BASE-NORMALITY AND TOTAL PARACOMPACTNESS OF SUBSPACES OF PRODUCTS OF TWO ORDINALS

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Abstract. In this paper, we study base-normality and total paracompactness of subspaces of products of two ordinals. We prove the following: (1) For every regular cardinal κ with $\kappa \geq \omega_1$, there exists a normal non-base-normal subspace X of $(\kappa+1)^2$ with $w(X) = \kappa$. (2) If A and B are subspaces of an ordinal, then $A \times B$ is base-normal if and only if $A \times B$ is normal. (3) Every normal subspace of ${\omega_1}^2$ is base-normal. (4) Every paracompact subspace of products of two ordinals is totally paracompact.

1 Introduction Throughout this paper, all spaces are assumed to be Hausdorff topological spaces. For a space X, $w(X)$ denotes the weight of X. For a subspace A of X, the closure of A in X is denoted by $\mathrm{cl}_X A$. For a collection $\mathcal{A}$ of subsets of X, $\{\mathrm{cl}_X A : A \in \mathcal{A}\}$ is denoted by $\mathrm{cl}_X \mathcal{A}$.

Yamazaki [10] defined a space X to be *base-normal* if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every binary open cover $\{U_1, U_2\}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that $\mathrm{cl}_X \mathcal{B}'$ refines $\{U_1, U_2\}$. A space X is said to be *totally paracompact* [3] if every open base for X contains a locally finite subcover.

In this paper, we discuss base-normality and total paracompactness of subspaces of products of two ordinals and show the results (1)-(4) stated in the abstract.

Answering Yamazaki's question in [11], Gruenhage [4] gave an example of a countably compact LOTS which is not base-normal. Our result (1) gives different examples in ZFC of a normal space which is not base-normal.

It is known that many familiar examples of paracompact spaces are not totally paracompact; for example, the space of all the irrationals, the Sorgenfrey line and the Michael line are not totally paracompact ([1] and [2]). Our result (4) shows that there is no difference between paracompactness and total paracompactness for a subspace of products of two ordinals.

Now we introduce some notations from [5].

Let $\mathrm{cf}(\mu)$ denote the cofinality of an ordinal μ. When $\mathrm{cf}(\mu) \geq \omega_1$, a subset S of μ is said to be *stationary* in μ if it intersects all cub (i.e., closed and unbounded) sets in μ. For $A \subseteq \mu$, let $\mathrm{Lim}_\mu(A) = \{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$. We consider $\sup(\emptyset) = -1$ if there is no special explanation. Assume that C is a cub in μ with $\mathrm{cf}(\mu) \geq \omega$, then $\mathrm{Lim}_\mu(C) \subseteq C$. We define $\mathrm{Succ}(C) = C \setminus \mathrm{Lim}_\mu(C)$, and $p_C(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Note that $p_C(\alpha) < \alpha$ if and only if $\alpha \in \mathrm{Succ}(C)$. Observe that $\mu \setminus \mathrm{Lim}_\mu(C)$ is the union of the pairwise disjoint collection $\{(p_C(\alpha), \alpha] : \alpha \in \mathrm{Succ}(C)\}$ of clopen intervals of μ.

For a limit ordinal μ, a strictly increasing function $M : \mathrm{cf}(\mu) \to \mu$ is said to be normal if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma < \mathrm{cf}(\mu)$ and $\mu = \sup\{M(\gamma) : \gamma < \mathrm{cf}(\mu)\}$. For convenience, we define $M(-1) = -1$. Clearly, M carries $\mathrm{cf}(\mu)$ homeomorphically

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to the range $\operatorname{ran}(M)$ of M and $\operatorname{ran}(M)$ is closed in μ. Note that for all $S \subseteq \mu$ with $\operatorname{cf}(\mu) \geq \omega_1$, S is stationary in μ if and only if $M^{-1}(S)$ is stationary in $\operatorname{cf}(\mu)$. If μ and ν are limit ordinals, let M and N denote the fixed normal functions for μ and ν, respectively.

The following Lemma will be used frequently throughout the paper (see [8]).

Lemma 1.1. (The PDL.) Let $\kappa > \omega$ be regular, S a stationary subset in κ, and $f : S \to \kappa$ such that $f(\gamma) < \gamma$ for each $\gamma \in S$; then for some $\alpha < \kappa$, $f^{-1}(\alpha)$ is stationary.

2 Normal non-base-normal subspaces of $(\kappa + 1)^2$ First, we show that for every regular cardinal κ with $\kappa \geq \omega_1$, there exists a normal non-base-normal subspace X of $(\kappa + 1)^2$ with $w(X) = \kappa$.

Theorem 2.1. *Let κ be a regular cardinal with $\kappa \geq \omega_1$, and let $X = \{\langle \alpha, \beta \rangle : \beta < \alpha < \kappa, \alpha$ and β are successor ordinals$\} \cup (\{\kappa\} \times \kappa)$. Then X is normal and not base-normal.*

Proof. To show that X is normal, let F_1 and F_2 be disjoint closed subsets of X. Since κ is normal, there exist disjoint open sets G_1 and G_2 of κ such that $F_i \cap (\{\kappa\} \times \kappa) \subseteq \{\kappa\} \times G_i$ for $i = 1, 2$. It is easy to show that $((((\kappa+1) \times G_1) \setminus F_2) \cup F_1) \cap X$ and $((((\kappa+1) \times G_2) \setminus F_1) \cup F_2) \cap X$ are disjoint open sets in X containing F_1 and F_2, respectively. Hence, X is normal.

Next, we show that X is not base-normal. Obviously, $w(X) = \kappa$. Suppose $\mathcal{B}$ is a base of X with $|\mathcal{B}| = \kappa$. We will show that $\mathcal{B}$ cannot witness base-normality of X.

Claim 1. Let $B \in \mathcal{B}$. If $\{\delta < \kappa : \langle \kappa, \delta \rangle \in B\}$ is stationary in κ, then there exist a cub set $C(B)$ in κ, a function $f(B, \cdot) : C(B) \to \kappa$ and an ordinal $g(B) < \min(C(B))$ such that $((f(B, \gamma), \kappa] \times (g(B), \gamma]) \cap X \subseteq B$ for each $\gamma \in C(B)$.

Proof of Claim 1. For each $\delta \in \kappa$ with $\langle \kappa, \delta \rangle \in B$, fix $p(B, \delta) < \kappa$ and $q(B, \delta) < \delta$ such that $((p(B, \delta), \kappa] \times (q(B, \delta), \delta]) \cap X \subseteq B$. Applying the PDL, we can find an ordinal $g(B) < \kappa$ and a stationary set S in κ such that $S \subseteq \{\delta < \kappa : \langle \kappa, \delta \rangle \in B\}$ and $q(B, \delta) = g(B)$ for each $\delta \in S$. Let $C(B) = \{\gamma \in \kappa : \gamma > \min(S)\}$. For each $\gamma \in C(B)$, let $\psi(\gamma) = \min\{\delta \in S : \gamma \leq \delta\}$, and $f(B, \gamma) = p(B, \psi(\gamma))$. Then $(f(B, \gamma), \kappa] \times (g(B), \gamma]) \cap X \subseteq ((p(B, \psi(\gamma)), \kappa] \times (g(B), \psi(\gamma)]) \cap X \subseteq B$. The proof of Claim 1 is complete. $\square$

Let $\mathcal{B}' = \{B \in \mathcal{B} : \{\delta < \kappa : \langle \kappa, \delta \rangle \in B\}$ is stationary in $\kappa\}$. Rewrite $\mathcal{B}' = \{B_\alpha : \alpha < \xi\}$, where ξ is a cardinal. By Claim 1, for each $\alpha < \xi$, there exist a cub set C_α in κ, a function $f(B_\alpha, \cdot) : C_\alpha \to \kappa$ and an ordinal $g(B_\alpha) < \min(C_\alpha)$ such that $((f(B_\alpha, \gamma), \kappa] \times (g(B_\alpha), \gamma]) \cap X \subseteq B_\alpha$ for each $\gamma \in C_\alpha$. If $\xi < \kappa$, let $C' = \bigcap_{\alpha < \xi} C_\alpha$. If $\xi = \kappa$, let $C' = \{\gamma \in \kappa : \forall \alpha < \gamma (\gamma \in C_\alpha)\}$. In any case, C' is a cub set in κ ([8], II, Lemma 6.8 and Lemma 6.14). Let $C = \operatorname{Lim}_\kappa(C')$. Then C is a cub set in κ and $C \subseteq C'$. For each $\gamma \in C$, take a limit ordinal $f(\gamma) < \kappa$ such that $f(\gamma) > \sup\{f(B_\alpha, \gamma) : \alpha < \gamma\}$. We may assume that $f(\gamma') < f(\gamma)$ if $\gamma' < \gamma$. Let $U_1 = \bigcup\{((f(\gamma), \kappa] \times [0, \gamma]) \cap X : \gamma \in C\}$. Then $\{\kappa\} \times \kappa \subseteq U_1$. Let $U_2 = X \setminus (\{\kappa\} \times \kappa)$. Then $\{U_1, U_2\}$ is a binary open cover of X. We will show that $\{U_1, U_2\}$ admits no locally finite refinement by members of $\mathcal{B}$. Suppose $\mathcal{B}^*$ is a refinement of $\{U_1, U_2\}$ by members of $\mathcal{B}$. To complete the proof, it suffices to show that $\mathcal{B}^*$ is not locally finite in X.

Claim 2. For each $\alpha < \xi$, $B_\alpha \setminus U_1 \neq \emptyset$.

Proof of Claim 2. Fix $\alpha < \xi$. Take $\gamma_1 \in C$ such that $\gamma_1 > \alpha$. Let $\gamma_2 = \min\{\gamma \in C : \gamma > \gamma_1\}$. By the definition of C, we have $\gamma_1 \in C_\alpha$ and $\gamma_2 \in C_\alpha$. Since $f(\gamma_2) > f(B_\alpha, \gamma_2)$ and $f(\gamma_2)$

is a limit ordinal, there exists a successor ordinal $\alpha' \in \kappa$ such that $f(B_\alpha, \gamma_2) < \alpha' < f(\gamma_2)$. Since $\gamma_2 > \gamma_1$ and γ_2 is a limit ordinal, there exists a successor ordinal $\beta' \in \kappa$ such that $\gamma_1 < \beta' < \gamma_2$. Since $g(B_\alpha) < \min(C_\alpha)$ and $\gamma_1 \in C_\alpha$, we have $\gamma_1 > g(B_\alpha)$. Hence, $\langle \alpha', \beta' \rangle \in ((f(B_\alpha, \gamma_2), \kappa] \times (\gamma_1, \gamma_2]) \cap X \subseteq ((f(B_\alpha, \gamma_2), \kappa] \times (g(B_\alpha), \gamma_2]) \cap X \subseteq B_\alpha$. Since $\{f(\gamma) : \gamma \in C\}$ is strictly increasing and γ_2 is the successor of γ_1 in C, it follows from the definition of U_1 that $\langle \alpha', \beta' \rangle \notin U_1$. The proof of Claim 2 is complete. □

Let $\mathcal{B}'' = \mathcal{B} \setminus \mathcal{B}'$. Rewrite $\mathcal{B}'' = \{B^\beta : \beta < \eta\}$, where η is a cardinal. For each $\beta < \eta$, since $\{\delta < \kappa : \langle \kappa, \delta \rangle \in B^\beta\}$ is not stationary in κ, there exists a cub set D_β in κ such that $D_\beta \cap \{\delta < \kappa : \langle \kappa, \delta \rangle \in B^\beta\} = \emptyset$. If $\eta < \kappa$, let $D = \bigcap_{\beta<\eta} D_\beta$. If $\eta = \kappa$, let $D = \{\sigma \in \kappa : \forall \beta < \sigma (\sigma \in D_\beta)\}$. In any case, D is a cub set in κ. Since $\mathcal{B}^*$ is a refinement of $\{U_1, U_2\}$, we can take $\sigma_0 \in D$ and $W_0 \in \mathcal{B}^*$ such that $\langle \kappa, \sigma_0 \rangle \in W_0 \subseteq U_1$. It follows from Claim 2 that $W_0 \in \mathcal{B}''$. Hence, $W_0 = B^{\beta_0}$ for some $\beta_0 \in \eta$. Since D is unbounded in κ, there exists $\sigma_1 \in D$ such that $\sigma_1 > \sigma_0$ and $\sigma_1 > \beta_0$. By the definition of D, we have $\sigma_1 \in D_{\beta_0}$. Hence, $\langle \kappa, \sigma_1 \rangle \notin B^{\beta_0} = W_0$. Take $W_1 \in \mathcal{B}^*$ such that $\langle \kappa, \sigma_1 \rangle \in W_1 \subseteq U_1$. Then, $W_1 \neq W_0$ and $W_1 \in \mathcal{B}''$. Proceeding by induction, we can choose a strictly increasing sequence $\{\sigma_i : i \in \omega\}$ in D and a sequence $\{W_i : i \in \omega\}$ in $\mathcal{B}^*$ such that $\langle \kappa, \sigma_i \rangle \in W_i$ for each $i \in \omega$ and $W_i \neq W_j$ whenever $i \neq j$. Let $\sigma = \sup\{\sigma_i : i \in \omega\}$. Then, $\{W_i : i \in \omega\}$ is not locally finite at $\langle \kappa, \sigma \rangle$ in X. Thus, $\mathcal{B}^*$ is not a locally finite refinement of $\{U_1, U_2\}$. The proof is complete. □

3 Some properties of τ-base-normality Throughout Sections 3 and 4, τ stands for an infinite cardinal. We introduce the notion of τ-base-normality, which is a generalization of base-normality, and use it to prove main theorems in Sections 4 and 5. For a space X and a cardinal τ with $w(X) \leq \tau$, we call X *τ-base-normal* if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| \leq \tau$ such that every binary open cover $\{U_1, U_2\}$ of X admits a locally finite cover $\mathcal{B}'$ of X by members of $\mathcal{B}$ such that $\mathrm{cl}_X \mathcal{B}'$ refines $\{U_1, U_2\}$. Note that for a space X with $w(X) \geq \omega$, X is $w(X)$-base-normal if and only if X is base-normal. Yamazaki[11] called a subspace A of X *base-normal relative to* X if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every binary open (in X) cover $\{U_1, U_2\}$ of A there is a locally finite (in X) family $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathrm{cl}_X \mathcal{B}'$ is a partial refinement of $\{U_1, U_2\}$ and $A \subseteq \bigcup \mathcal{B}'$. Similarly, we define *$\tau$-base-normality relative to* X by replacing the condition $|\mathcal{B}| = w(X)$ by $|\mathcal{B}| \leq \tau$. It is noted that if X is τ-base-normal, then every closed subspace of X is τ-base-normal and τ-base-normal relative to X.

The proof of the following lemma is straightforward and left to the reader.

Lemma 3.1. *For a space X and a cardinal τ with $\tau \geq w(X)$, the following statements hold*:

(1) *If X is the topological sum of a collection $\{A_t : t \in T\}$ of τ-base-normal subspaces of X with $|T| \leq \tau$, then X is τ-base-normal.*

(2) *If X is normal, then X is τ-base-normal if and only if there is a base $\mathcal{B}$ for X with $|\mathcal{B}| \leq \tau$ such that every binary open cover $\{U_1, U_2\}$ of X admits a locally finite refinement $\mathcal{B}'$ of X by members of $\mathcal{B}$.*

Yamazaki [11] showed that if a normal space X is the countable union of closed base-normal sets relative to X, then X is base-normal. Similarly, we can prove the following Lemma 3.2.

Lemma 3.2. *Let X be a normal space. If X is the countable union of closed τ-base-normal sets relative to X, then X is τ-base-normal.*

Lemma 3.3. *Let X be a normal space and $A \subseteq Y \subseteq X$ with A closed in X and Y open in X. Let $\tau \geq w(X)$. If Y is τ-base-normal, then A is τ-base-normal relative to X.*

Proof. Since X is normal, there exists an open subset V of X such that $X \setminus Y \subseteq V \subseteq \mathrm{cl}_X V \subseteq X \setminus A$. Let $\mathcal{B}_Y$ be an open base for Y witnessing τ-base-normality of Y. Since Y is open in X, we can take an open base $\mathcal{B}_X$ for X such that $|\mathcal{B}_X| \leq \tau$ and $\mathcal{B}_Y \subseteq \mathcal{B}_X$. Let $\{U_1, U_2\}$ be an open (in X) cover of A. Since X is normal and A is closed in X, there exist open subsets W_1 and W_2 of X such that $A \subseteq W_1 \cup W_2$ and $\mathrm{cl}_X W_i \subseteq U_i$ for $i = 1, 2$. Let $G_i = W_i \setminus \mathrm{cl}_X V$ for $i = 1, 2$. Then $A \subseteq G_1 \cup G_2 \subseteq Y$. Since A is a closed subset of τ-base-normal space Y, A is τ-base-normal relative to Y. Hence, there exists $\mathcal{B}' \subseteq \mathcal{B}_Y$ such that $\mathcal{B}'$ is locally finite in Y, $A \subseteq \bigcup \mathcal{B}'$ and $\mathrm{cl}_Y \mathcal{B}'$ is a partial refinement of $\{G_1, G_2\}$. Clearly, $\mathrm{cl}_X \mathcal{B}'$ is a partial refinement of $\{U_1, U_2\}$. Since Y is open in X and $X \setminus Y \subseteq V \subseteq X \setminus \bigcup \mathcal{B}'$, $\mathcal{B}'$ is locally finite in X. Thus, A is τ-base-normal relative to X. □

Proposition 3.4. *Let X be a normal space and $\tau \geq w(X)$. If X is the union of two τ-base-normal open subspaces of X, then X is τ-base-normal.*

Proof. Let Y and Z be two τ-base-normal open subsets of X with $Y \cup Z = X$. Since X is normal, there exist disjoint open subsets U and V of X such that $X \setminus Y \subseteq U$ and $X \setminus Z \subseteq V$. By Lemma 3.3, $X \setminus U$ and $X \setminus V$ are τ-base-normal relative to X. Since $X = (X \setminus U) \cup (X \setminus V)$, it follows from Lemma 3.2 that X is τ-base-normal. □

4 Base-normality of products of two subspaces of ordinals Let λ be an ordinal with the order topology and let $X \subseteq \lambda^2$. For $A \subseteq \lambda+1$ and $B \subseteq \lambda+1$, put $X_A = (A \times \lambda) \cap X$, $X^B = (\lambda \times B) \cap X$ and $X_A^B = X_A \cap X^B$. The proof of Lemma 4.1 is easy and omitted.

Lemma 4.1. *Let X be a subspace of λ^2 for some ordinal λ. Then $w(X) = |X|$.*

Proposition 4.2. *For every ordinal α, every subspace A of α is base-normal.*

Proof. We prove the proposition by induction on α. Assume that for all $\beta < \alpha$, every subspace of β is base-normal. Let A be a subspace of α. We separate the proof into the following two cases.

Case 1. $\alpha = \beta + 1$ for some ordinal β. If $\beta \notin A$, then $A \subseteq \beta$. By the assumption, A is base-normal. If $\beta \in A$ and β is an isolated point in A, then it is easy to show that A is base-normal. If $\beta \in A$ and $\beta \in \mathrm{Lim}_\alpha(A)$, then $w(A) = |A| \geq \mathrm{cf}(\beta)$. Let $f : \mathrm{cf}(\beta) \to \beta$ be a normal function for β. For each $\gamma < \mathrm{cf}(\beta)$, let $\mathcal{B}_\gamma$ be a base for $A \cap (f(\gamma) + 1)$ witnessing base-normality of $A \cap (f(\gamma) + 1)$. Let

$$\mathcal{B} = \Big(\bigcup_{\gamma < \mathrm{cf}(\beta)} \mathcal{B}_\gamma \Big) \cup \{ (f(\gamma), \beta] \cap A : \gamma < \mathrm{cf}(\beta) \}.$$

It is easy to check that $\mathcal{B}$ witnesses base-normality of A.

Case 2. α is a limit ordinal. If A is bounded in α, then $A \subseteq \beta$ for some $\beta < \alpha$ and therefore A is base-normal. If A is unbounded in α, we treat the following subcases (2a) and (2b) separately.

Subcase (2a). A is not stationary in α. Then there exists a cub set C in α such that $A \cap C = \emptyset$ and $|C| = \mathrm{cf}(\alpha)$. Hence, A can be represented as

$$A = \bigoplus_{\gamma \in \mathrm{Succ}(C)} ((p_C(\gamma), \gamma] \cap A),$$

where $p_C(\gamma)$ is defined in the introduction. For each $\gamma \in \mathrm{Succ}(C)$, since $(p_C(\gamma), \gamma] \cap A$ is a base-normal space with the weight $\leq w(A)$, it is $w(A)$-base-normal. Since $|\mathrm{Succ}(C)| \leq |C| = \mathrm{cf}(\alpha) \leq |A| = w(A)$, by Lemma 3.1(1), A is $w(A)$-base-normal. Hence, A is base-normal.

Subcase (2b). A is stationary in α. Let $g : \mathrm{cf}(\alpha) \to \alpha$ be a normal function for α. For each $\gamma < \mathrm{cf}(\alpha)$, let $\mathcal{W}_\gamma$ be a base for $A \cap (g(\gamma) + 1)$ witnessing base-normality of $A \cap (g(\gamma) + 1)$. Let $\mathcal{B} = (\bigcup_{\gamma < \mathrm{cf}(\alpha)} \mathcal{W}_\gamma) \cup \{(g(\gamma), \alpha) \cap A : \gamma < \mathrm{cf}(\alpha)\}$. Then $\mathcal{B}$ is a base for A and $|\mathcal{B}| = w(A)$. Using the PDL, it is easy to show that $\mathcal{B}$ witnesses base-normality of A. The proof is complete. □

Theorem 4.3. *For A, $B \subseteq \lambda$, $A \times B$ is base-normal if and only if $A \times B$ is normal.*

To prove Theorem 4.3, we need the following two lemmas.

Lemma 4.4 (Kemoto, Ohta and Tamano, [6], Lemma 4.3). *Let A, $B \subseteq \lambda$, μ, $\nu \in \lambda + 1$. Put $X = (A \cap (\mu + 1)) \times (B \cap (\nu + 1))$ and let $\mathcal{P}$ be a collection of subsets of X. Assume that $\mathcal{P}$ is closed under taking subsets and each point of X is contained in an open set in $\mathcal{P}$. Then there exist $\mu_0 < \mu$ and $\nu_0 < \nu$ such that*

$$X \cap ((\mu_0, \gamma] \times (\nu_0, \delta]) \in \mathcal{P}$$

for each $\gamma \in A \cap (\mu_0, \mu]$ and each $\delta \in B \cap (\nu_0, \nu]$, in each of the following cases (1) *and* (2):

(1) $\mu \notin A$, $\nu \notin B$, $\mathrm{cf}(\mu) \geq \omega_1$, $\mathrm{cf}(\nu) \geq \omega_1$ *and either* (1-1) or (1-2) in the following holds:

  (1-1) $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$, $A \cap \mu$ is stationary in μ and $B \cap \nu$ is stationary in ν.

  (1-2) $\mathrm{cf}(\mu) = \mathrm{cf}(\nu)$ and $M^{-1}(A) \cap N^{-1}(B)$ is stationary in $\mathrm{cf}(\mu)$, where $M : \mathrm{cf}(\mu) \to \mu$ and $N : \mathrm{cf}(\nu) \to \nu$ are normal functions defined in the introduction.

(2) $\mu \in A$, $\nu \notin B$, $\mathrm{cf}(\nu) \geq \omega_1$, $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$ and $B \cap \nu$ is stationary in ν.

Lemma 4.5 (Kemoto, Ohta and Tamano, [6], Theorem A). *For A, $B \subseteq \lambda$, $A \times B$ is normal if and only if for each μ, $\nu \in \lambda + 1$ with $\mathrm{cf}(\mu) = \mathrm{cf}(\nu) \geq \omega_1$, the following conditions hold*:

(1) *If $\mu \notin A$ and $\nu \notin B$, then $A \cap \mu$ is not stationary in μ or $B \cap \nu$ is not stationary in ν or $M^{-1}(A) \cap N^{-1}(B)$ is stationary in $\mathrm{cf}(\mu)$, where $M : \mathrm{cf}(\mu) \to \mu$ and $N : \mathrm{cf}(\nu) \to \nu$ are normal functions.*

(2) *If $\mu \in A$ and $\nu \notin B$, then $A \cap \mu$ is bounded in μ or $B \cap \nu$ is not stationary in ν.*

(3) *If $\mu \notin A$ and $\nu \in B$, then $A \cap \mu$ is not stationary in μ or $B \cap \nu$ is bounded in ν.*

Proof of Theorem 4.3. Assume that $A \times B$ is normal and let $\tau = w(A \times B)$. It suffices to show that $A \times B$ is τ-base-normal. We use the idea in the proof of Theorem A in [6]. Suppose that $A \times B$ is not τ-base-normal. Put

$$\mu = \min\{\xi \leq \lambda : (A \times B) \cap ((\xi + 1) \times \lambda)) \text{ is not } \tau\text{-base-normal}\}$$

and

$$\nu = \min\{\eta \leq \lambda : (A \times B) \cap ((\mu + 1) \times (\eta + 1)) \text{ is not } \tau\text{-base-normal}\}.$$

Let $X = (A \times B) \cap ((\mu + 1) \times (\nu + 1))$. Note that X is normal and not τ-base-normal. For each $\mu' < \mu$, since $X_{\mu'+1}$ is a closed subspace of τ-base-normal space $(A \times B) \cap ((\mu' + 1) \times \lambda)$, $X_{\mu'+1}$ is τ-base-normal. For each $\nu' < \nu$, by the definition of ν, $X^{\nu'+1}$ is τ-base-normal. It follows from Proposition 4.2 and Lemma 3.1(1) that μ and ν are limit ordinals. Observe that

(1) $A \cap \mu$ is unbounded in μ and $B \cap \nu$ is unbounded in ν;

(2) if $\mu \notin A$, then $\mathrm{cf}(\mu) \geq \omega_1$ and $A \cap \mu$ is stationary in μ; and

(3) if $\nu \notin B$, then $\mathrm{cf}(\nu) \geq \omega_1$ and $B \cap \nu$ is stationary in ν.

In fact, (1) follows from the definitions of μ and ν, Proposition 4.2 and Lemma 3.1(1). To see (2), suppose that $\mu \notin A$ and either $\mathrm{cf}(\mu) = \omega$ or $A \cap \mu$ is not stationary in μ. Then, in both cases, $A \cap (\mu + 1)$ can be expressed as the sum of $\leq \mathrm{cf}(\mu)$ many bounded, clopen subspaces of $A \cap \mu$. By Lemma 4.1 and (1) in the above, we have $\tau \geq w(X) = |X| = \max\{|\mathrm{A} \cap \mu|, |\mathrm{B} \cap \nu|\} \geq \max\{\mathrm{cf}(\mu), \mathrm{cf}(\nu)\}$. Thus, X is the topological sum of $\leq \tau$ many τ-base-normal subspaces of X. By Lemma 3.1(1), X is τ-base-normal, which is a contradiction. The proof of (3) is similar to that of (2).

For each $\mu' \in A \cap \mu$ and $\nu' \in B \cap \nu$, let $\mathcal{G}(\mu')$ be a base for $X_{\mu'+1}$ witnessing τ-base-normality of $X_{\mu'+1}$, and $\mathcal{H}(\nu')$ a base for $X^{\nu'+1}$ witnessing τ-base-normality of $X^{\nu'+1}$. Let

$$\mathcal{B} = \left(\bigcup_{\mu' \in A \cap \mu} \mathcal{G}(\mu') \right) \cup \left(\bigcup_{\nu' \in B \cap \nu} \mathcal{H}(\nu') \right) \cup \{X \cap ((\mu', \mu] \times (\nu', \nu]) : \mu' \in A \cap \mu \text{ and } \nu' \in B \cap \nu\}.$$

Then $\mathcal{B}$ is a base for X and $|\mathcal{B}| \leq \tau$. Let $\mathcal{U} = \{U_1, U_2\}$ be an open cover of X. We shall show that there exist $\mu_0 \in A \cap \mu$, $\nu_0 \in B \cap \nu$ and $i_0 \in \{1, 2\}$ such that

(4) $X \cap ((\mu_0, \mu] \times (\nu_0, \nu]) \subseteq U_{i_0}$.

If we can find such μ_0, ν_0 and i_0, then there exist $\mathcal{G}' \subseteq \mathcal{G}(\mu_0)$ such that $\mathcal{G}'$ is a locally finite (in X_{μ_0+1}) refinement of $\{U_1 \cap X_{\mu_0+1}, U_2 \cap X_{\mu_0+1}\}$, and $\mathcal{H}' \subseteq \mathcal{H}(\nu_0)$ such that $\mathcal{H}'$ is a locally finite (in X^{ν_0+1}) refinement of $\{U_1 \cap X^{\nu_0+1}, U_2 \cap X^{\nu_0+1}\}$. Put $\mathcal{B}' = \mathcal{G}' \cup \mathcal{H}' \cup \{X \cap ((\mu_0, \mu] \times (\nu_0, \nu])\}$. Then $\mathcal{B}'$ is a locally finite refinement of $\mathcal{U}$ by members of $\mathcal{B}$. It follows from Lemma 3.1(2) that $\mathcal{B}$ witnesses τ-base-normality of X, which is a contradiction.

To prove (4), let

$$\mathcal{P} = \{W \subseteq X : W \subseteq U_1 \text{ or } W \subseteq U_2\}.$$

Then $\mathcal{P}$ is closed under taking subsets and each point of X is contained in an open set in $\mathcal{P}$. Now we separate the proof into four cases.

Case 1. $\mu \notin A$ and $\nu \notin B$. In this case, it follows from (2) and (3) that $\mathrm{cf}(\mu) \geq \omega_1$, $\mathrm{cf}(\nu) \geq \omega_1$, $A \cap \mu$ is stationary in μ and $B \cap \nu$ is stationary in ν. Hence, by Lemma 4.5, we have two subcases: (1a) $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$, and (1b) $\mathrm{cf}(\mu) = \mathrm{cf}(\nu)$ and $M^{-1}(A) \cap N^{-1}(B)$ is stationary in $\mathrm{cf}(\mu)$, where $M : \mathrm{cf}(\mu) \to \mu$ and $N : \mathrm{cf}(\nu) \to \nu$ are normal functions. In both cases, (1) in Lemma 4.4 holds. Hence, there exist $\mu_0 < \mu$ and $\nu_0 < \nu$ such that

(5) $X \cap ((\mu_0, \gamma] \times (\nu_0, \delta]) \in \mathcal{P}$ for each $\gamma \in A \cap (\mu_0, \mu]$ and each $\delta \in B \cap (\nu_0, \nu]$.

Since $A \cap \mu$ and $B \cap \nu$ are unbounded in μ and ν respectively, we can assume that $\mu_0 \in A$ and $\nu_0 \in B$.

Subcase (1a). $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$. Put $B' = B \cap (\nu_0, \nu]$. For each $\delta \in B'$, define a map $f_\delta : \{1, 2\} \to [\mu_0, \mu]$ by

$$f_\delta(i) = \sup\{\gamma : \mu_0 < \gamma < \mu \text{ and } X \cap ((\mu_0, \gamma] \times (\nu_0, \delta]) \subseteq U_i\}$$

for $i \in \{1,2\}$, where the supremum is taken in $[\mu_0, \mu]$, thus $\sup(\emptyset) = \mu_0$. Since $A \cap \mu$ is unbounded in μ, by (5), either $f_\delta(1) = \mu$ or $f_\delta(2) = \mu$. Hence, there exist an unbounded set $D(\delta)$ in μ and $i(\delta) \in \{1,2\}$ satisfying

$$X \cap ((\mu_0, \gamma] \times (\nu_0, \delta]) \subseteq U_{i(\delta)} \text{ for each } \gamma \in D(\delta).$$

This implies that $X \cap ((\mu_0, \mu] \times (\nu_0, \delta]) \subseteq U_{i(\delta)}$. Since $B \cap \nu$ is unbounded in ν, there exist $i_0 \in \{1,2\}$ and $B'' \subseteq B'$ such that B'' is unbounded in ν and $i(\delta) = i_0$ for each $\delta \in B''$. Thus, $X \cap ((\mu_0, \mu] \times (\nu_0, \nu]) \subseteq U_{i_0}$.

Subcase (1b). $\mathrm{cf}(\mu) = \mathrm{cf}(\nu)$ and $M^{-1}(A) \cap N^{-1}(B)$ is stationary in $\mathrm{cf}(\mu)$. Choose $\alpha_0 < \mathrm{cf}(\mu)$ with $\mu_0 < M(\alpha_0)$ and $\nu_0 < N(\alpha_0)$. Define a map $g : \{1,2\} \to [\alpha_0, \mathrm{cf}(\mu)]$ by

$$g(i) = \sup\{\alpha : \alpha_0 < \alpha < \mathrm{cf}(\mu) \text{ and } X \cap ((\mu_0, M(\alpha)] \times (\nu_0, N(\alpha)]) \subseteq U_i\}$$

for $i \in \{1,2\}$, where the supremum is taken in $[\alpha_0, \mathrm{cf}(\mu)]$, thus $\sup(\emptyset) = \alpha_0$. By (5), there exists $i_0 \in \{1,2\}$ such that $g(i_0) = \mathrm{cf}(\mu)$. Hence, $X \cap ((\mu_0, \mu] \times (\nu_0, \nu]) \subseteq U_{i_0}$.

Case 2. $\mu \in A$ and $\nu \notin B$. In this case, it follows from (1) and (3) that $\mathrm{cf}(\nu) \geq \omega_1$, $A \cap \mu$ is unbounded in μ and $B \cap \nu$ is stationary in ν. Since $A \times B$ is normal, we have $\mathrm{cf}(\mu) \neq \mathrm{cf}(\nu)$ by Lemma 4.5. Since (2) in Lemma 4.4 holds, there exist $\mu_0 < \mu$ and $\nu_0 < \nu$ satisfying the condition (5) in the proof of Case 1 above. We may assume that $\mu_0 \in A$ and $\nu_0 \in B$. Define $h : \{1,2\} \to [\nu_0, \nu]$ by

$$h(i) = \sup\{\delta : \nu_0 < \delta < \nu \text{ and } X \cap ((\mu_0, \mu] \times (\nu_0, \delta]) \subseteq U_i\}$$

for $i \in \{1,2\}$, where the supremum is taken in $[\nu_0, \nu]$, thus $\sup(\emptyset) = \nu_0$. By (5), there exists $i_0 \in \{1,2\}$ such that $h(i_0) = \nu$. This means that μ_0, ν_0 and i_0 satisfy (4).

Case 3. $\mu \notin A$ and $\nu \in B$. The proof of this case is similar to that of Case 2.

Case 4. $\mu \in A$ and $\nu \in B$. In this case, it is obvious that (4) is satisfied by some $\mu_0 \in A \cap \mu$, $\nu_0 \in B \cap \nu$ and $i_0 \in \{1,2\}$.

The proof is complete. □

5 Base-normality of subspaces of $\omega_1{}^2$ By Theorem 2.1, not every normal subspace of $(\omega_1 + 1)^2$ is base-normal. By contrast, we show the following Theorem 5.1.

Theorem 5.1. *For $X \subseteq \omega_1{}^2$, X is base-normal if and only if X is normal.*

To prove Theorem 5.1, we need Lemma 5.2 below.

Let $X \subseteq \omega_1{}^2$, $\alpha \in \omega_1$ and $\beta \in \omega_1$. Put $V_\alpha(X) = \{\gamma < \omega_1 : \langle \alpha, \gamma \rangle \in X\}$, $H_\beta(X) = \{\gamma < \omega_1 : \langle \gamma, \beta \rangle \in X\}$ and $\Delta(X) = \{\gamma < \omega_1 : \langle \gamma, \gamma \rangle \in X\}$. For subsets C and D of $\omega_1 + 1$, put $X_C = X \cap (C \times \omega_1)$, $X^D = X \cap (\omega_1 \times D)$ and $X_C^D = X \cap (C \times D)$. Disjoint closed sets E and F of a space Y are said to be *separated* if there exist disjoint open sets U and V of Y such that $E \subseteq U$ and $F \subseteq V$.

Lemma 5.2 (N. Kemoto, T. Nogura, Kerry D. Smith and Y. Yajima, [5]). *Let $X \subseteq \omega_1{}^2$. Then X is normal if and only if the following conditions hold*:

(1) *If α is a limit ordinal in ω_1 and $V_\alpha(X)$ is not stationary in ω_1, then there is a cub set D in ω_1 such that $X_{\{\alpha\}}$ and X^D are separated.*

(2) *If β is a limit ordinal in ω_1 and $H_\beta(X)$ is not stationary in ω_1, then there is a cub set C in ω_1 such that $X_{\{\beta\}}$ and X^C are separated.*

(3) *If* $\Delta(X)$ *is not stationary in* ω_1*, then there is a cub set* C *in* ω_1 *such that* X_C *and* X^C *are separated.*

Proof of Theorem 5.1. Assume that X is normal. If $w(X) \leq \omega$, then X is metrizable and therefore is base-normal (see Theorem 3.3 in [9] and Corollary 2.3 in [10]). Suppose $w(X) = \omega_1$ and X is not base-normal. Then X is not ω_1-base-normal. Let

$$\mu = \min\{\xi \leq \omega_1 : X_{\xi+1} \text{ is not } \omega_1\text{-base-normal}\}$$

and

$$\nu = \min\{\eta \leq \omega_1 : X^{\eta+1}_{\mu+1} \text{ is not } \omega_1\text{-base-normal}\}.$$

Note that $X^{\nu+1}_{\mu+1}$ is normal and not ω_1-base-normal, but for each $\mu' < \mu$ and $\nu' < \nu$, $X^{\nu+1}_{\mu'+1}$ and $X^{\nu'+1}_{\mu+1}$ are ω_1-base-normal. Let $Y = X^{\nu+1}_{\mu+1}$. Then, for each $\mu' < \mu$ and $\nu' < \nu$, $Y_{\mu'+1} = X^{\nu+1}_{\mu'+1}$ and $Y^{\nu'+1} = X^{\nu'+1}_{\mu+1}$. For each $\mu' < \mu$ and $\nu' < \nu$, let $\mathcal{G}(\mu')$ be a base for $Y_{\mu'+1}$ witnessing ω_1-base-normality of $Y_{\mu'+1}$, and $\mathcal{H}(\nu')$ a base for $Y^{\nu'+1}$ witnessing ω_1-base-normality of $Y^{\nu'+1}$. Let

$$\mathcal{B} = \left(\bigcup_{\mu'<\mu} \mathcal{G}(\mu')\right) \cup \left(\bigcup_{\nu'<\nu} \mathcal{H}(\nu')\right) \cup \{Y \cap ((\mu', \mu] \times (\nu', \nu]) : \mu' < \mu, \nu' < \nu\}.$$

Then $\mathcal{B}$ is a base for Y and $|\mathcal{B}| \leq \omega_1$. Since Y is normal and not ω_1-base-normal, by Lemma 3.1(2), there exists a binary open cover $\mathcal{U} = \{U_1, U_2\}$ of Y admitting no locally finite refinement $\mathcal{B}'$ by members of $\mathcal{B}$. Similar to the proof of Theorem 4.3, μ and ν are limit ordinals.

Claim 1. $\langle \mu, \nu \rangle \notin Y$.

Proof of Claim 1. Suppose $\langle \mu, \nu \rangle \in Y$. Then there exist $\mu' < \mu$, $\nu' < \nu$ and $i_0 \in \{1, 2\}$ such that $Y \cap ((\mu', \mu] \times (\nu', \nu]) \subseteq U_{i_0}$. Take a subcollection $\mathcal{G}' \subseteq \mathcal{G}(\mu')$ such that $\mathcal{G}'$ is a locally finite (in $Y_{\mu'+1}$) refinement of $\{U_1 \cap Y_{\mu'+1}, U_2 \cap Y_{\mu'+1}\}$, and $\mathcal{H}' \subseteq \mathcal{H}(\nu')$ such that $\mathcal{H}'$ is a locally finite (in $Y^{\nu'+1}$) refinement of $\{U_1 \cap Y^{\nu'+1}, U_2 \cap Y^{\nu'+1}\}$. Put $\mathcal{B}' = \mathcal{G}' \cup \mathcal{H}' \cup \{Y \cap ((\mu', \mu] \times (\nu', \nu])\}$. Then $\mathcal{B}'$ is a locally finite refinement of $\mathcal{U}$, which is a contradiction. □

Claim 2. $\mu = \omega_1$ and $\nu = \omega_1$.

Proof of Claim 2. Suppose that it does not hold. We distinguish the following three cases (i), (ii) and (iii).

Case (i). $\mu < \omega_1$ and $\nu < \omega_1$. In this case, Y is metrizable. Hence, Y is base-normal and ω_1-base-normal, which is a contradiction.

Case (ii). $\mu < \omega_1$ and $\nu = \omega_1$. Since μ is a limit ordinal and $\mu < \omega_1$, $\mathrm{cf}(\mu) = \omega$. If $V_{\{\mu\}}(Y)$ is stationary in ω_1, then by the PDL, there exist $\mu_0 < \mu$, $\nu_0 < \omega_1$ and $i_0 \in \{1, 2\}$ such that $Y \cap ((\mu_0, \mu] \times (\nu_0, \omega_1)) \subseteq U_{i_0}$. Since $Y \setminus U_{i_0} \subseteq (Y_{\mu_0+1} \cup Y^{\nu_0+1})$, $\mathcal{U}$ has a locally finite refinement $\mathcal{B}' \subseteq \mathcal{B}$. This is a contradiction. If $V_{\{\mu\}}(Y)$ is not stationary in ω_1, then there exists a cub set D in ω_1 such that $D \cap V_{\{\mu\}}(Y) = \emptyset$. Hence, $\omega_1 \setminus D = \bigoplus_{\gamma \in \mathrm{Succ}(D)} (p_D(\gamma), \gamma]$. Thus, Y can be represented as the union

$$Y = \left(\bigoplus_{i<\omega} Y_{(M(i-1), M(i)]}\right) \cup \left(\bigoplus_{\gamma \in \mathrm{Succ}(D)} Y^{(p_D(\gamma), \gamma]}\right),$$

where $M : \omega \to \mu$ is a normal function. By Lemma 3.1(1) and Proposition 3.4, Y is ω_1-base-normal, which is a contradiction.

Case (iii). $\mu = \omega_1$ and $\nu < \omega_1$. The proof of this case is similar to that of Case (ii). The proof of Claim 2 is complete. □

By Claim 2, we have $Y = X$. To complete the proof, it suffices to consider the following two cases:

Case 1. $\Delta(X)$ is stationary. In this case, by the PDL, there exist $\mu' < \omega_1$, $\nu' < \omega_1$ and $i' \in \{1, 2\}$ such that $X \cap ((\mu', \omega_1) \times (\nu', \omega_1)) \subseteq U_{i'}$. This implies that $\mathcal{U}$ has a locally finite refinement $\mathcal{B}' \subseteq \mathcal{B}$, which is a contradiction.

Case 2. $\Delta(X)$ is not stationary. By Lemma 5.2, there is a cub set C in ω_1 such that $X_C \cap X^C = \emptyset$. Since Y can be represented as the union

$$Y = \left(\bigoplus_{\gamma \in \mathrm{Succ}(C)} Y_{(p_C(\gamma), \gamma]} \right) \cup \left(\bigoplus_{\gamma \in \mathrm{Succ}(C)} Y^{(p_C(\gamma), \gamma]} \right),$$

it follows from Lemma 3.1(1) and Proposition 3.4 that Y is ω_1-base-normal, which is a contradiction. The proof is complete. □

6 Total paracompactness of subspaces of products of two ordinals

J. E. Porter [9] called a space X *base-paracompact* if there is an open base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement by members of $\mathcal{B}$. Obviously, every totally paracompact space is base-paracompact and every base-paracompact space is paracompact. It remains unsolved whether or not every paracompact space is base-paracompact. Porter [9] proved that all metrizable spaces and all Lindelöf spaces are base-paracompact. Yamazaki [10] proved that a space is base-paracompact if and only if it is base-normal and paracompact. Gruenhage [4] proved that every paracompact *GO*-space is base-paracompact. In this section, we show that for a subspace of products of two ordinals, total paracompactness, base-paracompactness and paracompactness coincide.

We call a space X *locally totally paracompact* if each point x of X has an open neighborhood O_x such that $\mathrm{cl}_X O_x$ is totally paracompact.

The following Lemma seems to be known, but I could not find a suitable reference.

Lemma 6.1. *Paracompact, locally totally paracompact spaces are totally paracompact.*

Proof. Let X be paracompact and locally totally paracompact. For each $x \in X$, take an open neighborhood O_x of x such that $\mathrm{cl}_X O_x$ is totally paracompact. Since X is paracompact, the open cover $\{O_x : x \in X\}$ has a locally finite open refinement $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Omega\}$ be an open cover of X such that $\mathrm{cl}_X V_\alpha \subseteq U_\alpha$ for each $\alpha \in \Omega$. Let $\mathcal{B}$ be an arbitrary open base for X. For each $\alpha \in \Omega$, let $\mathcal{C}_\alpha = \{B \in \mathcal{B} : B \subseteq U_\alpha\} \cup \{B \cap \mathrm{cl}_X U_\alpha : B \in \mathcal{B} \text{ and } B \cap \mathrm{cl}_X V_\alpha = \emptyset\}$. Then $\mathcal{C}_\alpha$ is an open base for $\mathrm{cl}_X U_\alpha$. Since total paracompactness is hereditary to closed subspaces, $\mathrm{cl}_X U_\alpha$ is totally paracompact. Hence, there exists a subcollection $\mathcal{C}'_\alpha \subseteq \mathcal{C}_\alpha$ such that $\mathcal{C}'_\alpha$ is a locally finite open cover of $\mathrm{cl}_X U_\alpha$. Put $\mathcal{B}'_\alpha = \{C \in \mathcal{C}'_\alpha : C \cap V_\alpha \neq \emptyset\}$. Clearly, $\mathcal{B}'_\alpha$ covers V_α and $\mathcal{B}'_\alpha \subseteq \mathcal{B}$. Let $\mathcal{B}' = \bigcup_{\alpha \in \Omega} \mathcal{B}'_\alpha$. Then $\mathcal{B}'$ is a locally finite cover of X by members of $\mathcal{B}$. Hence, X is totally paracompact. □

The following Lemma 6.2 follows directly from Lemma 6.1.

Lemma 6.2. *If paracompact space X is the union of two totally paracompact open subspaces of X, then X is totally paracompact.*

Lemma 6.3 (N. Kemoto, K. Tamano and Y. Yajima, [7], Lemma 2.2). *Let X be a subspace of an ordinal λ. Then X is paracompact if and only if for every $\mu \in (\lambda+1) \setminus X$ with $\mathrm{cf}(\mu) \geq \omega_1$, $X \cap \mu$ is not stationary in μ.*

Proposition 6.4. *For a subspace X of an ordinal λ, the following are equivalent*:

(a) *X is totally paracompact.*

(b) *X is base-paracompact.*

(c) *X is paracompact.*

Proof. It suffices to show that (c) implies (a). The proof is similar to that of Lemma 2.2 in [7]. Suppose (c) holds and X is not totally paracompact. Put

$$\mu = \{\xi \leq \lambda : X \cap (\xi+1) \text{ is not totally paracompact}\}.$$

It follows from the minimality of μ that μ is a limit ordinal. Note that $X \cap (\mu+1)$ is not totally paracompact, and $X \cap (\mu'+1)$ is totally paracompact for each $\mu' < \mu$. Now we show that $\mu \notin X$. Suppose $\mu \in X$. Then for any open base $\mathcal{B}$ of $X \cap (\mu+1)$, there exists $\mu' < \mu$ and $B' \in \mathcal{B}$ such that $(\mu', \mu] \cap X \subseteq B'$. Since $X \cap (\mu'+1)$ is totally paracompact and $\mathcal{B}' = \{B \in \mathcal{B} : B \subseteq \mu'+1\}$ is an open base for $X \cap (\mu'+1)$, there exists a subcollection $\mathcal{B}'' \subseteq \mathcal{B}'$ such that $\mathcal{B}''$ is a locally finite open cover of $X \cap (\mu'+1)$. Hence, $\mathcal{B}'' \cup \{B'\}$ is a locally finite subcover of $\mathcal{B}$, which contradicts the fact that $X \cap (\mu+1)$ is not totally paracompact. Thus $\mu \notin X$. Let $M : \mathrm{cf}(\mu) \to \mu$ be a normal function for μ.

If $\mathrm{cf}(\mu) = \omega$, then $X \cap (\mu+1)$ can be represented as the sum

$$X \cap (\mu+1) = \bigoplus_{i<\omega} (X \cap (M(i-1), M(i)])$$

of totally paracompact subspaces. Hence, $X \cap (\mu+1)$ is totally paracompact, which is a contradiction.

If $\mathrm{cf}(\mu) \geq \omega_1$, then by Lemma 6.3, $X \cap \mu$ is not stationary in μ. Hence, there exists a cub set C in $\mathrm{cf}(\mu)$ such that $M(C) \cap X = \emptyset$. Since $X \cap (\mu+1)$ can be represented as the sum

$$X \cap (\mu+1) = \bigoplus_{\gamma \in \mathrm{Succ}(C)} (X \cap (M(p_C(\gamma)), M(\gamma)])$$

of totally paracompact subspaces, it is totally paracompact. This is a contradiction. □

Lemma 6.5 (N. Kemoto, K. Tamano and Y. Yajima, [7], Theorem 3.3). *Let X be subspace of λ^2 for some ordinal λ. Then X is paracompact if and only if for each $\langle \mu, \nu \rangle \in (\lambda+1)^2 \setminus X$,*

(1) *if $\mathrm{cf}(\mu) \geq \omega_1$, then there is a cub set C in $\mathrm{cf}(\mu)$ such that $X^{\{\nu\}}$ and $X_{M(C)\cup\{\mu\}}$ are separated,*

(2) *if $\mathrm{cf}(\nu) \geq \omega_1$, then there is a cub set D in $\mathrm{cf}(\nu)$ such that $X_{\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated,*

(3) *if $\mathrm{cf}(\mu) \geq \omega_1$ and $\mathrm{cf}(\nu) \geq \omega_1$, then there are two cub sets C in $\mathrm{cf}(\mu)$ and D in $\mathrm{cf}(\nu)$ such that $X_{M(C)\cup\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated, where $M : \mathrm{cf}(\mu) \to \mu$ and $N : \mathrm{cf}(\nu) \to \nu$ are normal functions.*

Theorem 6.6. *Let X be a subspace of λ^2 for some ordinal λ. Then the following are equivalent*:

(a) *X is totally paracompact.*

(b) *X is base-paracompact.*

(c) *X is paracompact.*

Proof. It suffices to show that (c) implies (a). Suppose (c) holds and X is not totally paracompact. Let

$$\mu = \min\{\xi \le \lambda : X_{\xi+1} \text{ is not totally paracomapct}\}$$

and

$$\nu = \min\{\eta \le \lambda : X_{\mu+1}^{\eta+1} \text{ is not totally paracompact}\}.$$

Note that $X_{\mu+1}^{\nu+1}$ is paracompact and not totally paracompact, but $X_{\mu'+1}^{\nu+1}$ and $X_{\mu+1}^{\nu'+1}$ are totally paracompact for each $\mu' < \mu$ and $\nu' < \nu$. It follows from Proposition 6.4 and the definitions of μ and ν that μ and ν are limit ordinals. Let $Y = X_{\mu+1}^{\nu+1}$. we show that $\langle \mu, \nu \rangle \notin Y$. Suppose $\langle \mu, \nu \rangle \in Y$. Let $\mathcal{B}$ be an arbitrary open base for Y. Then there exist $\mu' < \mu$, $\nu' < \nu$ and $B \in \mathcal{B}$ such that $\langle \mu, \nu \rangle \in (\mu', \mu] \times (\nu', \nu] \subseteq B$. Since $Y_{\mu'+1}$ and $Y^{\nu'+1}$are totally paracompact, this implies that $\mathcal{B}$ has a locally finite subcover, which contradicts the fact that Y is not totally paracompact. Let $M : \mathrm{cf}(\mu) \to \mu$ and $N : \mathrm{cf}(\nu) \to \nu$ be fixed normal functions.

To complete the proof, we only need to consider the following four cases.

Case 1. $\mathrm{cf}(\mu) = \omega$ and $\mathrm{cf}(\nu) = \omega$. In this case, Y can be represented as the union

$$Y = \left(\bigoplus_{i<\omega} Y_{(M(i-1),M(i)]} \right) \cup \left(\bigoplus_{i<\omega} Y^{(N(i-1),N(i)]} \right)$$

of two open subspaces which are the sum of totally paracompact subspaces. By Lemma 6.2, Y is totally paracompact, which is a contradiction.

Case 2. $\mathrm{cf}(\mu) \ge \omega_1$ and $\mathrm{cf}(\nu) = \omega$. In this case, by Lemma 6.5, there is a cub set C in $\mathrm{cf}(\mu)$ such that $Y^{\{\nu\}} \cap Y_{M(C)\cup\{\mu\}} = \emptyset$. Since Y can be represented as the union

$$Y = \left(\bigoplus_{\gamma \in \mathrm{Succ}(C)} Y_{(M(p_C(\gamma)),M(\gamma)]} \right) \cup \left(\bigoplus_{i<\omega} Y^{(N(i-1),N(i)]} \right),$$

Y is totally paracompact, which is a contradiction.

Case 3. $\mathrm{cf}(\mu) = \omega$ and $\mathrm{cf}(\nu) \ge \omega_1$. The proof of this case is quite similar to that of Case 2.

Case 4. $\mathrm{cf}(\mu) \ge \omega_1$ and $\mathrm{cf}(\nu) \ge \omega_1$. In this case, by Lemma 6.5, there are two cub sets C in $\mathrm{cf}(\mu)$ and D in $\mathrm{cf}(\nu)$ such that $Y_{M(C)\cup\{\mu\}} \cap Y^{N(D)\cup\{\nu\}} = \emptyset$. Since Y can be represented as the union

$$Y = \left(\bigoplus_{\gamma \in \mathrm{Succ}(C)} Y_{(M(p_C(\gamma)),M(\gamma)]} \right) \cup \left(\bigoplus_{\delta \in \mathrm{Succ}(D)} Y^{(N(p_D(\delta)),N(\delta)]} \right),$$

Y is totally paracompact, which is a contradiction. The proof is complete. □

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References

[1] H. H. Corson, T. J. McMinn, E. A. Michael and J. Nagata, *Bases and local finiteness*, Amer. Math. Soc. Notices Math. **6** (1959), 814.

[2] John O'Farrell, *The Sorgenfrey line is not totally metacompact*, Houston J. Math. **9** (1983), 271-273.

[3] R. M. Ford, *Basic properties in dimension theory*, Doctoral Dissertation, Auburn University, Auburn, Alabama 1963.

[4] Gary Gruenhage, *Base-paracompactness and base-normality of GO-spaces*, preprint.

[5] N. Kemoto, T. Nogura, Kerry D. Smith and Y. Yajima, *Normal subspaces in products of two ordinals*, Fund. Math. **151** (1996), 279-297.

[6] N. Kemoto, H. Ohta and K. Tamano, *Products of spaces of ordinal numbers*, Topology Appl. **45** (1992), 245-260.

[7] N. Kemoto, K. Tamano and Y. Yajima, *Generalized paracompactness of subspaces in products of two ordinals*, Topology Appl. **104** (2000), 155-168.

[8] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.

[9] John E. Porter, *Base-paracompact spaces*, Topology Appl. **128** (2003), 145-156.

[10] K. Yamazaki, *Base-normality and product spaces*, Topology Appl. **148** (2005), 123-142.

[11] K. Yamazaki, *Some theorems on base-normality*, preprint.

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