CONTROLLED CONVERGENCE THEOREM FOR NUCLEAR HILBERTIAN (UCs-N) SPACES VALUED HENSTOCK-KURZWEIL INTEGRALS

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Abstract. In [9], S. Nakanishi generalized the definition of Henstock-Kurzweil integral to functions with values in (UCs-N) spaces, and pointed out that the Saks-Henstock lemma holds for the case when the (UCs-N) spaces are nuclear Hilbertian (UCs-N) spaces, which include the spaces $\mathcal{S}, \mathcal{S}', \mathcal{D}$ and $\mathcal{D}'$ occurring in distribution theory of L. Schwartz as typical spaces. In [12], L. I. Paredes and T. S. Chew studied a controlled convergence theorem for Banach space valued *HL* integrals. The purpose of this paper is to study a controlled convergence theorem for Henstock-Kurzweil integrals of functions taking values in nuclear Hilbertian (UCs-N) spaces.

In [1], S. S. Cao studied the Henstock-Kurzweil integral for Banach space valued functions, and pointed out that the Saks-Henstock lemma holds for finite dimensional Banach space valued functions, but it does not always hold for the case of infinite dimension, and introduced a definition of *HL* integrability. In [9], S. Nakanishi generalized the definition of Henstock-Kurzweil integral to functions taking values in (UCs-N) spaces, and pointed out that the Saks-Henstock lemma holds for the case when the (UCs-N) spaces are nuclear Hilbertian (UCs-N) spaces, which include the spaces $\mathcal{S}, \mathcal{S}', \mathcal{D}$ and $\mathcal{D}'$ occurring in distribution theory of L. Schwartz as typical spaces(see [5-11]). In [12], L. I. Paredes and T. S. Chew studied a controlled convergence theorem for Banach space valued *HL* integrals. The purpose of this paper is to study a controlled convergence theorem for Henstock-Kurzweil integrals of functions taking values in nuclear Hilbertian (UCs-N) spaces.

1. Preliminaries.

Throughout this paper, "vector space" means a vector space over the field of real numbers, and we denote the set of all non-negative integers by $N = \{0, 1, 2, \cdots\}$.

First, according to Nakanishi, we recall the definitions of (UCs-N) spaces ([11, pp.1-3]) and *H-K* integrals ([9, p.320 and p.327]):

(1.1) (UCs-N) spaces. Let X be a vector space, and let $(X_\alpha, \{p_n^\alpha\}_{n=0}^\infty)$ $(\alpha \in \Xi)$ be a family of vector subspaces X_α of X such that a sequence of semi-norms $\{p_n^\alpha\}_{n=0}^\infty$ is defined on X_α for each $\alpha \in \Xi$. Suppose that they satisfy the following conditions (I)-(V):

(I) $\bigcup_{\alpha\in\Xi} X_\alpha = X$.

(II) Ξ is an upward directed set with the ordering $\leq$.

(III) $\alpha \leq \beta$ if and only if $X_\alpha \subset X_\beta$.

(IV) For each $\alpha \in \Xi$, $p_0^\alpha(x) \leq p_1^\alpha(x) \leq \cdots$ for every $x \in X_\alpha$.

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(V) If $\alpha \leq \beta$, then $p_n^\alpha(x) \geq p_n^\beta(x)$ for every $x \in X_\alpha$ and every $n \in N$.

In the space X mentioned in the above, the notion concerned with "convergence" is defined only for the countable sequence of points as follows.

(C_1) A sequence $\{x_i\}$ is said to be *convergent to* x in X if and only if there exists an $\alpha \in \Xi$ such that $x_i (i = 1, 2, \cdots)$ and x are contained in X_α and the sequence is convergent to x in the space X_α topologized by $\{p_n^\alpha\}_{n=0}^\infty$.

(C_2) A sequence $\{x_i\}$ is said to be a *Cauchy sequence* in X if and only if there exists an $\alpha \in \Xi$ such that $x_i (i = 1, 2, \cdots)$ are contained in X_α and the sequence is a Cauchy sequence in the space X_α topologized by $\{p_n^\alpha\}_{n=0}^\infty$.

(C_3) The space X is said to be *separated* if $x = y$ whenever $\lim x_i = x$ and $\lim x_i = y$.

By (C_1) and (C_2), we see that the space X is separated if and only if for every $\alpha \in \Xi$, the space X_α topologized by $\{p_n^\alpha\}_{n=0}^\infty$ is separated.

If X is a vector space endowed with $(X_\alpha, \{p_n^\alpha\}_{n=0}^\infty)(\alpha \in \Xi)$ satisfying (I)-(V) and if, on X, convergence, Cauchy sequence and separation axiom are defined by (C_1), (C_2) and (C_3), respectively, then the space X is called a (UCs-N) *space with component spaces* $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Xi)$.

In particular, when Ξ is a set consisting of a single element, say α, and $p_0^\alpha(x) \leq p_1^\alpha(x) \leq \cdots$ for every $x \in X$, the space X is called a (Cs-N) *space* and denoted by $(X, \{p_n^\alpha\})$.

(1.2) ***H-K*** **integrals.**

Two intervals are called *non-overlapping* if there are no common inner points. Let δ be a positive function defined on $[a, b]$, and let $\mathcal{P} = \{([c_i, d_i], \xi_i) : i = 1, 2, \cdots, h\}$ be a finite collection of interval-point pairs, where $[c_1, d_1], \cdots, [c_h, d_h]$ are non-overlapping intervals and $\xi_1, \cdots, \xi_h$ are real numbers. We say that $\mathcal{P}$ is a δ*-fine Perron partition* (abbr. *P-partition*) *in* $[a, b]$ if $\cup_{i=1}^h [c_i, d_i] \subset [a, b]$ and $\xi_i \in [c_i, d_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, \cdots, h$; if, in addition, $\cup_{i=1}^h [c_i, d_i] = [a, b]$, we say that $\mathcal{P}$ is a δ-fine P-partition *of* $[a, b]$.

Definition 1.1. Let (X, p) be a normed space endowed with a norm p and let f be an X-valued function defined on $[a, b]$. The function f is said to be *Henstock-Kurzweil* (abbr. *H-K*) *integrable* to a vector $z \in X$ on $[a, b]$ if for given $\varepsilon > 0$ there is a positive function δ_ε on $[a, b]$ such that for any δ_ε-fine P-partition $\mathcal{P} = \{([u_i, v_i], \xi_i) : i = 1, 2, \cdots, h\}$ of $[a, b]$, we have

$$p\left(\sum_{i=1}^h f(\xi_i)(v_i - u_i) - z\right) < \varepsilon,$$

or alternatively,

$$p\left(\sum_{\mathcal{P}} f(\xi)(v - u) - z\right) < \varepsilon,$$

where $([u, v], \xi)$ denotes a typical interval-point pair in $\mathcal{P}$ with $\xi \in [u, v] \subset (\xi - \delta_\varepsilon(\xi), \xi + \delta_\varepsilon(\xi))$.

It is easy to see that the vector z is uniquely determined. The *integral* of f on $[a, b]$ is given by the vector z, and it is written $\int_a^b f(t)dt$. The function f is said to be *H-K integrable* on a set $A \subset [a, b]$ if A is a Lebesgue measurable subset of $[a, b]$ and the function $\chi_A f$ is *H-K* integrable on $[a, b]$, where χ_A is the characteristic function of A.

Let f be an X-valued *H-K* integrable function defined on $[a, b]$. Then, f is also *H-K* integrable on any subinterval [c, d] of [a, b]. The *primitive* of f is the function F such that $F(x) = \int_a^x f(t)dt$ for each $x \in (a, b]$ and $F(a) = 0$. We say that the *Saks-Henstock Lemma* holds for f, if, given $\varepsilon > 0$, there is a positive function δ_ε on $[a, b]$ such that for any δ_ε-fine

P-partition $\{([c_i, d_i], \xi_i) : i = 1, 2, \cdots, h\}$ in $[a, b]$ we have

$$\sum_{i=1}^{h} p\left(f(\xi_i)(d_i - c_i) - (F(d_i) - F(c_i))\right) < \varepsilon.$$

Definition 1.2. Let $(X, \{p_n\})$ be a separated (Cs-N) space. An X-valued function f defined on $[a, b]$ is said to be *H-K integrable* to a vector $z \in X$ on $[a, b]$ if for every $n \in N$ there is a positive function $\delta_n(\xi)$ on $[a, b]$ such that for any δ_n-fine P-partition $\mathcal{P} = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$p_n\left(\sum_{\mathcal{P}} f(\xi)(v - u) - z\right) < 1/2^n,$$

It is easy to see that the vector z is uniquely determined. The definitions of the integral $\int_a^b f(t)dt$ and the primitive of f are similar to the normed space valued case.

Let X be a (Cs-N) space $(X, \{p_n\})$. Put $N(n) = \{x \in X : p_n(x) = 0\}$. Then, the quotient space $X/N(n)$ is a normed space. We denote the element of the quotient space with $x \in X$ as a representative by $[x]_n$. We denote the completion of the normed space $X/N(n)$ by $(\hat{X}_n, \hat{p}_n)$, where $\hat{p}_n$ denotes the norm on $\hat{X}_n$. In particular, we denote the element of $\hat{X}_n$ with a Cauchy sequence $\{[x]_n, [x]_n, \cdots\}$ $(x \in X)$ as a representative by $\{[x]_n\}^\wedge$. For an X-valued function f, we define $\hat{X}_n$-valued function $\hat{f}_n$ by $\hat{f}_n(t) = \{[f(t)]_n\}^\wedge$. (see [11, p.8]).

Then, the following proposition holds from [11, Proposition 3].

Proposition 1.3. *Let $(X, \{p_n\})$ be a separated complete* (Cs-N) *space, and f an X-valued function. Then, the function f is H-K integrable on $[a, b]$ as an $(X, \{p_n\})$-valued function if and only if for every $n \in N$, the function $\hat{f}_n$ is H-K integrable on $[a, b]$ as an $(\hat{X}_n, \hat{p}_n)$-valued function. In this case, $\int_a^b \hat{f}_n(t)dt = \{[\int_a^b f(t)dt]_n\}^\wedge$ for every $n \in N$.*

Definition 1.4. Let X be a separated (UCs-N) space with component spaces $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Xi)$. An X-valued function f defined on $[a, b]$ is said to be *H-K integrable* to a vector $z \in X$ on $[a, b]$ if there is a component space X_α such that:

(1) The image of $[a, b]$ by f is contained in X_α and $z \in X_\alpha$;

(2) f is *H-K* integrable to z on $[a, b]$ as an $(X_\alpha, \{p_n^\alpha\})$-valued function.

If it is necessary to indicate such an X_α explicitly, for convenience we will say that f is *H-K* integrable(X_α) to z on $[a, b]$. By [10, (0.13)] the vector z is determined uniquely independently of the choice of X_α. The definitions of the integral and the primitive are similar to the normed space valued case.

Next, according to Paredes and Chew([12]), we recall the controlled convergence theorem.

(1.3) *HL* integrals and the controlled convergence theorem.

An *interval function in* $[a, b]$ means a function defined on the family of all subintervals of $[a, b]$. An interval function F in $[a, b]$ is called *finitely additive* if $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any pair of non-overlapping intervals I_1 and I_2 in $[a, b]$ whose union is an interval(see [14, p.61]). Let F be a function defined on $[a, b]$. Then F can be treated as a function of intervals by defining $F([u, v]) = F(v) - F(u)$.

Definition 1.5. (cf. [1]) Let (X, p) be a Banach space with a norm p. An X-valued function f defined on $[a, b]$ is said to be *HL integrable* on $[a, b]$ if there is an X-valued interval function F in $[a, b]$ which is finitely additive and having the following property : for given $\varepsilon > 0$ there is a positive function δ_ε on $[a, b]$ such that for any δ_ε-fine P-partition $\mathcal{P} = \{([u, v], \xi)\}$ of $[a, b]$ we have

$$\sum_{\mathcal{P}} p\left(f(\xi)(v-u) - F([u, v])\right) < \varepsilon.$$

It is easy to see that the vector $F([a, b])$ is uniquely determined. The *HL integral* of f on $[a, b]$ is given by the vector $F([a, b])$, and it is denoted by $(HL) \int_a^b f(t)dt$. Setting $F(t) = F([a, t])$ when $t \in (a, b]$, and $F(a) = 0$, the function F is called the *HL-primitive* of f on $[a, b]$, or simply the *primitive*.

Definition 1.6. (cf. [4]) Let (X, p) be a normed space and let F be an X-valued function defined on $[a, b]$. Let E be a subset of $[a, b]$.

(1) F is said to be *absolutely continuous* (abbr. AC) on E if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every finite collection of non-overlapping intervals $\{[u_i, v_i] : i = 1, 2, \cdots, h\}$ with the endpoints belonging to E and with $\sum_{i=1}^h (v_i - u_i) < \eta$, we have

$$\sum_{i=1}^h p\left(F([u_i, v_i])\right) < \varepsilon.$$

(2) F is said to be *absolutely continuous in the restricted sense* (abbr. AC_*) on E if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every finite collection of non-overlapping intervals $\{[u_i, v_i] : i = 1, 2, \cdots, h\}$ with one of the endpoints belonging to E and with $\sum_{i=1}^h (v_i - u_i) < \eta$, we have

$$\sum_{i=1}^h p\left(F([u_i, v_i])\right) < \varepsilon.$$

(3) F is said to be *generalized absolutely continuous* (abbr. ACG) on E if E can be written as a countable union of sets on each of which F is AC. F is said to be *generalized absolutely continuous in the restricted sense* (abbr. ACG_*) on E if E can be written as a countable union of sets on each of which F is AC_*.

The following statement holds from the Theorem 3.1 in [12].

Theorem 1.7 (Controlled convergence theorem). *Let (X, p) be a Banach space, let $\{f_j\}$ be a sequence of X-valued functions which are HL integrable on $[a, b]$, and let F_j be the primitive of f_j for every j. Suppose that:*

(1) $\lim_{j\to\infty} f_j(t) = f(t)$ *almost everywhere on* $[a, b]$.
(2) $\{F_j\}$ *is* ACG_* *on* $[a, b]$ *uniformly in* j, *i.e.,* $[a, b]$ *is the union of a sequence* $\{E_s\}$ *of closed sets such that* $\{F_j\}$ *is* AC_* *on each* E_s *uniformly in* j.
(3) $\{F_j\}$ *converges uniformly on* $[a, b]$.

Then, f is also HL integrable on $[a, b]$ and

$$\lim_{j\to\infty} (HL) \int_a^b f_j(t)dt = (HL) \int_a^b f(t)dt.$$

2. Controlled convergence theorem for *H-K* integrals of functions with values in Hilbert spaces.

Throught this section, H_1 and H_2 are Hilbert spaces and T is a nuclear operator of H_1 into H_2.

The following lemma holds from [10, (0.7), and Lemmas 1, 2 and 9].

Lemma 2.1. *Let f be an H_1-valued function defined on $[a,b]$. If f is H-K integrable on $[a,b]$ and F is the primitive of f, then Tf has the following properties as an H_2-valued function.*

(1) *Tf is measurable on $[a,b]$.*
(2) *Tf is H-K integrable on $[a,b]$, and $\int_a^b Tf dt = T\int_a^b f dt$.*
(3) *TF is the primitive of Tf.*
(4) *Saks-Henstock Lemma holds for Tf.*
(5) *TF is continuous on $[a,b]$.*

Let $\{f_j\}$ be a sequence of H_1-valued functions which are H-K integrable on $[a,b]$, and F_j the primitive of f_j for every j. By Lemma 2.1, for every j, Tf_j is H-K integrable on $[a,b]$, TF_j is the primitive of Tf_j, and Saks-Henstock Lemma holds for Tf_j. Hence $\{Tf_j\}$ is a sequence of H_2-valued functions which are HL integrable on $[a,b]$. Therefore, the following statement holds from Theorem 1.7.

Theorem 2.2 (Controlled convergence theorem). *Let $\{f_j\}$ be a sequence of H_1-valued functions which are H-K integrable on $[a,b]$*

(1) *$\lim_{j\to\infty} Tf_j(t) = f(t)$ in H_2 almost everywhere on $[a,b]$.*
(2) *$\{TF_j\}$ is ACG_* on $[a,b]$ uniformly in j.*
(3) *$\{TF_j\}$ converges uniformly on $[a,b]$.*

Then, f is also H-K integrable on $[a,b]$ and

$$\lim_{j\to\infty}\int_a^b Tf_j(t)dt = \int_a^b f(t)dt \quad \text{in } H_2.$$

3. Generalized AC_* functions with values in (UCs-N) spaces.

Definition 3.1. Let $(X,\{p_n\})$ be a (Cs-N) space and let F be an X-valued function defined on $[a,b]$ and let E be a subset of $[a,b]$.

(1) F is said to be AC on E if for every $n\in N$ there exists an $\eta_n>0$ such that for every finite collection of non-overlapping intervals $\{[u_i,v_i] : i=1,2,\cdots,h\}$ with the endpoints belonging to E and with $\sum_{i=1}^h (v_i-u_i)<\eta_n$, we have

$$\sum_{i=1}^h p_n\left(F([u_i,v_i])\right) < 1/2^n.$$

(2) F is said to be AC_* on E if for every $n\in N$ there exists an $\eta_n>0$ such that for every finite collection of non-overlapping intervals $\{[u_i,v_i] : i=1,2,\cdots,h\}$ with one of the endpoints belonging to E and with $\sum_{i=1}^h (v_i-u_i)<\eta_n$, we have

$$\sum_{i=1}^h p_n\left(F([u_i,v_i])\right) < 1/2^n.$$

(3) F is said to be ACG(resp. ACG_*) on E if E can be written as a countable union of sets on each of which F is AC(resp. AC_*).

The proofs of the next two propositions are essentially similar to the real-valued case(see [4] or [3]).

Proposition 3.2. *Let X be a separated complete* (Cs-N) *space. Let E be a closed subset of $[a,b]$ and let $(a,b)\setminus E$ be the union of (a_k,b_k) for $k=1,2,\cdots$. Suppose that an X-valued function F is continuous on $[a,b]$. Then the following statements are equivalent:*

(1) *F is AC_* on E.*
(2) *F is AC on E and $\sum_{k=1}^{\infty}\omega_n(F;[a_k,b_k])<\infty$ for every $n\in N$.*
(3) *For every $n\in N$ there exists an $\eta_n>0$ such that for every finite collection $\{[u_i,v_i]: i=1,2,\cdots,h\}$ of non-overlapping intervals in $[a,b]$ with the endpoints belonging to E and with $\sum_{i=1}^{h}(v_i-u_i)<\eta_n$, we have*

$$\sum_{i=1}^{h}\omega_n(F;[u_i,v_i])<1/2^n$$

where $\omega_n(F;[u,v])=\sup\{p_n\left(F(x)-F(y)\right);x,y\in[u,v]\}$.

Proposition 3.3. *Let X be a separated complete* (Cs-N) *space. Let E be a subset of $[a,b]$. If an X-valued function F is AC_* on E and continuous on $[a,b]$, then F is AC_* on $\overline{E}$, where $\overline{E}$ is the closure of E.*

Definition 3.4. Let X be a (UCs-N) space with component spaces $(X_\alpha,\{p_n^\alpha\})$ $(\alpha\in\Xi)$. Let F be an X-valued function defined on $[a,b]$ and let E be a subset of $[a,b]$.

F is said to be AC (resp. AC_*, ACG, ACG_*) on E if there is a component space $(X_\alpha,\{p_n^\alpha\})$ such that the image of $[a,b]$ by F is contained in X_α and F is AC (resp. AC_*, ACG, ACG_*) on E as an $(X_\alpha,\{p_n^\alpha\})$-valued function.

Proposition 3.5. *Let X be a separated* (UCs-N) *space with complete component spaces $(X_\alpha,\{p_n^\alpha\})$ $(\alpha\in\Xi)$. Let E be a closed subset of $[a,b]$ and let $(a,b)\setminus E$ be the union of (a_k,b_k) for $k=1,2,\cdots$. Suppose that an X-valued function F defined on $[a,b]$ is continuous on $[a,b]$. Then the following statements are equivalent:*

(1) *F is AC_* on E.*
(2) *F is AC on E and there exists a $\beta\in\Xi$ such that $\sum_{k=1}^{\infty}\omega_n^\beta(F;[a_k,b_k])<\infty$ for every $n\in N$, where $\omega_n^\beta(F;[u,v])=\sup\{p_n^\beta\left(F(x)-F(y)\right);x,y\in[u,v]\}$.*
(3) *There is a component space X_α such that the image of $[a,b]$ by F is contained in X_α and for every $n\in N$ there exists an $\eta_n^\alpha>0$ such that for every finite collection $\{[u_i,v_i]: i=1,2,\cdots,h\}$ of non-overlapping intervals in $[a,b]$ with the endpoints belonging to E and with $\sum_{i=1}^{h}(v_i-u_i)<\eta_n^\alpha$, we have*

$$\sum_{i=1}^{h}\omega_n^\alpha(F;[u_i,v_i])<1/2^n.$$

Proof. (1) $\Rightarrow$ (2) : Since F is AC_* on E, there is a component space $(X_\beta,\{p_n^\beta\})$ such that the image of $[a,b]$ by F is contained in X_β and F is AC_* on E as an $(X_\beta,\{p_n^\beta\})$-valued function. Hence, by Proposition 3.2, F is AC on E as an $(X_\beta,\{p_n^\beta\})$-valued function and $\sum_{k=1}^{\infty}\omega_n^\beta(F;[a_k,b_k])<\infty$ for every $n\in N$.

(2) $\Rightarrow$ (3) : Let F be AC on E and there exists a $\beta\in\Xi$ such that $\sum_{k=1}^{\infty}\omega_n^\beta(F;[a_k,b_k])<\infty$ for every $n\in N$. Since F is AC on E, there is a component space $(X_\gamma,\{p_n^\gamma\})$ such

that the image of $[a,b]$ by F is contained in X_γ and F is AC on E as an $(X_\gamma, \{p_n^\gamma\})$-valued function. By (1.1) (I), choose an $\alpha \in \Xi$ such that $\beta \leq \alpha$ and $\gamma \leq \alpha$. Then, by (1.1) (III) and (V), $X_\gamma \subset X_\alpha$ and F is AC on E as an $(X_\alpha, \{p_n^\alpha\})$-valued function, and $\sum_{k=1}^\infty \omega_n^\alpha(F;[a_k,b_k]) \leqq \sum_{k=1}^\infty \omega_n^\beta(F;[a_k,b_k]) < \infty$ for every $n \in N$. Hence, (3) holds by Proposition 3.2.

(3) $\Rightarrow$ (1) : By Proposition 3.2, it is clear.

Proposition 3.6. *Let X be a separated* (UCs-N) *space with complete component spaces* $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Xi)$ *and let E be a subset of $[a,b]$. If an X-valued function F is AC_* on E and continuous on $[a,b]$, then F is AC_* on $\overline{E}$.*

Proof. Since an X-valued function F is AC_* on E, by definition, there is a component space $(X_\alpha, \{p_n^\alpha\})$ such that the image of $[a,b]$ by F is contained in X_α and F is AC_* on E as an $(X_\alpha, \{p_n^\alpha\})$-valued function. Hence, by Proposition 3.3, F is AC_* on $\overline{E}$ as an $(X_\alpha, \{p_n^\alpha\})$-valued function. Thus, F is AC_* on $\overline{E}$ as an X-valued function.

4. Controlled convergence theorem for *H-K* integrals of functions with values in nuclear Hilbertian (UCs-N) spaces.

According to Nakanishi [11, pp.5-6], we recall the definition of nuclear Hilbertian (UCs-N) spaces:

Let X be a separated (UCs-N) space with complete component spaces $(X_\alpha, \{p_n^\alpha\})$ $(\alpha \in \Xi)$ such that, on each component space $(X_\alpha, \{p_n^\alpha\})$, for every $n \in N$ there is defined a positive hermitian form $(\ ,\ )_n^\alpha$ and p_n^α is the semi-norm associated with $(\ ,\ )_n^\alpha$.

Put $N(\alpha,n) = \{x \in X_\alpha : p_n^\alpha(x) = 0\}$ and consider the quotient space $X_\alpha/N(\alpha,n)$. Then, we can regard $(\ ,\ )_n^\alpha$ as a nondegenerate positive hermitian form on $X_\alpha/N(\alpha,n)$, and therefore the quotient space $X_\alpha/N(\alpha,n)$, denoted by X_n^α, can be considered to be a prehilbert space with the scalar product $(\ ,\ )_n^\alpha$. We denote the element of X_n^α having $x \in X_\alpha$ as a representative by $[x]_n^\alpha$.

Let $\alpha \leq \beta$ and $m \geq n$. Since X is a (UCs-N) space, we have $X_\alpha \subset X_\beta$ and $p_m^\alpha(x) \geq p_n^\beta(x)$ for $x \in X_\alpha$. We denote the completion of prehilbert spaces X_m^α and X_n^β with respect to p_m^α and p_n^β by $\hat{X}_m^\alpha$ and $\hat{X}_n^\beta$, respectively. If $\{[x_i]_m^\alpha\}_{i=1}^\infty$ is a Cauchy sequence in X_m^α, then $\{[x_i]_n^\beta\}_{i=1}^\infty$ is a Cauchy sequence in X_n^β. Hence, the element of $\hat{X}_n^\beta$ having the Cauchy sequence $\{[x_i]_n^\beta\}_{i=1}^\infty$ as a representative is uniquely determined by the element of $\hat{X}_m^\alpha$ having the Cauchy sequence $\{[x_i]_m^\alpha\}_{i=1}^\infty$ as a representative. We denote the correspondence by $\hat{T}_{\beta n}^{\alpha m}$. Then, $\hat{T}_{\beta n}^{\alpha m}$ is a continuous linear mapping of $\hat{X}_m^\alpha$ into $\hat{X}_n^\beta$ such that

$$\hat{p}_m^\alpha(\hat{x}_m^\alpha) \geq \hat{p}_n^\beta(\hat{T}_{\beta n}^{\alpha m}(\hat{x}_m^\alpha)) \text{ for } \ \hat{x}_m^\alpha \in \hat{X}_m^\alpha,$$

where $\hat{p}_m^\alpha$ and $\hat{p}_n^\beta$ are the norms associated with the scalar products on $\hat{X}_m^\alpha$ and $\hat{X}_n^\beta$, respectively.

Now, suppose that, for every $\alpha \in \Xi$, corresponding to α we can find

(†) a β and two increasing sequences of non-negative integers $\{m(0) < m(1) < \cdots\}$ and $\{n(0) < n(1) < \cdots\}$ such that:

(4.1) $\beta \geq \alpha$,

(4.2) $m(i) \geq n(i)$ for every $i \in N$, and

(4.3) $\hat{T}_{\beta,n(i)}^{\alpha,m(i)}$ is nuclear for every $i \in N$, where $\hat{T}_{\beta,n(i)}^{\alpha,m(i)}$ is the continuous linear mapping of $\hat{X}_{m(i)}^\alpha$ into $\hat{X}_{n(i)}^\beta$ defined in the above.

Then we call such a space X a *nuclear Hilbertian* (UCs-N) *space with component spaces* $(X_\alpha, \{p_n^\alpha\})(\alpha \in \Xi)$.

Let X be a nuclear Hilbertian (UCs-N) space with component spaces $(X_\alpha, \{p_n^\alpha\})(\alpha \in \Xi)$. We denote the element of $\hat{X}_n^\alpha$ with a Cauchy sequence $\{[x]_n^\alpha, [x]_n^\alpha, \cdots\}$ $(x \in X_\alpha)$ as a representative by $\{[x]_n^\alpha\}^\wedge$. For an X_α-valued function f defined on $[a,b]$, we define an $\hat{X}_n^\alpha$-valued function $\hat{f}_n^\alpha$ by $\hat{f}_n^\alpha(t) = \{[f(t)]_n^\alpha\}^\wedge$.

Now, we obtain the following convergence theorem.

Theorem 4.1 (Controlled convergence theorem). *Let X be a nuclear Hilbertian* (UCs-N) *space with component spaces $(X_\alpha, \{p_n^\alpha\})(\alpha \in \Xi)$. Let $\{f_j\}$ be a sequence of X-valued functions which are H-K integrable(X_α) on $[a,b]$ for some α, and let F_j be the primitive of f_j for every j. Suppose that there is a β such that:*

(1) *The image of $[a,b]$ by f_j is contained in X_β for every j, and $\lim_{j\to\infty} f_j(t) = f(t)$ in $(X_\beta, \{p_n^\beta\})$ almost everywhere on $[a,b]$.*
(2) *$\{F_j\}$ is ACG_* on $[a,b]$ uniformly in j as $(X_\beta, \{p_n^\beta\})$-valued functions.*
(3) *$\{F_j\}$ converges uniformly to F on $[a,b]$ as $(X_\beta, \{p_n^\beta\})$-valued functions.*

Then, f is H-K integrable on $[a,b]$ and

$$\lim_{j\to\infty} \int_a^b f_j(t)dt = \int_a^b f(t)dt \qquad \text{in} \quad X.$$

Proof. In the theorem we can suppose that β is the β associated with α by (†). In addition to β, take $\{m(i)\}$ and $\{n(i)\}$ associated with α by (†), i.e., for α, we can find a β and two increasing sequences of non-negative integers $\{m(0) < m(1) < \cdots\}$ and $\{n(0) < n(1) < \cdots\}$ so that $\beta \geq \alpha$, $m(i) \geq n(i)$ for every $i \in N$, and $\hat{T}_{\beta,n(i)}^{\alpha,m(i)}$ is nuclear for every $i \in N$.

Given $n \in N$, choose an $i \in N$ with $n \leq n(i)$. Then, since each f_j is *H-K* integrable on $[a,b]$ as an $(X_\alpha, \{p_n^\alpha\})$-valued function, by Proposition 1.3 $(\hat{f}_j)_{m(i)}^\alpha$ is *H-K* integrable on $[a,b]$ as an $(\hat{X}_{m(i)}^\alpha, \hat{p}_{m(i)}^\alpha)$-valued function and $(\hat{F}_j)_{m(i)}^\alpha$ is the primitive of $(\hat{f}_j)_{m(i)}^\alpha$ for every j.

From the assumptions (1), (2) and (3), it is easy to see that the following three conditions hold:

1) $\lim_{j\to\infty} (\hat{f}_j)_{n(i)}^\beta(t) = \hat{f}_{n(i)}^\beta(t)$ in $(\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta)$ a.e. on $[a,b]$.
2) $\{(\hat{F}_j)_{n(i)}^\beta\}$ is ACG_* on $[a,b]$ uniformly in j as $(\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta)$-valued functions.
3) $\{(\hat{F}_j)_{n(i)}^\beta\}$ converges uniformly to $\hat{F}_{n(i)}^\beta$ on $[a,b]$ as $(\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta)$-valued functions.

Hence, by Theorem 2.2 $\hat{f}_{n(i)}^\beta$ is *H-K* integrable on $[a,b]$ as an $(\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta)$-valued function and

$$\lim_{j\to\infty} \int_a^b (\hat{f}_j)_{n(i)}^\beta(t)dt = \int_a^b \hat{f}_{n(i)}^\beta(t)dt \quad \text{in} \quad (\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta).$$

Then, since $\lim_{j\to\infty} \int_a^b (\hat{f}_j)_{n(i)}^\beta(t)dt = \lim_{j\to\infty} (\hat{F}_j)_{n(i)}^\beta([a,b]) = \hat{F}_{n(i)}^\beta([a,b])$, we have

$$\int_a^b \hat{f}_{n(i)}^\beta(t)dt = \hat{F}_{n(i)}^\beta([a,b]) \quad \text{in} \quad (\hat{X}_{n(i)}^\beta, \hat{p}_{n(i)}^\beta).$$

Moreover, since $n \leq n(i)$, we have

$$\int_a^b \hat{f}_n^\beta(t)dt = \hat{F}_n^\beta([a,b]) \quad \text{in} \quad (\hat{X}_n^\beta, \hat{p}_n^\beta).$$

Consequently, by Proposition 1.3 f is H-K integrable(X_β) and

$$\int_a^b f(t)dt = F([a,b]) \quad \text{in} \quad (X_\beta, \{p_n^\beta\}).$$

Since the right side of this equality is $\lim_{j\to\infty} F_j([a,b]) = \lim_{j\to\infty} \int_a^b f_j(t)dt$, we have the conclusion immediately.

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