$ for every $j$, and $\lim_{j \to \infty} f_j(t) = f(t)$ in $(X_\beta, \{p^\beta_n\})$ almost everywhere on $[a, b]$.

2) $\{F_j\}$ is ACG* on $[a, b]$ uniformly in $j$ as $(X_\beta, \{p^\beta_n\})$-valued functions.

3) $\{F_j\}$ converges uniformly to $F$ on $[a, b]$ as $(X_\beta, \{p^\beta_n\})$-valued functions.

Then, $f$ is H-K integrable on $[a, b]$ and

$$\lim_{j \to \infty} \int_a^b f_j(t)dt = \int_a^b f(t)dt \quad \text{in } X.$$  

**Proof.** In the theorem we can suppose that $\beta$ is the $\beta$ associated with $\alpha$ by (†). In addition to $\beta$, take $\{m(i)\}$ and $\{n(i)\}$ associated with $\alpha$ by (†), i.e., for $\alpha$, we can find a $\beta$ and two increasing sequences of non-negative integers $\{m(0) < m(1) < \cdots\}$ and $\{n(0) < n(1) < \cdots\}$ so that $\beta \geq \alpha$, $m(i) \geq n(i)$ for every $i \in N$, and $\tilde{F}^\alpha_{\beta, m(i)}$ is nuclear for every $i \in N$.

Given $n \in N$, choose an $i \in N$ with $n \leq n(i)$. Then, since each $f_j$ is H-K integrable on $[a, b]$ as an $(X_\alpha, \{p^\alpha_n\})$-valued function, by Proposition 1.3 $(\hat{f}_j)_{m(i)}^\alpha$ is H-K integrable on $[a, b]$ as an $(\hat{X}^\alpha_m, \hat{p}^\alpha_n)$-valued function and $(\hat{F}_j)_{m(i)}^\alpha$ is the primitive of $(\hat{f}_j)_{m(i)}^\alpha$ for every $j$.

From the assumptions (1), (2) and (3), it is easy to see that the following three conditions hold:

1) $\lim_{j \to \infty} (\hat{f}_j)_{m(i)}^\beta(t) = \hat{f}^\beta_{m(i)}(t)$ in $(\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$ a.e. on $[a, b]$.

2) $\{(\hat{F}_j)_{m(i)}^\beta\}$ is ACG* on $[a, b]$ uniformly in $j$ as $(\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$-valued functions.

3) $\{(\hat{F}_j)_{m(i)}^\beta\}$ converges uniformly to $\hat{F}_{m(i)}^\beta$ on $[a, b]$ as $(\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$-valued functions.

Hence, by Theorem 2.2 $\hat{f}^\beta_{m(i)}$ is H-K integrable on $[a, b]$ as an $(\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$-valued function and

$$\lim_{j \to \infty} \int_a^b (\hat{f}_j)_{m(i)}^\beta(t)dt = \int_a^b \hat{f}_{m(i)}^\beta(t)dt \quad \text{in } (\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$$  

Therefore, $\int_a^b \hat{f}^\beta_{m(i)}(t)dt = \lim_{j \to \infty} \int_a^b (\hat{F}_j)_{m(i)}^\beta(t)dt = \int_a^b \hat{F}_{m(i)}^\beta(t)dt \quad \text{in } (\hat{X}^\beta_{m(i)}, \hat{p}^\beta_{m(i)})$.

Moreover, since $n \leq n(i)$, we have

$$\int_a^b \hat{f}^\beta_n(t)dt = \hat{F}^\beta_n([a, b]) \quad \text{in } (\hat{X}^\beta_n, \hat{p}^\beta_n).$$
Consequently, by Proposition 1.3 \( f \) is \( H-K \) integrable \((X_{\beta})\) and
\[
\int_a^b f(t)dt = F([a, b]) \quad \text{in} \quad (X_{\beta}, \{p^\alpha_n\}).
\]

Since the right side of this equality is \( \lim_{j \to \infty} F_j([a, b]) = \lim_{j \to \infty} \int_a^b f_j(t)dt \), we have the conclusion immediately.

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**References**


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