STABLE RANK OF THE SEMIGROUP CROSSED PRODUCTS BY NATURAL NUMBERS

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Abstract. We estimate the stable rank of crossed products by actions of the additive semigroups of natural numbers under certain conditions on actions. As an application we estimate the stable rank of canonical subalgebras of the Hecke C^*-algebra of Bost and Connes, which is isomorphic to the crossed product by the multiplicative semigroup of natural numbers of Laca and Raeburn, and also estimate that of the crossed products by the additive semigroups of natural numbers of Larsen and Raeburn.

1 Introduction The (topological) stable rank for C^*-algebras was introduced by Rieffel [Rf] to study the dimension theory of C^*-algebras and the (non-stable) K-theory of C^*-algebras (cf. [Bl] and [BP] for the real rank of C^*-algebras). As one of interesting results in [Rf] the stable rank of the ordinary crossed products $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ of C^*-algebras $\mathfrak{A}$ by actions α of $\mathbb{Z}$ the group of integers by automorphisms was estimated as: $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ (see [Pd] for the general theory of crossed products of C^*-algebras). On the other hand, crossed products of C^*-algebras by semigroups have also been of great interest, and their representation theory and structures have been investigated by Murphy [Mp1], [Mp3-6], Laca and Raeburn [LR1-2] and some many others (cf. [A-R], [ALR], [Lc], [Sc]). In particular, Laca and Raeburn [LR2] studied the Hecke C^*-algebra of Bost and Connes as the crossed product by actions of the multiplicative semigroup of natural numbers (cf. [LsR] for certain crossed products by actions of the products of the additive semigroup of natural numbers).

Under the situation given above, it should be interesting and useful to obtain the similar stable rank estimate for the case of semigroup crossed products as the case of ordinary crossed products. Our first motivation is in fact to estimate the stable rank of the Hecke C^*-algebra of Bost-Connes described by Laca-Raeburn as a semigroup crossed product, and it is found by us that existence of a left inverse of an action and its certain conditions (a sort of right inverse of an action) are useful for calculating the stable rank of canonical subalgebras of the Hecke C^*-algebra. As the first step we consider the case of crossed products by the additive semigroup of natural numbers. It would be possible to generalize this case to the case of crossed products by some general abelian or non-abelian semigroups. In particular, it would be an interesting problem to consider the stable rank of the C^*-algebras of semigroups (cf. Remarks 2.2.2, 2.4.1, 2.5.2 and 2.5.3 below). Refer to Sheu [Sh], Takai and the author [ST] and [Sd1-5] for some works on the stable rank of group C^*-algebras.

In this paper we first estimate the stable rank and connected stable rank of crossed products of C^*-algebras by the additive semigroup $\mathbb{N}$ of natural numbers. We next consider the case of crossed products by the products $\mathbb{N}^k$ similarly. Our technique for the proofs is based on the Rieffel's proof for the case of crossed products by the integers $\mathbb{Z}$ [Rf, Theorem 7.1]. However, there are some differences between group and semigroup cases so that we

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need to be more careful about generators in crossed products and the definition of lengths of elements of dense parts of crossed products. Consequently, we obtain the stable rank estimates of canonical subalgebras (certain crossed products by $\mathbb{N}^k$) of the Hecke C^*-algebra of Bost-Connes (or the semigroup crossed product of Laca-Raeburn), and obtain those of the crossed products by $\mathbb{N}^k$ of Larsen and Raeburn.

Notation Let $\mathbb{N}$ denote the additive semigroup of all natural numbers. Let $\mathbb{N}^*$ be the multiplicative semigroup of all natural numbers. For a C^*-dynamical system $(\mathfrak{A}, S, \alpha)$ of a C^*-algebra $\mathfrak{A}$, an abelian semigroup S and an action α of S by endomorphisms of $\mathfrak{A}$, its covariant representation (π, V, H) (or (π, V) in what follows) consists of a nondegenerate representation π of $\mathfrak{A}$ and an isometric representation V of S on the same Hilbert space H such that $\pi(\alpha_n(a)) = V_n \pi(a) V_n^*$ for $n \in S$ and $a \in \mathfrak{A}$ (the covariance of (π, V)). The crossed product $\mathfrak{A} \rtimes_\alpha S$ of $(\mathfrak{A}, S, \alpha)$ is the universal C^*-algebra generated by the universal covariant representation $(\pi^{(u)}, V^{(u)}, H^{(u)})$ of $(\mathfrak{A}, S, \alpha)$ in the sense that for any covariant representation (π, V, H) of $(\mathfrak{A}, S, \alpha)$ there exists a $*$-homomorphism $\pi \times V$ from $\mathfrak{A} \rtimes_\alpha S$ to $\mathbb{B}(H)$ the C^*-algebra of all bounded operators on H such that $\pi = (\pi \times V) \circ \pi^{(u)}$ and $V = (\pi \times V) \circ V^{(u)}$ (compositions). Therefore, the crossed product $\mathfrak{A} \rtimes_\alpha S$ may be assumed to be generated by $\pi(\mathfrak{A})$ and V_n for $n \in S$ if the representation $\pi \times U$ of $\mathfrak{A} \rtimes_\alpha S$ associated with (π, V) is faithful although existence of such faithful representations is non-trivial in general (cf. [LR1-2]).

For a C^*-algebra $\mathfrak{A}$ (or its unitization $\mathfrak{A}^+$), we denote by $\mathrm{sr}(\mathfrak{A})$, $\mathrm{csr}(\mathfrak{A})$ the (topological) stable rank and connected stable rank of $\mathfrak{A}$ respectively [Rf]. By definition, for $n \in \mathbb{N}$, $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if the set $L_n(\mathfrak{A})$ of all elements $(a_j) \in \mathfrak{A}^n$ with $\sum_{j=1}^n a_j^* a_j$ invertible in $\mathfrak{A}$ is dense in $\mathfrak{A}^n$, and $\mathrm{csr}(\mathfrak{A}) \leq n$ if and only if $L_m(\mathfrak{A})$ is connected for all $m \geq n$ (note that $\sum_{j=1}^n a_j^* a_j$ is invertible if and only if there exists $(b_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n b_j a_j$ is invertible in $\mathfrak{A}$). If no such $n \in \mathbb{N}$, set $\mathrm{sr}(\mathfrak{A}) = \infty$ and $\mathrm{csr}(\mathfrak{A}) = \infty$. See [Bl], [Pd] and [Mp2] for some other related topics.

2 Stable rank of semigroup crossed products

First of all, we check the following conditions which are used for the stable rank estimates below (cf. [Mp6, Section 4]):

Proposition 2.1 *Let $(\mathfrak{A}, S, \alpha)$ be a C^*-dynamical system and (π, V) its covariant representation in the sense of Notation. Suppose that π is faithful on $\mathfrak{A}$ and the property of β : $V_n^* \pi(a) = \pi(\beta_n(a)) V_n^*$ for $n \in S$ and $a \in \mathfrak{A}$ where β is an action of S by endomorphisms of $\mathfrak{A}$. Then β_n is a left inverse for α_n. Hence α_n is injective.*

In addition, if $\mathfrak{A}$ is unital, then $V_n V_n^* \in \pi(\mathfrak{A})$, $\beta_n(V_n V_n^*)$ is the identity and $\alpha_n(\mathfrak{A})$ is the corner $p_n \mathfrak{A} p_n$ with $p_n = V_n V_n^*$.

Proof. For each $n \in S$ and $a \in \mathfrak{A}$, using the property of β we have

$$\pi(\beta_n(\alpha_n(a)) = V_n^* \pi(\alpha_n(a)) V_n = V_n^* V_n \pi(a) V_n^* V_n = \pi(a).$$

Since π is faithful, β_n is a left inverse for α_n so that α_n is injective.

If $\mathfrak{A}$ is unital, then $\pi(\alpha_n(1)) = V_n \pi(1) V_n^* = V_n V_n^* \in \pi(\mathfrak{A})$ by the covariance. Since π is faithful, we identify $\pi(\mathfrak{A})$ with $\mathfrak{A}$ in the following. Set $p_n = V_n V_n^* \in \mathfrak{A}$. For $a \in \mathfrak{A}$,

$$\begin{aligned} \alpha_n(\beta_n(p_n)a) &= V_n \beta_n(p_n) a V_n^* = V_n \beta_n(p_n) V_n^* (V_n a V_n^*) \\ &= V_n V_n^* p_n (V_n a V_n^*) = V_n a V_n^* = \alpha_n(a). \end{aligned}$$

Since α_n is injective, $\beta_n(p_n)a = a$ for all $a \in \mathfrak{A}$. Thus, $\beta_n(p_n)$ is the identity for $\mathfrak{A}$. Furthermore, we have $\alpha_n(a) = V_n a V_n^* = V_n a V_n^* V_n V_n^* = \alpha_n(a) p_n$. Thus, we have $p_n \alpha_n(a) p_n = p_n \alpha_n(a) = V_n V_n^* V_n a V_n^* = \alpha_n(a)$. □

Remark 2.1.1**.** Conversely, if α_n is injective, under the same situation as above note that

$$\beta_n(\alpha_n(a))V_n^* = aV_n^* = V_n^*(V_n a V_n^*) = V_n^* \alpha_n(a).$$

Thus, the action β_n satisfying the property on $\alpha_n(\mathfrak{A})$ can be defined to be the left inverse of α_n on the restriction to $\alpha_n(\mathfrak{A})$, but it would be nontrivial to have β_n defined on $\mathfrak{A}$ satisfying the property.

Remark 2.1.2**.** The property of β: $V_n^*\pi(a) = \pi(\beta_n(a))V_n^*$ is equivalent to the equality $\pi((\alpha_n \circ \beta_n)(a)) = V_n V_n^* \pi(a)$. By taking conjugation, it is also equivalent to $\pi((\alpha_n \circ \beta_n)(a)) = \pi(a)V_n V_n^*$, which suggests that β is a sort of a right inverse of α. The condition $V_n V_n^* \in \pi(\mathfrak{A})$ with π faithful implies that $\mathfrak{A}$ can not be projectionless. Non-unitarity of isometries V_n is closely related to faithfulness of its covariant representation (cf. [LR2, Proposition 1.1]).

Theorem 2.2 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{N}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by $\mathbb{N}$ and (π, V) a covariant representation of $(\mathfrak{A}, \mathbb{N}, \alpha)$ with π and $\pi \times V$ faithful. Suppose that there exists an action β of $\mathbb{N}$ by endomorphisms of $\mathfrak{A}$ such that $V_n^*\pi(a) = \pi(\beta_n(a))V_n^*$ for any $n \in \mathbb{N}$ and $a \in \mathfrak{A}$. Then $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}) \le \mathrm{sr}(\mathfrak{A}) + 1$.*

In addition, if $\mathfrak{A}$ is unital, then $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}) \le \mathrm{sr}(\mathfrak{A}) + 1$. Also, if V_n for some $n \in \mathbb{N}$ is non-unitary, then $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}) \ge 2$ and $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}) \ge 2$.

Proof. The lower estimate $\mathrm{sr}(\mathfrak{A} \rtimes \mathbb{N}) \ge 2$ is easily deduced from that $\mathfrak{A} \rtimes \mathbb{N}$ contains a non-unitary isometry V_n. In fact, if $\mathrm{sr}(\mathfrak{A} \rtimes \mathbb{N}) = 1$, then there exists an element W_n of $\mathfrak{A} \rtimes \mathbb{N}$ such that $W_n V_n^* = 1$ for $n \in \mathbb{N}$, which is impossible. Also, $\mathrm{csr}(\mathfrak{A} \rtimes \mathbb{N}) = 1$ means that left invertibles of $\mathfrak{A} \rtimes \mathbb{N}$ are invertible (cf. [Rf, p.312]), which is impossible when $\mathfrak{A} \rtimes \mathbb{N}$ contains a proper isometry.

When $\mathfrak{A}$ is non-unital, we consider the exact sequence: $0 \to \mathfrak{A} \rtimes_\alpha \mathbb{N} \to \mathfrak{A}^+ \rtimes_{\alpha^+} \mathbb{N} \to \mathbb{C} \rtimes \mathbb{N} \to 0$, where the extended action α^+ of $\mathbb{N}$ on $\mathbb{C}1$ of $\mathfrak{A}^+$ is trivial. Then $\mathrm{sr}(\mathfrak{A} \rtimes \mathbb{N}) \le \mathrm{sr}(\mathfrak{A}^+ \rtimes \mathbb{N})$ by [Rf, Theorem 4.4]. Note that $\mathfrak{A}^+ \rtimes \mathbb{N} = \mathfrak{A}^+ \rtimes \mathbb{Z}$ by the covariance since α^+ is trivial on $\mathbb{C}1$. By [Rf, Theorem 7.1], we obtain $\mathrm{sr}(\mathfrak{A}^+ \rtimes \mathbb{Z}) \le \mathrm{sr}(\mathfrak{A}) + 1$. Thus we may assume that $\mathfrak{A}$ is unital in the following. By Proposition 2.1, we have $V_n V_n^* \in \pi(\mathfrak{A})$ for all $n \in \mathbb{N}$. We also identify $\mathfrak{A}$ with $\pi(\mathfrak{A})$ in the following since π is faithful.

By a part of assumptions, $\mathfrak{A} \rtimes \mathbb{N}$ may be assumed to be generated by the linear span of elements of the form $aV_x V_y^*$ for $x, y \in \mathbb{N} \cup \{0\}$ and $a \in \mathfrak{A}$, where we set $V_0 = 1 \in \mathfrak{A}$. In fact, using the property of β and the covariance of (π, V) we have

$$\begin{aligned}(aV_{x_1}V_{y_1}^*)(bV_{x_2}V_{y_2}^*) &= aV_{x_1}\beta_{y_1}(b)V_{y_1}^* V_{x_2} V_{y_2}^* \\ &= a(V_{x_1}\beta_{y_1}(b)V_{x_1}^*)V_{x_1}V_{y_1}^*V_{x_2}V_{y_2}^* = (a\alpha_{x_1}(\beta_{y_1}(b)))V_{x_1}V_{y_1}^*V_{x_2}V_{y_2}^*.\end{aligned}$$

Moreover, when $y_1 < x_2$, $V_{y_1}^* V_{x_2} = V_{x_2 - y_1}$, and when $y_1 > x_2$, $V_{y_1}^* V_{x_2} = V_{y_1 - x_2}^*$, and

$$(aV_x V_y^*)^* = V_y V_x^* a^* = V_y \beta_x(a^*) V_x^* = (V_y \beta_x(a^*) V_y^*) V_y V_x^* = \alpha_y(\beta_x(a^*)) V_y V_x^*.$$

Denote by $\mathfrak{B}$ the set of all linear spans of elements of the the form $aV_x V_y^*$. Moreover, since $V_n V_n^* \in \mathfrak{A}$, observe that $aV_x V_y^* = (aV_x V_x^*)V_{y-x}^*$ when $y > x$, and

$$aV_x V_y^* = aV_{x-y}(V_y V_y^*) = (aV_{x-y}(V_y V_y^*)V_{x-y}^*)V_{x-y} = a\alpha_{x-y}(V_y V_y^*)V_{x-y}$$

when $y < z$. Therefore, $\mathfrak{B}$ is in fact equal to the set of all linear spans of elements of the the forms aV_x and bV_y^* for $a, b \in \mathfrak{A}$.

Define the length of a nonzero finite sum $d = \sum_{k=1}^{l} a_k V_{x_k} + \sum_{k=1}^{m} b_k V_{y_k}^*$ with $0 \le x_1 < \cdots < x_l$ and $0 < y_1 < \cdots < y_m$ by $L(d) = x_l + y_m + 1$ when both x_l and y_m exist, and

$L(d) = x_l - x_1 + 1$ when no y_k exists, and $L(d) = y_m - y_1 + 1$ when no x_k exists. Set $L(0) = 0$. Note that $L(V_1 d) = L(d)$ and $L(V_1^* d) = L(d)$. In fact, observe that

$$V_1 a V_x = (V_1 a V_1^*) V_{x+1}, \quad V_1 a V_y^* = (V_1 a V_1^*) V_1 V_y^* = (\alpha_1(a) V_1 V_1^*) V_{y-1}^*, \text{ and}$$
$$V_1^* a V_x = \beta_1(a) V_1^* V_x = \beta_1(a) V_{x-1}, \quad V_1^* a V_y^* = \beta_1(a) V_{y+1}^*.$$

To have the length $L(d)$ well-defined, we of course use the cancellations in the expression of d such as $V_n a V_n^* = \alpha_n(a)$ and $V_n^* V_n = 1$ for $n \in \mathbb{N}$.

Now suppose that $\mathrm{sr}(\mathfrak{A}) \leq n$. Let $(c_j)_{j=1}^{n+1} \in (\mathfrak{A} \rtimes \mathbb{N})^{n+1}$. Since $\mathfrak{A} \rtimes \mathbb{N}$ is generated by elements of the forms aV_x and bV_y^*, each c_j $(1 \leq j \leq n+1)$ is approximated closely by a finite sum $d_j = \sum_{k=1}^{l_j} a_{jk} V_{x_{jk}} + \sum_{k=1}^{m_j} b_{jk} V_{y_{jk}}^*$ such that $0 \leq x_{j1} < \cdots < x_{jl_j}$ and $0 < y_{j1} < \cdots < y_{jm_j}$. Set $L((d_j)_{j=1}^{n+1}) = \sum_{j=1}^{n+1} L(d_j)$.

Now consider the left multiplication to $D = (d_j)_{j=1}^{n+1}$ by elements of the set $EL_{n+1}(\mathfrak{B})$ of all elementary matrices over $\mathfrak{B}$. Note that $EL_{n+1}(\mathfrak{B}) \subset GL_{n+1}(\mathfrak{A} \rtimes \mathbb{N})$. We now may assume that for X a fixed (sufficiently small) open neighborhood of D, $L(D)$ is smallest among $\{L(WD) \mid D \in X, W \in EL_{n+1}(\mathfrak{B})\}$ by replacing D with WD for some W if necessary.

Suppose that $d_j \neq 0$ for any j. We then show a contradiction in the following. We may assume that $L(d_1) \leq L(d_2) \leq \cdots \leq L(d_{n+1})$ by a permutation by elementary matrices if necessary. When $x_{n+1,l_{n+1}}$ exists, consider the multiplication as follows: for $1 \leq j \leq n$,

$$S_j d_j \equiv \begin{cases} V_{x_{n+1,l_{n+1}} - x_{jl_j}} d_j & \text{if } x_{jl_j} \text{ exists and } x_{jl_j} \leq x_{n+1,l_{n+1}}, \\ V^*_{x_{jl_j} - x_{n+1,l_{n+1}}} d_j & \text{if } x_{jl_j} \text{ exists and } x_{jl_j} > x_{n+1,l_{n+1}}, \\ V_{x_{n+1,l_{n+1}} - y_{j1}} d_j & \text{if no } x_{jk} \text{ exists.} \end{cases}$$

Note that the highest term of $S_j d_j$ with respect to V is $V_{x_{n+1,l_{n+1}}}$. When no $x_{n+1,k}$ exists, consider the multiplication as follows: for $1 \leq j \leq n$,

$$S_j d_j \equiv \begin{cases} V_{y_{n+1,1} - y_{j1}} d_j & \text{if no } x_{jk} \text{ exists and } y_{j1} \geq y_{n+1,1}, \\ V^*_{y_{n+1,1} - y_{j1}} d_j & \text{if no } x_{jk} \text{ exists and } y_{j1} < y_{n+1,1}, \\ V^*_{x_{j,l_j} + y_{n+1,1}} d_j & \text{if } x_{j,l_j} \text{ exists.} \end{cases}$$

Note that the lowest term of $S_j d_j$ with respect to V^* is $V^*_{n+1,1}$. In both cases, let $h_j \in \mathfrak{A}$ $(1 \leq j \leq n+1)$ be the coefficients of $S_j d_j$ $(1 \leq j \leq n)$ and d_{n+1} at $V_{x_{n+1,l_{n+1}}}$ or $V^*_{n+1,1}$. Since $\mathrm{sr}(\mathfrak{A}) \leq n$, there exists $(f_j)_{j=1}^n \in \mathfrak{A}^n$ such that $h_{n+1} = \sum_{j=1}^n f_j h_j$ if necessary by replacing h_j with elements obtained by small perturbation. Then, consider the following operation:

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ -f_1 S_1 & \cdots & -f_n S_n & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ d'_{n+1} \end{pmatrix}$$

where $d'_{n+1} = d_{n+1} - \sum_{j=1}^n f_j S_j d_j$. Then we obtain $L(d_{n+1}) > L(d'_{n+1})$, which is the contradiction.

From the above argument we can assume that $d_j = 0$ for some j. By permutation by elementary matrices, we may assume $d_1 = 0$. Then we may replace d_1 with $\varepsilon 1$ for small $\varepsilon > 0$. By subtraction by elementary matrices, $(d_j)_{j=1}^{n+1}$ can be mapped to $(1, 0, \cdots, 0) \in L_{n+1}(\mathfrak{A} \rtimes \mathbb{N})$. Since $L_{n+1}(\mathfrak{A} \rtimes \mathbb{N})$ is open and stable under the left multiplication by elementary matrices ([Rf, Propositions 8.2 and 4.1]), any element $(d_j)_{j=1}^{n+1}$ can be approximated by elements of $L_{n+1}(\mathfrak{A} \rtimes \mathbb{N})$. Hence $\mathrm{sr}(\mathfrak{A} \rtimes \mathbb{N}) \leq n + 1$.

Also we have shown from the above argument that $EL_{n+1}(\mathfrak{B})(1,0,\cdots,0)$ is dense in $(\mathfrak{A}\rtimes\mathbb{N})^{n+1}$. Since $EL_{n+1}(\mathfrak{B})$ is a subset of the connected component of $GL_{n+1}(\mathfrak{A}\rtimes\mathbb{N})$ with the identity matrix, it follows that $L_{n+1}(\mathfrak{A}\rtimes\mathbb{N})$ is connected. Therefore, $\mathrm{csr}(\mathfrak{A}\rtimes\mathbb{N})\leq n+1$ (cf. [Rf, Corollary 8.6] for the same reasoning). □

Remark 2.2.1**.** The Cuntz algebra O_n $(2\leq n<\infty)$ generated by n isometries with their range projections orthogonal and their sum equal to 1 ([Ct]) is regarded as a crossed product $\mathfrak{U}_n\rtimes_\alpha\mathbb{N}$ of the UHF algebra $\mathfrak{U}_n$ of type n^∞ by $\mathbb{N}$ (the shift endomorphism) (cf. [Ct], [Sc]). Then $\mathrm{sr}(O_n)=\infty$ by [Rf, Proposition 6.5]. Thus it is not true that $\mathrm{sr}(\mathfrak{U}_n\rtimes\mathbb{N})\leq\mathrm{sr}(\mathfrak{U}_n)+1$ since $\mathrm{sr}(\mathfrak{U}_n)=1$ by [Rf, Proposition 3.5]. Also, $\mathrm{csr}(O_n)=\infty$ by [Eh, Proposition 1.4]. Therefore, the formula in Theorem 2.2 is not true if no condition to β and no left or right inverse of α as in Proposition 2.1 and Remark 2.1.2. This consequence might be of some interest. See [Rd, Example 2.5] for another description of O_n as a C^*-algebra by a corner endomorphism. See also [Rd, Theorem 3.1] from which the crossed product $\mathfrak{B}\rtimes_\alpha\mathbb{N}$ with $\mathfrak{B}$ a certain simple unital C^*-algebra of real rank zero and α a proper corner endmorphism is purely infinite so that $\mathrm{sr}(\mathfrak{B}\rtimes_\alpha\mathbb{N})=\infty$ and $\mathrm{csr}(\mathfrak{B}\rtimes_\alpha\mathbb{N})=\infty$ by [Rf, Proposition 6.5] and [Eh, Proposition 1.4].

Remark 2.2.2**.** When $\mathfrak{A}=\mathbb{C}$, we have $\mathfrak{A}\rtimes\mathbb{N}\cong C^*(\mathbb{Z})$ by the covariance since the action of $\mathbb{N}$ on $\mathbb{C}$ is trivial. Then $C^*(\mathbb{Z})\cong C(\mathbb{T})$ the C^*-algebra of continuous functions on the torus $\mathbb{T}$. Thus we have $\mathrm{sr}(C^*(\mathbb{C}\rtimes\mathbb{N}))=1$ by [Rf, Proposition 1.7] while $\mathrm{csr}(C^*(\mathbb{C}\rtimes\mathbb{N}))=2$ by [Sh, p.381]. Note that the C^*-algebra $C^*(\mathbb{N})$ of $\mathbb{N}$ generated by a non-unitary isometry is regarded as the Toeplitz algebra $\mathfrak{T}(\mathbb{N})$ generated by all Toeplitz operators with continuous symbols on the usual Hardy space (cf. [Mp3, p.324] and [Mp5, Introduction]). Then it is well known that $C^*(\mathbb{N})=\mathfrak{T}(\mathbb{N})$ is an extension of $C(\mathbb{T})$ by the C^*-algebra of compact operators (cf. [Mp2]). Hence $\mathrm{sr}(C^*(\mathbb{N}))=2$ (cf. [Rf, Examples 4.13]) and $\mathrm{csr}(C^*(\mathbb{N}))=2$ by using [Sh, Theorem 3.9 and p.381] and [Eh, Proposition 1.15]. If the K_1-group of $\mathfrak{A}\rtimes_\alpha\mathbb{N}$ is nontrivial, then $\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N})\geq 2$ [Eh, Corollary 1.6]. Note that $K_1(\mathfrak{T}(\mathbb{N}))$ is trivial (cf. [Wo, 9.L]) but $\mathfrak{T}(\mathbb{N})$ is not stably finite.

Remark 2.2.3**.** As for the connected stable rank of $\mathfrak{A}\rtimes\mathbb{N}$ with $\mathfrak{A}$ nonunital, we just know the estimate: $\mathrm{csr}(\mathfrak{A}^+\rtimes_{\alpha^+}\mathbb{N})\leq\max\{\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}),\mathrm{csr}(\mathbb{C}\rtimes\mathbb{N})\}=\max\{\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}),2\}$ obtained by [Sh, Theorem 3.9].

Next we consider the case of crossed products of C^*-algebras by actions of $\mathbb{N}^2$. We say that for a covariant representation (π,V) of the system $(\mathfrak{A},\mathbb{N}^2,\alpha)$ for a C^*-algebra $\mathfrak{A}$, the representation V of $\mathbb{N}^2$ is $*$-commuting (or covariant) if $V_xV_y^*=V_y^*V_x$ with $V_x=V_{(x,0)}$, $V_y=V_{(0,y)}$ for $x,y\in\mathbb{N}$ (cf. [LR1, Definition 1.2 and its equivalent condition], and [Lc, 2.3 Definition 3] in which it says that the condition $\alpha_m(1)\alpha_n(1)=\alpha_{m\vee n}(1)$ for $m,n\in\mathbb{N}^2$ with an partial order and $m\vee n$ the least upper bound of m and n holds only if the associated representation of $\mathbb{N}^2$ is $*$-commuting).

Theorem 2.3 *Let $\mathfrak{A}\rtimes_\alpha\mathbb{N}^2$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by $\mathbb{N}^2$ and (π,V) a covariant representation of $(\mathfrak{A},\mathbb{N}^2,\alpha)$ with π and $\pi\times V$ faithful and V $*$-commuting. Suppose that there exists an action β of $\mathbb{N}^2$ by endomorphisms of $\mathfrak{A}$ such that $V_n^*\pi(a)=\pi(\beta_n(a))V_n^*$ for any $n\in\mathbb{N}^2$ and $a\in\mathfrak{A}$. Then* $\mathrm{sr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}^2)\leq\mathrm{sr}(\mathfrak{A})+2$.

In addition, if $\mathfrak{A}$ is unital, then $\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}^2)\leq\mathrm{sr}(\mathfrak{A})+2$. *Also, if V_n for some $n\in\mathbb{N}^2$ is non-unitary,* $\mathrm{sr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}^2)\geq 2$ *and* $\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}^2)\geq 2$.

Proof. The lower estimates $\mathrm{sr}(\mathfrak{A}\rtimes\mathbb{N}^2)\geq 2$ and $\mathrm{csr}(\mathfrak{A}\rtimes_\alpha\mathbb{N}^2)\geq 2$ are deduced by the same way with the proof of Theorem 2.2.

When $\mathfrak{A}$ is non-unital, we consider the exact sequence: $0\to\mathfrak{A}\rtimes_\alpha\mathbb{N}^2\to\mathfrak{A}^+\rtimes_{\alpha^+}\mathbb{N}^2\to\mathbb{C}\rtimes\mathbb{N}^2\to 0$, where the extended action α^+ of $\mathbb{N}^2$ on $\mathbb{C}1$ of $\mathfrak{A}^+$ is trivial. Then $\mathrm{sr}(\mathfrak{A}\rtimes\mathbb{N}^2)\leq$

$\mathrm{sr}(\mathfrak{A}^+ \rtimes \mathbb{N}^2)$ by [Rf, Theorem 4.4]. Note that $\mathfrak{A}^+ \rtimes \mathbb{N}^2 = \mathfrak{A}^+ \rtimes \mathbb{Z}^2$ by the covariance since α^+ is trivial on $\mathbb{C}1$. By [Rf, Theorem 7.1], we obtain $\mathrm{sr}(\mathfrak{A}^+ \rtimes \mathbb{Z}^2) \leq \mathrm{sr}(\mathfrak{A}) + 2$.

Next we may assume that $\mathfrak{A}$ is unital. By Proposition 2.1, we have $V_n V_n^* \in \pi(\mathfrak{A})$ for all $n \in \mathbb{N}^2$. We identify $\mathfrak{A}$ with $\pi(\mathfrak{A})$ since π is faithful. Set $V_{(x,0)} = V_x$ and $V_{(0,y)} = V_y$ and use x for the first variable and y for the second one. Note that $\mathfrak{A} \rtimes \mathbb{N}^2 \cong (\mathfrak{A} \rtimes \mathbb{N}) \rtimes \mathbb{N}$ the iterated crossed product by $\mathbb{N}$ since $V_y(\pi(a)V_x)V_y^* = V_y\pi(a)V_y^* V_x = \pi(\alpha_x(a))V_x$ for $a \in \mathfrak{A}$, but we cannot use Theorem 2.2 repeatedly. We need to check the dense part of $(\mathfrak{A} \rtimes \mathbb{N}) \rtimes \mathbb{N}$ involving the second action of $\mathbb{N}$ on $\mathfrak{A} \rtimes \mathbb{N}$ and the property of β as the proof of Theorem 2.2 in the following. Since any element of $\mathfrak{A} \rtimes \mathbb{N}$ is approximated by finite sums of elements of the form $aV_{x_1}V_{x_2}^*$ for $a \in \mathfrak{A}$, $x_1, x_2 \in \mathbb{N}$, any element of $\mathfrak{A} \rtimes \mathbb{N}^2$ is approximated by finite sums of elements of the form $(aV_{x_1}V_{x_2}^*)V_{y_1}V_{y_2}^*$. In fact, using the property of β and the covariance of (π, V) we observe that for $b \in \mathfrak{A}$, $x_j, y_j \in \mathbb{N}$ $(1 \leq j \leq 4)$,

$$
\begin{aligned}
&(aV_{x_1}V_{x_2}^*)V_{y_1}V_{y_2}^*(bV_{x_3}V_{x_4}^*)V_{y_3}V_{y_4}^* \\
&= (aV_{x_1}V_{x_2}^*)V_{y_1}\beta_{y_2}(b)V_{y_2}^*(V_{x_3}V_{x_4}^*)V_{y_3}V_{y_4}^* \\
&= (aV_{x_1}V_{x_2}^*)\alpha_{y_1}(\beta_{y_2}(b))V_{y_1}V_{y_2}^*(V_{x_3}V_{x_4}^*)V_{y_3}V_{y_4}^*.
\end{aligned}
$$

Moreover, when $x_4 \geq x_3$, we compute

$$
\begin{aligned}
&V_{y_1}V_{y_2}^*(V_{x_3}V_{x_4}^*)V_{y_3}V_{y_4}^* \\
&= V_{y_1}V_{y_2}^*(V_{x_3}V_{x_3}^*V_{x_4-x_3}^*)V_{y_3}V_{y_4}^* \\
&= V_{y_1}\beta_{y_2}(V_{x_3}V_{x_3}^*)V_{y_2}^*V_{x_4-x_3}^*V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(V_{x_3}V_{x_3}^*))V_{y_1}V_{(x_4-x_3,y_2)}^*V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(V_{x_3}V_{x_3}^*))V_{y_1}V_{x_4-x_3}^*V_{y_2}^*V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(V_{x_3}V_{x_3}^*))V_{y_1}V_{x_4-x_3}^*V_{y_1}^*V_{y_1}V_{y_2}^*V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(V_{x_3}V_{x_3}^*))V_{y_1}V_{(x_4-x_3,y_1)}^*V_{y_1}V_{y_2}^*V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(V_{x_3}V_{x_3}^*))V_{x_4-x_3}^*(V_{y_1}V_{y_2}^*V_{y_3}V_{y_4}^*),
\end{aligned}
$$

and when $x_4 < x_3$, we compute

$$
\begin{aligned}
&V_{y_1}V_{y_2}^*(V_{x_3}V_{x_4}^*)V_{y_3}V_{y_4}^* \\
&= V_{y_1}V_{y_2}^*(V_{x_3-x_4}V_{x_4}V_{x_4}^*)V_{y_3}V_{y_4}^* \\
&= V_{y_1}V_{y_2}^*(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*)V_{x_3-x_4})V_{y_3}V_{y_4}^* \\
&= V_{y_1}\beta_{y_2}(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*))V_{y_2}^*V_{x_3-x_4}V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*)))V_{y_1}V_{y_2}^*V_{x_3-x_4}V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*)))V_{y_1}V_{y_2}^*V_{y_1}^*V_{y_1}V_{x_3-x_4}V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*)))V_{y_1}V_{y_1}^*V_{y_2}^*V_{x_3-x_4}V_{y_1}V_{y_3}V_{y_4}^* \\
&= \alpha_{y_1}(\beta_{y_2}(\alpha_{x_3-x_4}(V_{x_4}V_{x_4}^*)))V_{y_1}V_{y_1}^*V_{x_3-x_4}(V_{y_2}^*V_{y_1+y_3}V_{y_4}^*)
\end{aligned}
$$

(Note that $*$-commuting property of (π, V) is necessary at the last equality only). For transforming $V_{y_1}V_{y_2}^*V_{y_3}V_{y_4}^*$ and $V_{y_2}^*V_{y_1+y_3}V_{y_4}^*$ to the form $cV_{y_5}V_{y_6}^*$ for $c \in \mathfrak{A}$, $y_5, y_6 \in \mathbb{N}$, use the properties observed in the proof of Theorem 2.2. Moreover, note that

$$\begin{aligned}((aV_{x_1}V_{x_2}^*)V_{y_1}V_{y_2}^*)^* &= V_{y_2}V_{y_1}^*V_{x_2}V_{x_1}^*a^* \\ &= V_{y_2}V_{y_1}^*V_{x_2}\beta_{x_1}(a^*)V_{x_1}^* \\ &= V_{y_2}V_{y_1}^*\alpha_{x_2}(\beta_{x_1}(a^*))V_{x_2}V_{x_1}^* \\ &= V_{y_2}\beta_{y_1}(\alpha_{x_2}(\beta_{x_1}(a^*)))V_{y_1}^*V_{x_2}V_{x_1}^* \\ &= \alpha_{y_2}(\beta_{y_1}(\alpha_{x_2}(\beta_{x_1}(a^*))))V_{y_2}V_{y_1}^*V_{x_2}V_{x_1}^*\end{aligned}$$

Then use the above observation of $V_{y_2}V_{y_1}^*V_{x_2}V_{x_1}^*$.

By the same reason as the proof of Theorem 2.2, define the length of a finite sum $d = \sum_{k=1}^{l} a_k V_{y_k^1} + \sum_{k=1}^{m} b_k V_{y_k^2}^*$ for a_k, b_k finite sums of elements of the form $c_s V_{x_s^1} V_{x_s^2}^*$ for $c_s \in \mathfrak{A}$ by the same say. Note that $L(V_{(0,1)}d) = L(d)$ and $L(V_{(0,1)}^*d) = L(d)$, and this property is used to show the length of d well-defined as in the proof of Theorem 2.2. In fact, observe that

$$\begin{aligned}V_{(0,1)}(aV_{x_1}V_{x_2}^*)V_y &= V_{(0,1)}(aV_{x_1}V_{x_2}^*)V_{(0,1)}^*V_{y+1} \\ &= V_{(0,1)}aV_{(0,1)}^*V_{(0,1)}V_{x_1}V_{x_2}^*V_{(0,1)}^*V_{y+1} \\ &= V_{(0,1)}aV_{(0,1)}^*V_{x_1}V_{(0,1)}V_{(x_2,1)}^*V_{y+1} \\ &= V_{(0,1)}aV_{(0,1)}^*V_{x_1}V_{(0,1)}V_{(0,1)}^*V_{x_2}^*V_{y+1} \\ &= [V_{(0,1)}aV_{(0,1)}^*\alpha_{x_1}(V_{(0,1)}V_{(0,1)}^*)V_{x_1}V_{x_2}^*]V_{y+1} \quad \text{and} \\ V_{(0,1)}(aV_{x_1}V_{x_2}^*)V_y^* &= V_{(0,1)}(aV_{x_1}V_{x_2}^*)V_{(0,1)}^*V_{(0,1)}V_y^* \\ &= [V_{(0,1)}(aV_{x_1}V_{x_2}^*)V_{(0,1)}^*V_{(0,1)}V_{(0,1)}^*]V_{y-1}^*.\end{aligned}$$

Moreover, when $x_1 \leq x_2$

$$\begin{aligned}V_{(0,1)}^*(aV_{x_1}V_{x_2}^*)V_y &= \beta_{(0,1)}(a)V_{(0,1)}^*V_{x_1}V_{x_2}^*V_y \\ &= \beta_{(0,1)}(a)V_{(0,1)}^*V_{x_1}V_{x_1}^*V_{x_2-x_1}^*V_y \\ &= \beta_{(0,1)}(a)\beta_{(0,1)}(V_{x_1}V_{x_1}^*)V_{(0,1)}^*V_{x_2-x_1}^*V_y \\ &= \beta_{(0,1)}(a)\beta_{(0,1)}(V_{x_1}V_{x_1}^*)V_{x_2-x_1}^*V_{(0,1)}^*V_y \\ &= [\beta_{(0,1)}(a)\beta_{(0,1)}(V_{x_1}V_{x_1}^*)V_{x_2-x_1}^*]V_{y-1}\end{aligned}$$

and when $x_1 > x_2$,

$$\begin{aligned}V_{(0,1)}^*(aV_{x_1}V_{x_2}^*)V_y &= \beta_{(0,1)}(a)V_{(0,1)}^*V_{x_1}V_{x_2}^*V_y \\ &= \beta_{(0,1)}(a)V_{x_1}V_{(0,1)}^*V_{x_2}^*V_y \\ &= \beta_{(0,1)}(a)V_{x_1}V_{x_2}^*V_{(0,1)}^*V_y \\ &= [\beta_{(0,1)}(a)V_{x_1}V_{x_2}^*]V_{y-1}\end{aligned}$$

(Note that $*$-commuting property is necessary in the second step of the last calculation), and it follows from the above calculations that

$$\begin{aligned}V_{(0,1)}^*(aV_{x_1}V_{x_2}^*)V_y^* &= \begin{cases} [\beta_{(0,1)}(a)\beta_{(0,1)}(V_{x_1}V_{x_1}^*)V_{x_2-x_1}^*]V_{(0,1)}^*V_y^* & \text{if } x_1 \leq x_2 \\ [\beta_{(0,1)}(a)V_{x_1}V_{x_2}^*]\, V_{(0,1)}^*V_y^* & \text{if } x_1 > x_2 \end{cases} \\ &= \begin{cases} [\beta_{(0,1)}(a)\beta_{(0,1)}(V_{x_1}V_{x_1}^*)V_{x_2-x_1}^*]V_{y+1}^* & \text{if } x_1 \leq x_2 \\ [\beta_{(0,1)}(a)V_{x_1}V_{x_2}^*]\, V_{y+1}^* & \text{if } x_1 > x_2 \end{cases}\end{aligned}$$

Now assume $\mathrm{sr}(\mathfrak{A} \rtimes \mathbb{N}) \le n$. Then the rest of the proof is the same as that of Theorem 2.2. □

We say that for a covariant representation (π, V) of the system $(\mathfrak{A}, \mathbb{N}^k, \alpha)$ for a C^*-algebra $\mathfrak{A}$, the representation V of $\mathbb{N}^k$ is $*$-commuting (or covariant) if $V_{x_i}V_{x_j}^* = V_{x_j}^*V_{x_i}$ for $i \ne j$, $1 \le i, j \le k$ with $x_i = x_i e_i$ for $x_i \in \mathbb{N}$ and e_i the i-th basis of $\mathbb{N}^k$ (cf. [LR1, Definition 1.2] and [Lc, 2.3 Definition 3]).

Corollary 2.4 *Let $\mathfrak{A} \rtimes_\alpha \mathbb{N}^k$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by $\mathbb{N}^k$ and (π, V) a covariant representation of $(\mathfrak{A}, \mathbb{N}^k, \alpha)$ with π and $\pi \times V$ faithful and V $*$-commuting. Suppose that there exists an action β of $\mathbb{N}^k$ by endomorphisms of $\mathfrak{A}$ such that $V_n^*\pi(a) = \pi(\beta_n(a))V_n^*$ for any $n \in \mathbb{N}^k$ and $a \in \mathfrak{A}$. Then $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}^k) \le \mathrm{sr}(\mathfrak{A}) + k$.*

In addition, if $\mathfrak{A}$ is unital, then $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}^k) \le \mathrm{sr}(\mathfrak{A}) + k$. Also, if V_n for some $n \in \mathbb{N}^k$ is non-unitary, $\mathrm{sr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}^k) \ge 2$ and $\mathrm{csr}(\mathfrak{A} \rtimes_\alpha \mathbb{N}^k) \ge 2$.

Proof. Use the argument of the proof of Theorem 2.3 repeatedly since we have $\mathfrak{A} \rtimes_\alpha \mathbb{N}^k \cong (\cdots((\mathfrak{A} \rtimes \mathbb{N}) \rtimes \mathbb{N})\cdots) \rtimes \mathbb{N}$ the k-times iterated crossed product by $\mathbb{N}$. □

Remark 2.4.1**.** When $\mathfrak{A} = \mathbb{C}$, we have $\mathbb{C} \rtimes_\alpha \mathbb{N}^k \cong C^*(\mathbb{Z}^k)$ by the covariance since α is trivial on $\mathbb{C}$. Since $C^*(\mathbb{Z}^k) \cong C(\mathbb{T}^k)$ by the Fourier transform, we obtain $\mathrm{sr}(\mathbb{C} \rtimes_\alpha \mathbb{N}^k) = \mathrm{sr}(C(\mathbb{T}^k)) = [k/2] + 1$ and $\mathrm{csr}(\mathbb{C} \rtimes_\alpha \mathbb{N}^k) = \mathrm{csr}(C(\mathbb{T}^k)) = [(k+1)/2] + 1$ by [Rf, Proposition 1.7] and [Sh, p.381], where $[x]$ means the maximum integer $\le x$. On the other hand, the C^*-algebra $C^*(\mathbb{N}^k)$ of $\mathbb{N}^k$ is isomorphic to the tensor product $\otimes^k C^*(\mathbb{N})$. Thus it follows from Remark 2.2.2 that $C^*(\mathbb{N}^k) \cong \otimes^k \mathfrak{T}(\mathbb{N})$. Therefore, $C^*(\mathbb{N}^k)$ has a quotient map to $C(\mathbb{T}^k)$ and the kernel $\mathfrak{I}$ by this map has a finite composition series with its subquotients having the C^*-algebra of compact operators as a tensor factor, that is, stable C^*-algebras. By using Remark 2.2.2, [Rf, Theorems 4.3, 4.11 and 6.4] and [Sh, Theorem 3.9] repeatedly, we deduce that $\mathrm{sr}(\mathfrak{I}) \le 2$, $\mathrm{csr}(\mathfrak{I}) \le 2$ and

$$\begin{aligned} \mathrm{sr}(C(\mathbb{T}^k)) \le \mathrm{sr}(C^*(\mathbb{N}^k)) &\le \max\{2, \mathrm{sr}(C(\mathbb{T}^k)), \mathrm{csr}(C(\mathbb{T}^k))\}, \\ 2 \le \mathrm{csr}(C^*(\mathbb{N}^k)) &\le \max\{2, \mathrm{csr}(C(\mathbb{T}^k))\}. \end{aligned}$$

By [Rf, Proposition 1.7] and [Sh, p.381], we obtain $[k/2]+1 \le \mathrm{sr}(C^*(\mathbb{N}^k)) \le [(k+1)/2]+1$ and $2 \le \mathrm{csr}(C^*(\mathbb{N}^k)) \le [(k+1)/2]+1$.

Remark 2.4.2**.** Since the C^*-algebra $C^*(S)$ of the positive cone S of an partially ordered abelian group G is just the Toeplitz algebra $\mathfrak{T}(S)$ (or $\mathfrak{T}(G)$) (cf. [Mp3, p.338] and Remark 2.5.2 below), we could obtain the similar stable rank estimates of $C^*(S)$ as above, and the structure of $\mathfrak{T}(S)$ obtained in [Mp1] and [Mp4] would be useful for computing the stable ranks of $\mathfrak{T}(S)$.

Now recall from [LR2] that the Bost-Connes' Hecke C^*-algebra denoted by $\mathfrak{H}_\mathbb{Q}$ is the C^*-enveloping algebra of the Hecke algebra defined to be the universal involutive algebra over $\mathbb{C}$ generated by elements μ_n for $n \in \mathbb{N}^*$ and $e(r)$ for $r \in \mathbb{Q}/\mathbb{Z}$ such that

$$\begin{aligned} &\mu_n^*\mu_n = 1, \quad \mu_m\mu_n = \mu_{mn} \quad \text{for } m, n \in \mathbb{N}^*, \quad \mu_m\mu_n^* = \mu_n^*\mu_m \quad \text{if } (m,n) = 1, \\ &\text{and} \quad e(0) = 1, \quad e(r)^* = e(-r), \quad e(r+s) = e(r)e(s) \quad \text{for } r, s \in \mathbb{Q}/\mathbb{Z}, \\ &e(r)\mu_n = \mu_n e(nr), \quad \mu_n e(r)\mu_n^* = (1/n)\sum_{j=1}^n e(r/n + j/n) \quad \text{for } n \in \mathbb{N}^*, r \in \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

It is shown by [LR2] that the Hecke C^*-algebra $\mathfrak{H}_\mathbb{Q}$ is in fact the semigroup crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an action α of $\mathbb{N}^*$ on $C^*(\mathbb{Q}/\mathbb{Z})$ defined by $\alpha_n(i(r)) = (1/n)\sum_{j=1}^n i(r/n + j/n)$ for $r \in \mathbb{Q}$, where $i : \mathbb{Q}/\mathbb{Z} \to C^*(\mathbb{Q}/\mathbb{Z})$ is the canonical embedding (unitary representation) and $i(r)$ means $i(r + \mathbb{Z})$. As an interesting application, we obtain

Theorem 2.5 *Let $\mathfrak{H}_{\mathbb{Q}} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ be the Hecke C^*-algebra of Bost-Connes or the semigroup crossed product of Laca-Raeburn. For the canonical subalgebras $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k$ of $\mathfrak{H}_{\mathbb{Q}}$, we obtain*

$$[k/2] + 1 \le \mathrm{sr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k) \le 1 + k \quad \textit{and} \quad 2 \le \mathrm{csr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k) \le 1 + k.$$

On the other hand, we have $\mathrm{sr}(\mathfrak{H}_{\mathbb{Q}}) = \infty$.

Proof. Note that $C^*(\mathbb{Q}/\mathbb{Z})$ is isomorphic to the tensor product $\otimes_{p\in\mathfrak{P}} C^*(G_p)$, where $G_p = \{r + \mathbb{Z} : r = n/p^k \text{ for some } k \in \mathbb{N}^+ \text{ and } n \in \mathbb{Z}\} = \mathbb{Z}[p^{-1}]/\mathbb{Z}$ and $\mathfrak{P}$ is the set of all prime numbers, and also note that $\mathbb{N}^* \cong \oplus_{p\in\mathfrak{P}}\mathbb{N}$ (cf. [LR2, Introduction]). Moreover, we have $C^*(\mathbb{Q}/\mathbb{Z})$ is isomorphic to the C^*-algebra $C((\mathbb{Q}/\mathbb{Z})^\wedge)$ of all continuous functions on the compact space $(\mathbb{Q}/\mathbb{Z})^\wedge$, where $(\mathbb{Q}/\mathbb{Z})^\wedge$ is the dual of $\mathbb{Q}/\mathbb{Z}$, which is an inverse limit of $\mathbb{Z}/n\mathbb{Z}$, so that $C((\mathbb{Q}/\mathbb{Z})^\wedge)$ is an inductive limit of $C(\mathbb{Z}/n\mathbb{Z})$ ([LR2, p.336]). Hence $C^*(\mathbb{Q}/\mathbb{Z})$ is unital and $\mathrm{sr}(C^*(\mathbb{Q}/\mathbb{Z})) = 1$. Since $\mathbb{N}^* \cong \mathbb{N}^k \times H_k$ with $H_k = \oplus_{p\in\mathfrak{P}_k}\mathbb{N}$ for $\mathfrak{P}_k$ the subset of $\mathfrak{P}$ obtained by removing k prime numbers, then $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k$ is a C^*-subalgebra of $\mathfrak{H}_{\mathbb{Q}}$. By [LR2, Proposition 2.1, Remark 2.2, Lemmas 3.1 and 3.2 and Theorem 3.7], the algebra $\mathfrak{H}_{\mathbb{Q}}$ satisfies the conditions in Corollary 2.4. Thus $\mathrm{sr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k) \le 1 + k$ and $\mathrm{csr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k) \le 1 + k$.

On the other hand, we have the quotients: $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k \to \mathbb{C} \rtimes \mathbb{N}^k \to 0$ and $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^* \to \mathbb{C} \rtimes \mathbb{N}^* \to 0$ which are deduced from their covariant representations corresponding to the trivial representation of $C^*(\mathbb{Q}/\mathbb{Z})$. Note that $\mathbb{C} \rtimes \mathbb{N}^k \cong C^*(\mathbb{Z}^k) \cong C(\mathbb{T}^k)$ and $\mathbb{C} \rtimes \mathbb{N}^* \cong C^*(\oplus_{k=1}^\infty \mathbb{Z}) \cong C(\Pi_{k=1}^\infty \mathbb{T})$. By [Rf, Proposition 1.7 and Theorem 4.3], we obtain $\mathrm{sr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^k) \ge \mathrm{sr}(\mathbb{C} \rtimes \mathbb{N}^k) = [k/2]+1$ and $\mathrm{sr}(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*) \ge \mathrm{sr}(\mathbb{C} \rtimes \mathbb{N}^*) = \infty$. □

Remark 2.5.1**.** The action α of $\mathfrak{H}_{\mathbb{Q}}$ has a left inverse β of endomorphisms which is also a sort of a right inverse as in Remark 2.1.1 (cf. [LR2, Proposition 2.1]). There exists a covariant representation (V, π) of $\mathfrak{H}_{\mathbb{Q}}$ with $\pi \times V$ faithful such that the covariant condition $V_n V_n^* V_m V_m^* = V_{[n,m]} V_{[n,m]}^*$ holds for any $n, m \in \mathbb{N}^*$, where $[n, m]$ is the least common multiple of n and m, and $V_n V_n^* \in \pi(C^*(\mathbb{Q}/\mathbb{Z}))$ for all $n \in \mathbb{N}^*$, and that there exists an action β of $\mathbb{N}^*$ by endomorphisms of $C^*(\mathbb{Q}/\mathbb{Z})$ such that $V_n^*\pi(a) = \pi(\beta_n(a))V_n^*$ for any $n \in \mathbb{N}^*$ and $a \in C^*(\mathbb{Q}/\mathbb{Z})$. Note that the covariant condition is equivalent to the relation $V_n^* V_m = V_{n^{-1}[n,m]} V_{m^{-1}[n,m]}^*$ (cf. [LR1, Definition 1.2]).

Remark 2.5.2**.** Also, the covariant condition $V_n V_n^* V_m V_m^* = V_{[n,m]} V_{[n,m]}^*$ is generalized to the notion of covariant representations of certain semigroups P of quasi-lattice ordered groups (cf. [LR1]). Note that all totally ordered groups and lattice orders are quasi-lattice ordered (cf. [LR1] and [Nc]). The semigroup crossed product $\mathfrak{B}_P \rtimes_\alpha P$ associated with P of [LR1] has a covariant representation (π, V) satisfying the covariant condition, where $\mathfrak{B}_P$ is a C^*-subalgebra of $l^\infty(P)$ and α is a left translation. In this case, $V_x V_x^* \in \pi(\mathfrak{B}_P)$ for any $x \in P$ ([LR1, p. 423 and Proposition 2.3]). Thus our methods might be applicable to this situation. See also [A-R] for the case of P totally ordered abelian semigroups and $C^*(P) = \mathfrak{B}_P \rtimes_\alpha P = \mathfrak{T}(P)$.

Remark 2.5.3**.** The C^*-algebra $C^*(\mathbb{N}^*)$ of $\mathbb{N}^*$ is in fact the Toeplitz C^*-algebra of $\mathbb{N}^*$, $\mathfrak{T}(\mathbb{N}^*)$ ([LR2, Section 1]) and $\mathfrak{T}(\mathbb{N}^*) = \mathfrak{B}_{\mathbb{N}^*} \rtimes_\alpha \mathbb{N}^*$ in the sense of Remark 2.5.1. Moreover, the algebra $C^*(\mathbb{N}^*) = \mathfrak{T}(\mathbb{N}^*)$ is isomorphic to the infinite tensor product of the usual Toeplitz algebra $C^*(\mathbb{N}) = \mathfrak{T}(\mathbb{N})$ over the set of prime numbers. There exist canonical quotients from $C^*(\mathbb{N}^*)$ to any finite tensor products of $C^*(\mathbb{N})$, and thus to $C(\mathbb{T}^n)$ for any $n \in \mathbb{T}$. Hence it follows from [Rf, Theorem 4.3 and Proposition 1.7] that $\mathrm{sr}(C^*(\mathbb{N}^*)) = \mathrm{sr}(\mathfrak{T}(\mathbb{N}^*)) = \infty$.

Remark 2.5.4**.** As a large generalization of the Hecke C^*-algebra above, Arledge, Laca and Raeburn [ALR] have studied semigroup crossed products of the form $C^*(K/\mathfrak{O}) \rtimes_\alpha \mathfrak{O}^\times$, where K is a finite extension of $\mathbb{Q}$, $\mathfrak{O}$ is the ring of integers in K, and $\mathfrak{O}^\times$ is the multiplicative semigroup of nonzero integers of $\mathfrak{O}$. By using [ALR, Proposition 1.2, Lemmas 1.5 and 1.8 and Section 5] and that the dual of $K/\mathfrak{O}$ is a topological inverse limit of finite groups (cf. [ALR, Proof of Lemma 4.2]), we could deduce the similar results as Theorem 2.5.

Now recall from [LsR] that the semigroup crossed product $C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k$ of Larsen and Raeburn is induced by the C^*-dynamical system $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$, where G is a (discrete) abelian group and G_∞ is the direct limit of the system (G, η_{n-m}) with η an action of $\mathbb{N}^k$ by injective endomorphisms of G for $m = (m_i), n = (n_i) \in \mathbb{N}^k$ with $m_i \le n_i$ for $1 \le i \le k$, and the action α of $\mathbb{N}^k$ by endomorphisms of the group C^*-algebra $C^*(G_\infty/G)$ is defined by $\alpha_m(\delta_r) = (1/|G : \eta_m(G)|) \sum_{s \in G_\infty/G : \beta_m(s) = r} \delta_s$ for $r \in G_\infty/G$ and $m \in \mathbb{N}^k$, where β_m is an endomorphism of the quotient G_∞/G such that $\beta_m(s) = \beta_m(g+G) = \eta_m^\infty(g) + G$ for $s = g + G$ and $g \in G_\infty$ and η_m^∞ is an automorphism of G_∞ induced by the map η_m, and δ_r, δ_s are unitaries by the unitary representation $\delta : G_\infty/G \to C^*(G_\infty/G)$. Similarly as Theorem 2.5,

Theorem 2.6 *Let $C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k$ be the semigroup crossed product of Larsen and Raeburn. Then* $[k/2]+1 \le \mathrm{sr}(C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k) \le 1+k$ *and* $2 \le \mathrm{csr}(C^*(G_\infty/G) \rtimes_\alpha \mathbb{N}^k) \le 1+k$.

Proof. Use [LsR, Proposition 1.3, Theorem 2.1, Lemma 2.2 and the first paragraph of the proof of Proposition 2.4] and Corollary 2.4. In particular, note that $C^*(G_\infty/G)$ is isomorphic to the C^*-algebra $C((G_\infty/G)^\wedge)$ of all continuous functions on the (compact) space $(G_\infty/G)^\wedge$, where $(G_\infty/G)^\wedge$ is the dual group of G_∞/G and it is an inverse limit of finite discrete groups. Thus, $C^*(G_\infty/G)$ is unital and $C((G_\infty/G)^\wedge)$ is an inductive limit of finite dimensional C^*-algebras. Hence, $\mathrm{sr}(C^*(G_\infty/G)) = \mathrm{sr}(C((G_\infty/G)^\wedge)) = 1$ by [Rf, Theorem 5.1]. □

Remark 2.6**.** The action α of $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$ satisfies $\alpha_n(1)\alpha_m(1) = \alpha_{n\vee m}(1)$, which implies that for any covariant representation (π, V) of $(C^*(G_\infty/G), \mathbb{N}^k, \alpha)$, V is covariant, i.e. $V_n V_n^* V_m V_m^* = V_{n\vee m} V_{n\vee m}^*$ for $n, m \in \mathbb{N}^k$ ([LsR, Remark 1.4]), where $n \vee m$ is the least upper bound of n and m. Then the covariant condition is equivalent to $V_n^* V_m = V_{(n\vee m)-n} V_{(n\vee m)-m}^*$ (cf, [LR1, Definition 1.2]). Hence, V is $*$-commuting in our sense as above.

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