ON N-SEMIHEREDITARY RINGS

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Abstract. A ring R is said to be left n-semihereditary, if every n-generated left ideal of R is projective. It is shown that for a ring R, the following statements are equivalent: (1) R is left n-semihereditary; (2)every n-generated submodule of a projective left R-module is projective; (3)every torsion-less right R-module is n-flat; (4)R is left n-coherent and every n-generated right ideal of R is flat; (5)R is left n-coherent and every right ideal of R is n-flat; (6)every factor module of an n-injective left R-module is n-injective; (7)the sum of an arbitrary family of n-injective submodules of a left R-module is n-injective. Moreover, some new characterizations of Prüfer rings are given.

Throughout this paper R denotes an associative ring with identity, and all modules are unitary R-modules.

Let m, n be two positive integers. An R-module M is said to be n-generated if it has a generating set of cardinality at most n[5]. A left R-module M is called n-injective, if for every n-generated left ideal I of R, each R-homomorphism from I to M can be extended to R[10],[11]. A left R-module M is said to be (m, n)-injective, if for every n-generated submodule I of the left R-module R^m, each R-homomorphism from I to M can be extended to R^m[3]. Clearly, M is n-injective iff M is $(1, n)$-injective. Following[11], R is said to be left n-coherent, if every n-generated left ideal of R is finitely presented. A left(right) R-module M is called n-flat if for every n-generated right(left) ideal I of R, the canonical map $I \otimes_R M \to M(M \otimes_R I \to M)$ is monic. n-flat modules have been studied in [5] and [11]. In this paper, we extend the concept of left semihereditary rings and introduce the concept of left n-semihereditary rings. Several characterizations of left n-semihereditary rings are given by means of n-injective modules, (m, n)-injective modules, n-flat modules, projective modules, injective modules, flat modules, torsion-less modules and left n-coherent rings.

Definition 1. A ring R is said to be left n-semihereditary, if every n-generated left ideal of R is projective.

We note that this definition is at odds with another definition of left n-semihereditary rings, see[11]. It is obvious that R is left semihereditary if and only if R is left n-semihereditary for each positive integer n, R is a left p.p. ring if and only if R is left l-semihereditary.

Example. Let n be any natural number and let R be the K-algebra (K is any field) on the $2(n+1)$ generators $X_i, Y_i (i = 1, \cdots, n+1)$ and defining relations $\sum_{i=1}^{n+1} X_i Y_i = 0$. It follows from [8] that R is left n-semihereditary, but R is not left $(n+1)$-semihereditary.

Lemma 1. If R is left n-semihereditary, then every n-generated submodule A of a free left R-module F is isomorphic to a direct sum of finitely many n-generated left ideals.

Proof: Let F have basis $\{x_k \mid k \in K\}$. Since A is finitely generated, A is contained in a free summand of F generated by finitely many $x_k's$. We may, therefore, assume F is free with

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basis $\{x_1, \cdots, x_m\}$. We prove by induction on m that A is isomorphic to a finite direct sum of n-generated left ideals. If $m = 1$, then A is isomorphic to an n-generated left ideal. If $m > 1$, define $B = A \cap (Rx_1 \oplus \cdots \oplus Rx_{m-1})$. Each $a \in A$ has a unique expression $a = b + rx_m$, where $b \in Rx_1 \oplus \cdots \oplus Rx_{m-1}, r \in R$. If $\varphi : A \to R$ is defined by $a \mapsto r$, then there is an exact sequence $0 \to B \to A \xrightarrow{\varphi} I \to 0$, where $I = im\varphi$ is an n-generated left ideal. Since I is projective, $A \cong B \oplus I$ and so B is n-generated. Since B is contained in $Rx_1 \oplus \cdots \oplus Rx_{m-1}$, the induction hypothesis gives B, hence A, is isomorphic to a finite direct sum of n-generated left ideals.

Theorem 1. A ring R is left n-semihereditary if and only if every n-generated submodule of a projective R-module is projective.

Proof: Suppose R is left n-semihereditary. Let A be an n-generated submodule of a projective module. Then A is an n-generated submodule of a free module. By Lemma 1, A is isomorphic to a direct sum of n-generated left ideals, each of which is projective since R is left n-semihereditary. Therefore, A is projective.

The converse is obvious.

Lemma 2. The following statements are equivalent for a ring R:

(1) All n-generated left ideals of R are flat;
(2) All n-generated right ideals of R are flat;
(3) Submodules of n-flat left R-modules are n-flat;
(4) Submodules of n-flat right R-modules are n-flat.

Proof (1)$\Leftrightarrow$ (2). See [5], Theorem 2.2.
(2)$\Leftrightarrow$ (3) and (1)$\Leftrightarrow$ (4). See [11], §5,(f).

Lemma 3. For a ring R, the following statements are equivalent:

(1) Every direct product of n-flat right R-modules is n-flat;
(2) R_R^A is n-flat for every set A;
(3) R is left n-coherent.

Proof: By virtue of Proposition 4.1 in [11], the proof is similar to that of [1, Theorem 19.20].

Lemma 4 [11, §5(a)]. If M is an n-generated n-flat R-module, then it is flat.

Now we give following characterizations of left n-semihereditary rings.

Theorem 2. The following statements are equivalent for a ring R:

(1) R is left n-semihereditary;
(2) R is left n-coherent and every n-generated right ideal of R is flat;
(3) Every torsion-less right R-module is n-flat;
(4) R is left n-coherent and every right ideal of R is n-flat;
(5) R is left n-coherent and every submodule of an n-flat right R-module is n-flat.

Proof: (2)$\Leftrightarrow$ (5). By Lemma 2.

(1) $\Rightarrow$ (2). Let R be left n-semihereditary. Then each n-generated left ideal of R is projective, and hence is flat and finitely presented. By Lemma 2, every n-generated right ideal of R is flat.

(5)$\Rightarrow$ (3). Let X be a torsion-less right R-module. Then by [6, Proposition 23.4], there exists an R-monomorphism $X \to R_R^A$ in a direct product of copies of R. Since R is left n-coherent, R_R^A is n-flat by Lemma 3. Hence it follows from (5) that X is n-flat.

(3)$\Rightarrow$ (4). Since submodules of torsion-less R-modules are torsion-less and R_R is torsion-less, so each right ideal of R is torsion-less. It follows from (3) that each right ideal of R is n-flat. For the other, let R_R^A be any product of copies of R. Since R_R^A is torsion-less, by (3), R_R^A is n-flat. Hence by Lemma 3, R is left n-coherent.

(4)$\Rightarrow$ (1). Let I be any n-generated right ideal of R. Then I is flat by (4) and Lemma 4. Hence every n-generated left ideal is flat by lemma 2. Moreover, since R is left n-coherent, every n-generated left ideal of R is finitely presented, and hence is projective.

Theorem 3. For a ring R, the following statements are equivalent:

(1) R is left n-semihereditary;

(2) For every positive integer m, each factor module of an (m,n)-injective left R-module is (m,n)-injective;

(3) Each factor module of an n-injective left R-module is n-injective;

(4) Each factor module of an injective left R-module is n-injective;

(5) For every positive integer m and every left R-module A, the sum of an arbitrary family of (m,n)-injective submodules of A is (m,n)-injective;

(6) For every left R-module A, the sum of an arbitrary family of n-injective submodules of A is n-injective.

Proof: (1)$\Rightarrow$ (2). Consider the diagram 1. Here E is an (m,n)-injective left R-module, E' is a homomorphic image of E, α is an epimorphism from E to E', I is an n-generated submodule of R^m and $f \in Hom_R(I,E')$.

$$\begin{array}{ccccc} 0 & \longleftarrow & E' & \overset{\alpha}{\longleftarrow} & E \\ & & \Big\uparrow f & \gamma\nearrow & \Big\uparrow \delta \\ 0 & \longrightarrow & I & \overset{i}{\longrightarrow} & R^m \end{array}$$

diagram 1

$$\begin{array}{ccccc} 0 & \longleftarrow & E' & \overset{\alpha}{\longleftarrow} & E \\ & & \Big\uparrow f & \nwarrow\beta & \Big\uparrow \gamma \\ 0 & \longrightarrow & I & \overset{i}{\longrightarrow} & R \end{array}$$

diagram 2

Since R is left n-semihereditary, by Theorem 1, I is projective. Hence there exists a $\gamma \in Hom_R(I,E)$ such that $f = \alpha\gamma$. But E is (m,n)-injective, so there exists a $\delta \in Hom_R(R^m,E)$ with $\gamma = \delta i$. Therefore $\alpha\delta \in Hom_R(R^m,E')$ and $f = (\alpha\delta)i$, and thus E' is (m,n)-injective.

(2)$\Rightarrow$ (3)$\Rightarrow$ (4) and (5)$\Rightarrow$ (6) are clear.

(4)$\Rightarrow$ (1). Consider the diagram 2, where E is injective, α is epic and I is an n-generated left ideal, $f \in Hom_R(I,E')$. Assume (4). Then E' is n-injective and there exists a $\beta \in Hom_R(R,E')$ such that $f = \beta i$. Since R is projective, there is a $\gamma \in Hom_R(R,E)$ such that $\beta = \alpha\gamma$. Then $f = \alpha(\gamma i)$ and so I is projective.

(2)$\Rightarrow$ (5). Let $\{A_i \mid i \in I\}$ be an arbitrary family of (m,n)-injective submodules of A. Since the direct sum of (m,n)-injective modules is (m,n)-injective and $\sum_{i\in I} A_i$ is a homomorphic image of $\oplus_{i\in I}A_i$, by (2), $\sum_{i\in I} A_i$ is (m,n)-injective.

(6)$\Rightarrow$ (4). Let E be an injective left R-module and $K \leq E$. Take $E_1 = E_2 = E, N = E_1 \oplus E_2, D = \{(x,-x) \mid x \in K\}$. Define $f_1 : E_1 \to N/D$ by $e_1 \mapsto (e_1,0)+D, f_2 : E_2 \to N/D$ by $e_2 \mapsto (0,e_2)+D$ and write $\overline{E}_i = f_i(E_i), i = 1,2$. Then $\overline{E}_i \cong E_i$ is injective, $i = 1,2$, and hence $N/D = \overline{E}_1 + \overline{E}_2$ is n-injective. By the injectivity of $\overline{E}_i, (N/D)/\overline{E}_i$ is isomorphic to a summand of N/D and thus it is n-injective.

Now we define $f : E \to (N/D)/\overline{E}_1$ by $e \mapsto f_2(e) + \overline{E}_1$. Then f is epic and $ker f = K$, therefore $E/K \cong (N/D)/\overline{E}_1$ is n-injective.

Observing that an R-module M is FP-injective if and only if M is (m, n)-injective for each pair of positive integers m and n, and that M is F-injective if and only if M is n-injective for each positive integer n, by Theorem 3 and Theorem 2, we have immediately the following corollary.

Corollary. The following statements are equivalent for a ring R:

(1) R is left semihereditary;

(2) Factor modules of FP-injective left R-modules are FP-injective;

(3) Factor modules of F-injective left R-modules are F-injective;

(4) Factor modules of injective left R-modules are F-injective;

(5) For every left R-module A, the sum of an arbitrary family of FP-injective submodules of A is FP-injective;

(6) For every left R-module A, the sum of an arbitary family of F-injective submodules of A is F-injective;

(7) R is left coherent and $\mathrm{wD}(R) \leq 1$;

(8) Every torsion-less right R-module is flat.

Lemma 5[4, Theorem 3.3]. Let R be an integral domain. Then an R-module A is l-flat if and only if A is torsion-free.

Finally, we give some new characterizations of Prüfer rings.

Theorem 4. The following statements are equivalent for an integral domain R:

(1) R is a Prüfer ring;

(2) R is 2-semihereditary;

(3) Every 2-generated torsion-free R-module is projective;

(4) Every torsion-free R-module is 2-flat;

(5) Every 1-flat R-module is 2-flat;

(6) Every divisible R-module is 2-injective.

Proof: (1)⇒ (2) and (3)⇒ (2) are trivial.

(2)⇒ (1). Immediate consequence of [7, Theorem 22.1].

(2)⇒ (3). Suppose A is a 2-generated torsion-free R-module. Then A embeds in a f.g. free R-module because R is an integral domain. Since R is 2-semihereditary, by Theorem 1, A is projective.

(2)⇒ (4). Since R is an integral domain, by [2], each f.g. torsion-free R-module is torsion-less. As R is 2-semihereditary, by Theorem 2, each torsion-less R-module is 2-flat. Hence each f.g. torsion-free R-module is 2-flat. Recall that submodules of torsion-free R-modules are torsion-free and every R-module is the direct limit of its f.g. submodules, so by [11, Proposition 4.1(iii)], each torsion-free R-module is 2-flat.

(4)⇒ (2). Since R is an integral domain, every torsion-less R-module A is torsion-free. It follows from (4) that A is 2-flat. By Theorem 2, R is 2-semihereditary.

(4)⇔ (5). By Lemma 5.

(1)⇒ (6). Every divisible module over a Prüfer ring is FP-injective (see[9], Theorem 6), so (6) follows from (1).

(6)⇒ (2). Suppose every divisible R-module is 2-injective. Noting that 2-injective modules are divisible, factor modules of 2-injective R-modules are 2-injective. Therefore, by Theorem 3, R is 2-semihereditary.

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