MINIMAL QUASI-INJECTIVE MODULES

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Abstract. Let R be a ring. A right R-module M is called minimal quasi-injective if every homomorphism from a simple submodule of M to M can be extended to an endomorphism of M. Some characterizations and properties of minimal quasi-injective modules are given. Some results of Nicholson and Yousif on mininjective rings are extended to these modules. Besides, V-rings are characterized by minimal quasi-injective modules.

A ring R is called right mininjective if every homomorphism from a simple right ideal of R to R can be extended to an endomorphism of R. These rings were first introduced by Harade [2], who studied the Artinian case in [2] and [3]. In their paper [5], Nicholson and Yousif studied the general case and some particular case. The nice structure of right mininjective rings have led us to extend this notion to modules. In this paper, we extend the notion of mininjective rings to minimal quasi-injective modules and many properties of mininjective rings are extended to these modules.

Throught this paper, R is an associative ring with identity and all modules are unitary . All standard notations can be found in the text book of Anderson and Fuller[1].

1. Minimal quasi-injectivity

We start with the following definition.

Definition 1.1 A right R-module M is called minimal quasi-injective if every homomorphism from a simple submodule of M to M can be extended to an endomorphism of M.

Clearly, a ring R is right mininjective if and only if R_R is minimal quasi-injective. Each principally quasi-injective module [6] is minimal quasi-injective.

Theorem 1.2 Let M be a right R-module with $S = end(M_R)$. Then the following conditions are equivalent:

(1) M is minimal quasi-injective;

(2) If mR is simple, where $m \in M$, then $l_M r_R(m) = Sm$;

(3) If mR is simple and $r_R(m) \subseteq r_R(n)$, where $m, n \in M$ and $n \neq 0$, then $Sm = Sn$;

(4) If mR is simple and $\gamma : mR \to M$ is a homomorphism, where $m \in M$, then $\gamma(m) \in Sm$;

(5) If mR is simple, where $m \in M$, then $l_M[aR \cap r_R(m)] = l_M(a) + Sm$ for all $a \in R$. .

Proof. (1)⇒(2). Since $Smr_R(m) = 0$, we always have $Sm \subseteq l_M r_R(m)$. Conversely, if $n \in l_M r_R(m)$, then $\gamma : mR \to M$ is well defined by $\gamma(mr) = nr$, so let $\hat{\gamma} \in S$ extend γ. Then $n = \gamma(m) = \hat{\gamma}(m) \in Sm$, proving (2).

(2)⇒(3). If $r_R(m) \subseteq r_R(n)$ and mR is simple, $n \neq 0$, then $r_R(m) = r_R(n)$ and nR is also simple. Hence $Sm = Sn$ by (2).

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(3)⇒(4). It is obvious that $r_R(m) \subseteq r_R(\gamma(m))$, therefore $\gamma(m) \in Sm$ by (3).

(4)⇒(1) and (5)⇒(2) are clear.

(3)⇒(5). It is always the case that $l_M(a) + Sm \subseteq l_M[aR \cap r_R(m)]$. Let $x \in l_M[aR \cap r_R(m)]$. Then $r_R(ma) \subseteq r_R(xa)$ (in fact, if $(ma)r = 0$ then $ar \in aR \cap r_R(m)$, so $(xa)r = 0$). If $ma = 0$, then $xa = 0$, so $x \in l_M(a)$ and (5) follows. If $ma \neq 0$, then maR is simple, and hence $xa = sma$ for some $s \in S$ by (3). This means that $x - sm \in l_M(a)$, so $x \in l_M(a) + Sm$. Thus $l_M[aR \cap r_R(m)] \subseteq l_M(a) + Sm$, and again (5) follows.

Let M be a right R-module. We call a right R-module N minimal M-injective, if for each simple submodule K of M, every R-homomorphism $\gamma : K \to N$ extends to M. Clearly, M is minimal quasi-injective if and only if M is minimal M-injective. It is easy to see that $\bigoplus_{i=1}^{n} N_i$ is minimal M-injective if and only if each N_i is minimal M-injective.

Our following theorem shows that min-C_2 and min-C_3 conditions which are weaker than C_2 and C_3 conditions [4] hold for minimal quasi-injective modules.

Theorem 1.3 Let M_R be a minimal quasi-injective module with $S = end(M_R)$ and let e, f be idempotents in S.

$(Min - C_2)$ If K is a simple submodule of M_R and $K \cong eM$, then $K = gM$ for some $g^2 = g \in S$.

$(Min - C_3)$ If fM is simple and $eM \cap fM = 0$, then $eM \oplus fM = gM$ for some $g^2 = g \in S$.

Proof. $(Min - C_2)$ Let K be a simple submodule of M and let $K \cong eM$. Since eM is a summand of M and M is minimal M-injective, so eM is minimal M-injective, thus K is a minimal M-injective simple submodule of M, and whence K is a summand of M.

$(Min - C_3)$ We have $eM \oplus fM = eM \oplus (1-e)fM$. Clearly, $(1-e)fM \cong fM$ because fM is simple, so $(1-e)fM = hM$ for some $h^2 = h \in S$ by $(Min - C_2)$. Hence $eh = 0, g = e + h - he$ is an idempotent such that $eg = e = ge$ and $hg = h = gh$. It follows that $eM \oplus fM = gM$.

For any module $M_R, soc(M_R)$ stands for the socle of M_R. For a simple submodule K of $M_R, soc_K(M_R)$ denotes the homogeneous component of M_R generated by K.

Our next result extends Theorem 1.14 in [5].

Theorem 1.4 Let M_R be a minimal quasi-injective module with $S = end(M_R)$, and let $m, n \in M$.

(1) If mR is simple, then Sm is also simple.

(2) If nR is simple and $nR \cong mR$, then $Sn \cong Sm$.

(3) If mR is simple, then $soc_{mR}(M_R) = SmR$ is a simple submodule of ${}_SM_R$ contained in $soc_{Sm}({}_SM)$.

(4) $soc(M_R) \subseteq soc({}_SM)$.

Proof. (1) If $0 \neq sm \in Sm$, define $\gamma : mR \to smR$ by $\gamma(x) = sx$. Then γ is a right R-isomorphism, and hence γ^{-1} extends to an endomorphism of M. Thus $m = \gamma^{-1}(sm) = \alpha(sm)$ for some $\alpha \in S$, and (1) follows.

(2) Let $\sigma : nR \to mR$ be an isomorphism. Write $\sigma(n) = ma, a \in R$. Obviously $r_R(n) = r_R(\sigma(n))$. As $\sigma(n)R = mR$ is simple, Theorem 1.2 gives $Sn = S\sigma(n) = S(ma) = (Sm)a$. Now we define $\tau : Sm \to Sn$ by $\tau(sm) = (sm)a$. Then τ is a left S-isomorphism.

(3) Write $T = soc_{mR}(M_R)$. We always have $SmR \subseteq T$. Suppose $K_R \leq M_R$ and $\sigma : mR \to K$ is an R-isomorphism. Then $r_R(m) = r_R(\sigma(m))$, so $Sm = S\sigma(m)$ by Theorem 1.2. Hence $K = \sigma(m)R \subseteq SmR$, so $T \subseteq SmR$, and thus $T = SmR$. Now let $0 \neq {}_SA_R \leq {}_ST_R$. If B_R is a simple submodule of A_R, then $B \cong mR$. So, if X_R is any submodule of M_R isomorphic to mR, let $\gamma : B \to X$ be an R-isomorphism. Then γ extends to an endomorphism s of M, so $X = \gamma(B) = s(B) \subseteq A$. This means that $T \subseteq A$, therefore T is a

simple submodule of ${}_SM_R$. Finally, for any $r \in R$, we define $\phi_r : Sm \to {}_SM$ by $sm \mapsto smr$. Then ϕ_r is a left S-homomorphism, so $Smr \subseteq soc_{Sm}({}_SM)$, and thus $SmR \subseteq soc_{Sm}({}_SM)$.

(4) This follows from (1).

Recall that *a* ring R is said to be a right V-ring if every simple right R-module is injective. Our following theorem gives a new characterization of right V-rings.

For any right R-module $M, E(M)$ denotes the injective hull of M.

Theorem 1.5 A ring R is a right V-ring if and only if every right R-module is minimal quasi-injective.

Proof. We need only to prove the sufficiency. Let K be any simple right R-module. Since $K \oplus E(K)$ is minimal quasi-injective, K is minimal $K \oplus E(K)$-injective, and hence K is minimal $E(K)$-injective. Therefore, $K = E(K)$ is injective. This proves the theorem.

2.Duality

Let M be a right R-module with $S = end(M_R)$. If N is a right R-module, then $hom_R(N_R, {}_SM_R)$ is a left S-module. Here, if $s \in S$ and $f \in hom_R(N_R, {}_SM_R)$, the map sf is defined by $(sf)(n) = s(f(n))$. We call the left S-module $hom_R(N_R, {}_SM_R)$ the M-dual of N_R.

Lemma 2.1 Let $N = nR(n \in N)$ be a cyclic module and let $T = r_R(n)$. If M is a right R-module with $S = end(M_R)$, then $hom_R(N_R, {}_SM_R) \cong l_M(T)$.

Proof. For any $m \in l_M(T)$, let $f_m : N \to M$ by $f_m(nr) = mr$. Then f_m is a right R-homomorphism. Now we define $\sigma : l_M(T) \to hom_R(N_R, {}_SM_R)$ by $\sigma(m) = f_m$. Then σ is a left S-isomorphism.

The next result gives an important characterization of minimal quasi-injective modules in terms of duality.

Theorem 2.2 The following conditions are equivalent for a module M_R with $S = end(M_R)$:

(1) M_R is minimal quasi-injective;

(2) $hom_R(N_R, {}_SM_R)$ is a simple or zero left S-module for all simple right R-module N;

(3) $l_M(T)$ is simple or zero left S-module for all maximal right ideals T of R.

Proof. (1)$\Rightarrow$ (2). Let $\gamma, \delta \in hom_R(N_R, {}_SM_R)$, where N_R is simple, and assume that $\gamma \neq 0$. Then $\delta\gamma^{-1} : \gamma(N) \to M$ is a homomorphism. Since $\gamma(N)$ is simple, $\delta\gamma^{-1}$ can be extended to an endomorphism α of M by (1). Thus $\delta = \alpha\gamma$, proving (2).

(2)$\Rightarrow$(3). Let T be a maximal right ideal. Take $N = R/T, n = 1 + T$. Then $T = r_R(n)$, and $l_M(T) \cong hom_R(R/T, M)$ by Lemma 2.1. Consequently, $l_M(T)$ is simple or zero by (2).

(3)$\Rightarrow$ (1). Let $\gamma : mR \to M$ be an R-homomorphism, where mR is simple, and let $i : mR \to M$ be the inclusion map. Write $T = r_R(m)$. Then T is a maximal right ideal of R, so $l_M(T) \cong hom_R(mR, M)$ by Lemma 2.1. Thus $hom_R(mR, M)$ is simple, and whence $\gamma = \beta i$ for some $\beta \in S$, proving(1).

Recall that a right R-module M is called a Kasch module if every simple subquotient of M embeds in M[6]. We call M_R strongly Kasch if every simple right R-module embeds in M. Clearly, if M is a generator, then M is Kasch if and only if M is strongly Kasch.

The following theorem gives important properties of minimal quasi-injective strongly Kasch modules.

Theorem 2.3 Let M_R be a minimal quasi-injective strongly Kasch module with $S = end(M_R)$, and consider the map

$$\theta : T \longmapsto l_M(T)$$

from the set of maximal right ideals T of R to the set of minimal submodules of ${}_SM$.

(1) θ is one-to-one.

(2) θ is a bijection if and only if $l_M r_R(K) = K$ for all minimal submodules K of ${}_S M$. In this case θ^{-1} is given by $K \mapsto r_R(K)$.

Proof. (1) If T is a maximal right ideal, and $\phi : R/T \to M$ is a monomorphism, then $0 \neq \phi(1+T) \in l_M(T)$, and so $l_M(T) \neq 0$. This implies that $l_M(T)$ is simple by Theorem 2.2. Since $T \subseteq r_R l_M(T) \neq R$, we have $T = r_R l_M(T)$ because T is maximal. Now (1) follows.

(2) Suppose θ is onto and K is a minimal submodule of ${}_S M$. If $K = l_M(T)$, where T is a maximal right ideal of R, then $l_M r_R(K) = l_M r_R l_M(T) = l_M(T) = K$. Conversely, assume that $l_M r_R(K) = K$ for all minimal submodules K of ${}_S M$.

Claim. If K is a minimal submodule of ${}_S M$, then $r_R(K)$ is a maximal right ideal.

Proof. Let $r_R(K) \subseteq T$, where T is a maximal right ideal. Then $K = l_M r_R(K) \supseteq l_M(T) \neq 0$ by the proof of (1), so $K = l_M(T)$ by the minimality of K. Thus $r_R(K) = r_R l_M(T) \supseteq T$, and whence $r_R(K) = T$. This proves the claim.

By the claim we have a map ϕ given by $K \mapsto r_R(K)$, which we assert is the inverse of θ. Indeed, $(\phi\theta)(T) = (\phi(\theta(T)) = r_R l_M(T) = T$ by the calculation in (1), while $(\theta\phi)(K) = (\theta(\phi(K)) = l_M r_R(K) = K$. This completes the proof of (2).

Motivated by Theorem 2.3, we call a module M_R with $S = end(M_R)$ a minannihilator module if, for every minimal submodule K of ${}_S M$, there exists a subset $X \subseteq R$ such that $K = l_M(X)$, equivalently, if $l_M r_R(K) = K$. Motivated by Theorem 1.4, we call a module M_R with $S = end(M_R)$ minsymmetric if mR is simple, where $m \in M$, implies that Sm is also simple.

Theorem 2.4 The following are equivalent for a minannihilator module M_R:

(1) M_R is minimal quasi-injective;

(2) M_R is minsymmetric;

(3) $soc(M_R) \subseteq soc({}_S M)$.

Proof. We have (1)⇒(2) by Theorem 1.4; (2)⇒(3) always holds.

(3)⇒(1). Given (3), let mR be simple. Then $m \in soc({}_S M)$ by (3), so Sm contains a simple submodule Sn, thus $r_R(m) \subseteq r_R(n)$ and so $r_R(m) = r_R(n)$ because $r_R(m)$ is maximal. Since M_R is a minannihilator module and Sn is simple, $Sm \subseteq l_M r_R(Sm) = l_M r_R(Sn) = Sn$. It follows that $Sm = l_M r_R(Sm) = l_M r_R(m)$, proving (1).

The proof of the implication (3)⇒(1) in Theorem 2.4 also yields the following

Corollary 2.5 If M_R is a minannihilator module such that $soc({}_S M)$ is essential in ${}_S M$, where $S = end(M_R)$, then M_R is minimal quasi-injective.

Finally , we give a characterization of minsymmetric modules.

Theorem 2.6 The following are equivalent for a module M_R :

(1) M_R is minsymmetric;

(2) If mR is simple, then $l_S[mR \cap ker(s)] = l_S(m) + Ss$ for all $s \in S$, where $S = end(M_R)$.

Proof. (1)⇒(2). Assume mR is simple and let $s \in S$. Clearly, $l_S(m) + Ss \subseteq l_S[mR \cap ker(s)]$. If $sm = 0$, then $mR \cap ker(s) = mR$, and so $l_S[mR \cap ker(s)] = l_S(mR) = l_S(m) \subseteq l_S(m) + Ss$. If $sm \neq 0$, then $mR \cap ker(s) = 0$, and hence $l_S[mR \cap ker(s)] = S = l_S(m) + Ss$ because Sm is simple. Therefore (2) follows.

(2)⇒(1). Let mR be simple. If $s \overline{\in} l_S(m)$, then $mR \cap ker(s) = 0$, so $l_S(m) + Ss = S$ by (2). This shows that $l_S(m)$ is maximal, proving (1).

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