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ANALYSIS OF A METHOD FOR CALCULATING AN ECONOMIC ORDER QUANTITY IN INVENTORY MODELS

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Abstract. The author studies probabilistic inventory models of multi-period with discrete demand, in which a method is shown to get an optimal policy and the influence of demand distribution on the optimal policy has also been investigated. On the other hand we have developed the theory of probabilistic inventory models with a piecewise cost function under continuous demand. A lot of properties of an optimal policy in its inventory system are obtained. As this application a method is presented in this paper to get an approximation of an economic order quantity in multi-period inventory problems provided the demand subjects to a continuous distribution. For that sake we assume some conditions on the total cost function of single period. The amount of an economic order quantity depends heavily on the properties of a demand distribution and we shall show it numerically in the inventory examples.

1 Introduction. Probabilistic inventory models of multi-period have been developed in [1], [2], [3] and [4], in which some conditions are searched to help getting an optimal policy provided that the total cost function of single period is known very well. Standard inventory models with continuous demand are constructed and they are converted into mathematical models in order to make an analysis of the functions $F_{n-1}(z)$ that are useful to decide the amount of order on a problem of n-period. Though we obtain a lot of result and study the economic order quantity in many examples, it is difficult in essence to capture the precise value of the economic order quantity because demand is a continuous random variable.

Recently a fundamental theory of a probabilistic mathematical model of multi-period with discrete demand is shown in [5] and its applications in inventory models are stated in [6] and a method to get an optimal policy is presented in [7] that are all a generalization of the former one. It is not so difficult in this case to decide the economic order quantity and a consideration how much the values of the purchasing cost, the holding cost and the shortage cost influence our decision is given in [8].

The concept of [8] is attempted to generalize to the case demand is a continuous random variable B in this paper. There are elaborate results in [1] how to get the economic order quantity (EOQ) with a given demand distribution. We present another method to catch the optimal policy in the dynamic inventory problem. Let x be the amount on hand before an order is placed and let h and p be the holding cost, the shortage costs per unit per period respectively. Let c be the purchasing cost per unit and let denote by z the amount on hand in initial period after an order is received, which means that the initial regular order is $z - x$. Put $\kappa = (p-c)/(h+p)$. Our main tool is the function $w_n(z)$ $(n = 1, 2, \dots)$ that is constructed inductively and our aim is achieved by solving an equation

$$w_n(z) = \kappa + \frac{\alpha c}{h+p}$$

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where α $(0 < \alpha < 1)$ is the discount factor. Numerical results are shown for solving this equation by a computer soft Mathematica in some examples with a particular distribution. However it is not so easy to find a solution and a secondary method to support the method should be studied. The decision criterion is the minimization of expected costs which include the ordering, holding, and shortage costs. That is, let $C(B, z)$ be total cost per period where B is a demand random variable. Then the expectation of total cost $E\{C(B, z)\}$ is

$$E\{C(B, z)\} = c \cdot \{\text{purchasing quantity}\} + hE\{\text{holding quantity}\} + pE\{\text{shortage quantity}\}$$

and our purpose is to obtain a value z at which $E\{C(B, z)\}$ is minimized.

In section 2 the inventory model which is searched will be shown and some fundamental results including an analysis of a mathematical model of dynamic inventory problems are reviewed. A method to seek the optimum policy is gotten in section 3. Some examples and numerical results are discussed in section 5 and 6.

2 Inventory model.

The following inventory model is studied in this article. That is a probabilistic multi-period inventory model with zero delivery lag, backlogging of demand.

2.1 Model and notations.

1. The multi-period model with backlogging of demand will be investigated under general demand without setup cost. The stock replenishment occurs instantaneously.

2. Regular ordering takes at the beginning of each period, purchasing cost per unit c is charged and the period length is t. Let x be the initial stock level and let z be the amount on hand in initial period after an order is received. That means that the amount of a regular order is $z - x$.

3. Let h and p be the holding and shortage costs per unit per period, respectively. We assume $c < p$.

4. Demand B in each period is a non-negative continuous random variable with a known distribution $\Phi(b)$ and its probability density function $\phi(b)$ with $\phi(b) = 0$ $(b < 0)$. The functions $\Phi(b)$ and $\phi(b)$ remain unchanged from period to period and demand in each period are independent.

5. Let b be demand during the period t. Demand occurs according to a general function $g(T/t)b$ at time T $(0 \leq T \leq t)$, where $g(x)$ is a real valued function such that $g(x) = 0$, $g(1) = 1$ and $g(x)$ has a continuous derivative on $[0,\ 1]$ with $dg(x)/dx > 0$ $(0 \leq x \leq 1)$. That is, we assume that the amount in inventory at time T is $z - g(T/t)b$ $(0 \leq T \leq t)$.

6. The total cost is the sum of the purchasing cost, the holding cost and the shortage cost. We search the amount of a regular order at which the expectation of the total cost is minimal through N-period.

7. α $(0 < \alpha < 1)$ denotes the discount factor. Let $f_n(x)$ be the discount minimal expected loss for n-period inventory model provided that an optimal policy is used at each purchasing opportunity, where x is the initial stock level.

Since the unfilled demand is backlogged, it is necessary to investigate in the case when z is negative.

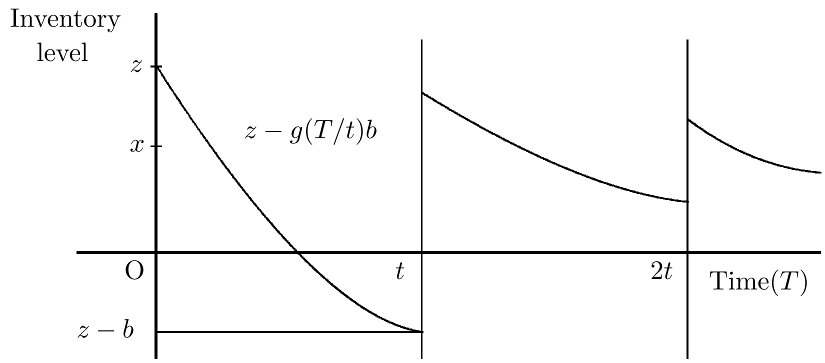

Figure 1 Inventory Level.

2.2 Minimal expectation of the total cost. First let us define the function $H(z)$. Let $I_1(z,b)$ be the average inventory per cycle and $I_2(z,b)$ the shortage quantity average per cycle. We let $E\{C(z,B)\}$ be the expectation of the total cost per period. Then we have

$$E\{C(z,B)\} = c(z-x) + hE\{I_1(z,B)\} + pE\{I_2(z,B)\}. \tag{2.1}$$

The function $H(z)$ is defined by the equality

$$E\{C(z,B)\} = -cx + H(z). \tag{2.2}$$

Let x be an initial stock level and let $f(x)$ be the minimal expectation of the total cost of single period. Then

$$f_1(x) = \min_{z \geq x}\{-cx + H(z)\}. \tag{2.3}$$

Let $f_n(x)$ be as one in assumption 7 on $H(z)$. Then

$$f_n(x) = \min_{z \geq x}\left\{-cx + H(z) + \alpha \int_0^\infty f_{n-1}(z-b)\phi(b)db\right\}. \tag{2.4}$$

Put functions $F_k(z)$ $(k = 1, 2, \ldots, N-1)$ as follows :

$$F_k(z) = H(z) + \alpha \int_0^\infty f_k(z-b)\phi(b)db. \tag{2.5}$$

where $f_0(x) = 0$. Note $F_0(z) = H(z)$. By (2.4) and (2.5) we may write

$$f_n(x) = \min_{z \geq x}\{-cx + F_{n-1}(z)\}. \tag{2.6}$$

2.3 Optimal policy. It needs to search properties of the function $H(z)$ in order to get the optimal policy in our inventory model. In general situation some assumptions are made on the function $H(z)$ and the optimal policy of inventory models are discussed as follows in [4].

Assumption on $H(z)$:

1. $H(z)$ is a piecewise continuous function on $\mathbf{R}$ and $H(z)$ has a minimal value at $z = \bar{x}_1$. More precisely if $z < \bar{x}_1$, then $H(z) > H(\bar{x}_1)$, and if $z \geq \bar{x}_1$ then $H(z) \geq H(\bar{x}_1)$.

2. Let $R_1, \ldots, R_m$ be a sequence of real numbers such that $R_1 < \cdots < R_m$ and $R_1 < \bar{x}_1 < R_2$. There are real valued functions $H_i(z)$ $(1 \le i \le m-1)$ defined on $[R_i,\ R_{i+1}]$ and $H_m(z)$ defined on $[R_m,\ \infty)$. We suppose that $H_i(z)$ $(1 \le i \le m)$ has a continuous derivative on $[R_i,\ R_{i+1}]$ and we assume that
$$H(z) = H_i(z) \ \text{ if } \ z \in [R_i,\ R_{i+1}] \ \ (1 \le i \le m),$$
which leads us that $H(z)$ is continuous on $[R_1, \infty)$.
3. $H'(z)$ is non-decreasing on $[R_1,\ \infty)$.
4. We have $\lim_{z\to\infty} H'(z) > c$.

Calculating the function $H(z)$ in a lot of natural inventory models we are able to find a sequence $R_1, \ldots, R_m$. Under these assumptions it is shown that

$$\bar{x}_1 = \inf\{\ z \mid H'_+(z) \ge 0\} = \inf\{\ z \mid F'_{0+}(z) \ge 0\ \} \tag{2.7}$$

since $F_0(z) = H(z)$. Moreover by these assumptions the following fundamental results are obtained.

Theorem 2.1 (Theorem 1.8 and Theorem 1.9 in [4]) *There is a number $\bar{x}_{n+1}$ such that $\bar{x}_{n+1} = \inf\{\ z \mid F'_{n+}(z) \ge 0\ \}$. The optimal policy in the inventory problem of N-period is that if the initial stock x is less than $\bar{x}_N$, then order $\bar{x}_N - x$ and otherwise do not order.*

2.4 Function $H(z)$. Assume that the inventory model is one in 2.1. We shall check the conditions on $H(z)$ above. First put $G(y) = \int_0^\infty$ if $0 \le y \le 1$ and let m be the mean of the distribution $\Phi(b)$. The following proposition holds in [1].

Proposition 2.2 *If $z \le 0$, then $H(z) = (c-p)z + pG(1)m$. If $z > 0$, then*

$$\begin{aligned} H(z) =& (c-p)z + (h+p)\left\{ z\int_0^z \phi(b)db + z\int_z^\infty g^{-1}(z/b)\phi(b)db \right. \\ & \left. - \int_z^\infty bG\left(g^{-1}(z/b)\right)\phi(b)db - G(1)\int_0^z b\phi(b)db \right\} + pG(1)m. \end{aligned}$$

We obtain a derivative of the function $H(z)$.

Proposition 2.3 *We have*

$$H'(z) = \begin{cases} c-p & \text{if} \quad z < 0, \\ c - p + (h+p)\left\{ \int_0^z \phi(b)db + \int_z^\infty g^{-1}(z/b)\phi(b)db \right\} & \text{if} \quad z > 0. \end{cases}$$

Proposition 2.4 *We have*

$$H''(z) = \begin{cases} 0 & \text{if} \quad z < 0, \\ (h+p)\int_z^\infty \dfrac{\phi(b)}{bg'(z/b)}db & \text{if} \quad z > 0. \end{cases}$$

By proposition 2.3 it is shown that

$$\lim_{z\to\infty} H'(z) = c + h. \tag{2.8}$$

It is clear $H'_-(0) = c - p$. If we could prove the existence of $H'_+(0) = c - p$ and assumption 3, then the conditions on $H(z)$ hold by setting $m = 1$ and $R_1 = 0$, and we may use Theorem 2.1. It follows from Corollary to Theorem 1.10 in [4] that if a number R_2 satisfies $H'_+(R_2) \geq \alpha c$, then $\bar{x}_n \leq R_2$. By (2.8) it could be able to find a number R_2 with $H'_+(R_2) \geq \alpha c$, and the examples we are going to search in this paper follows theses conditions. Therefore we add new assumptions on $H(z)$ as the way in [8].

Assumption on $H(z)$:

5. Let $R_1 = 0$. There is a number R_2 such that $0 < R_2$ and $H'_+(R_2) \geq \alpha c$.

6. $H'_+(0)$ exists.

Then the following theorem holds.

Theorem 2.5 (Theorem 2.6 in [2] and Theorem 1.8 in [4]) *Under the assumptions on $H(z)$ we have the following statements in the model 2.1.*

1. $0 < \bar{x}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_n \leq \cdots \leq R_2$.

2. $F'_{n+}(0) = H_+(0) \quad (n = 1, 2, \ldots, N-1)$.

3. $F'_n(z) = H'(z) - \alpha c \quad for \quad z < \bar{x}_n \quad (n = 1, 2, \ldots, N-1)$.

4. $F'_n(z) = H'(z) - \alpha c + \alpha \int_0^{z-\bar{x}_n} F'_{n-1}(z-b)\phi(b)db \quad for \quad \bar{x}_n \leq z \quad (n = 1, 2, \ldots, N-1)$.

3 Approximation of EOQ

3.1 Method. Under the assumptions on $H(z)$ we present a method to seek an approximation of $\bar{x}_n$. For the simplicity let introduce a function $w(z)$ as follows :

$$w(z) = \int_0^z \phi(b)db + \int_z^\infty g^{-1}(z/b)\phi(b)db \tag{3.1}$$

If $z > 0$, then $w'(z) = \displaystyle\int_z^\infty \frac{\phi(b)}{bg'(z/b)}db \geq 0$, and assumption 5 of 2.1 leads us

$$\lim_{z \to \infty} w(z) = 1. \tag{3.2}$$

It is known by Proposition 2.3 that if $z > 0$, then

$$H'(z) = c - p + (h + p)w(z). \tag{3.3}$$

Set

$$\kappa = \frac{p - c}{h + p}. \tag{3.4}$$

Then $0 < \kappa < 1$ and we see that $H'(z) \geq 0$ if and only if $w(z) \geq \kappa$. Thus

$$\bar{x}_1 = \inf\{ z \mid w(z) \geq \kappa\}. \tag{3.5}$$

It is also shown that $H'(z) \geq \alpha c$ if and only if $w(z) \geqq \kappa + \alpha c/(h+p)$. Since $0 < \kappa + \alpha c/(h+p) < 1$, it is sufficient to find the positive number R_2 which satisfies the inequality

$$w(R_2) \geq \kappa + \frac{\alpha c}{h + p}. \tag{3.6}$$

Next in order to get $\bar{x}_n$ $(n = 2,\ 3, \ldots,\ N)$ we need another expression of $F_n'(z)$ that is able by Theorem 2.5. Define functions $w_i(z)$ $(i = 1, 2,\ \ldots, N-1)$ as follows:

$$w_i(z) = w(z) + \alpha \int_{\bar{x}_i}^{z} \left(w_{i-1}(b) - \kappa - \frac{\alpha c}{h+p} \right) \phi(z-b)db \tag{3.7}$$

where $w_0(z) = w(z) + \alpha c/(h+p)$.

Proposition 3.1 *If $z \geq \bar{x}_n$, then*

$$F_n'(z) = c - p - \alpha c + (h+p)w_n(z).$$

Proof. In fact these equalities may be proved by induction. We see by Theorem 2.5, (3.3) and (3.7) that

$$\begin{aligned}
F_1'(z) =& H'(z) - \alpha c + \alpha \int_0^{z-\bar{x}_1} H'(z-b)\phi(b)db \\
=& c - p + (h+p)w(z) - \alpha c + \alpha \int_0^{z-\bar{x}_1} \left(c - p + (h+p)w(z-b)\right) \phi(b)db \\
=& c - p - \alpha c + (h+p) \left\{ w(z) + \alpha \int_{\bar{x}_1}^{z} \left(w(b) - \kappa\right) \phi(z-b)db \right\} \\
=& c - p - \alpha c + (h+p)w_1(z).
\end{aligned}$$

Assume the equality

$$F_i(z) = c - p - \alpha c + (h+p)w_i(z)$$

holds. Then it follows that

$$\begin{aligned}
F_{i+1}'(z) =& H'(z) - \alpha c + \alpha \int_0^{z-\bar{x}_{i+1}} F_i'(z-b)\phi(b)db \\
=& c - p + (h+p)w(z) - \alpha c + \alpha \int_{\bar{x}_{i+1}}^{z} F_i'(z)\phi(z-b)db \\
=& c - p + (h+p)w(z) - \alpha c + \alpha \int_{\bar{x}_{i+1}}^{z} \left(c - p - \alpha c + (h+p)w_i(z)\right) \phi(z-b)db \\
=& c - p - \alpha c + (h+p) \left\{ w(z) + \alpha \int_{\bar{x}_{i+1}}^{z} \left(w_i(z) - \kappa - \frac{\alpha c}{h+p} \right) \phi(z-b)db \right\} \\
=& c - p - \alpha c + (h+p)w_{i+1}(z).
\end{aligned}$$

We complete the proof.

By Proposition 3.1, $F_n'(\bar{x}_{n+1}) = 0$ if and only if

$$w_n(\bar{x}_{n+1}) = \kappa + \frac{\alpha c}{h+p}. \tag{3.8}$$

It is also seen that

$$\bar{x}_{n+1} = \inf \left\{ z \mid w_n(z) \geq \kappa + \frac{\alpha c}{h+p} \right\}. \tag{3.9}$$

Proposition 3.2 *Assume that $w'(z) > 0$ for $0 < z < R_2$. Then we have $w_n'(z) > 0$ for $\bar{x}_n < z < R_2$.*

Proof. This is also proved by induction. Suppose $\bar{x}_n < z < R_2$. Because $0 < b < z - \bar{x}$ if and only if $\bar{x}_n < z - b < z$, we see that $w_{n-1}'(z-b) > 0$ for $0 < b < z - \bar{x}_n$ by induction hypothesis. The equation (3.7) implies that

$$w_n(z) = w(z) + \alpha \int_0^{z-\bar{x}_n} \left(w_{n-1}(z-b) - \kappa - \frac{\alpha c}{h+p} \right) \phi(b) db$$

and hence

$$w_n'(z) = w'(z) + \alpha \int_0^{z-\bar{x}_n} w_{n-1}'(z-b)\phi(b)db + \alpha \left(w_{n-1}(\bar{x}_n) - \kappa - \frac{\alpha c}{h+p} \right) \phi(z - \bar{x}_n).$$

It follows from (3.9) that $w_{n-1}(\bar{x}_n) - \kappa - \alpha c/(h+p) \geq 0$ and therefore $w_n'(z) > 0$. We complete the proof.

Suppose that $w'(z) > 0$ for $0 < z < R_2$. Then it follows from Proposition 3.2 that a number R_2 is obtain by solving the equation

$$w(z) = \kappa + \frac{\alpha c}{h+p} \tag{3.10}$$

and (3.9) is stated as following. $\bar{x}_{n+1}$ is an solution of equation

$$w_n(z) = \kappa + \frac{\alpha c}{h+p} \tag{3.11}$$

which is in $[0, R_2]$.

3.2 Algorithm. Suppose that $w'(z) > 0$ for $0 < z < R_2$. Summarizing the steps to get $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N$ we have:

1. Solve the equation $w(z) = \kappa$ and its root in $[0, R_2]$ is $\bar{x}_1$.
2. Let $i = 1$.
3. Put $w_0(z) = w(z) + \dfrac{\alpha c}{h+p}$.
4. Calculate $w_i(z)$ by the equation
$$w_i(z) = w(z) + \alpha \int_{\bar{x}_i}^{z} \left(w_{i-1}(b) - \kappa - \frac{\alpha c}{h+p} \right) \phi(z-b) db$$
5. Solve the equation $w_i(z) = \kappa + \dfrac{\alpha c}{h+p}$ and its root in $[0, R_2]$ is $\bar{x}_{i+1}$.
6. If $i < N - 1$, set $i = i + 1$ and go to 4. Otherwise stop.

4 Demand distribution. We make use of the following two distributions in examples of the inventory system.

4.1 Uniform distribution. Let d be a positive number and let $\phi(b)$ be

$$\phi(b) = \begin{cases} \dfrac{1}{d} & \text{if} \quad 0 \leq b \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

4.2 Exponential distribution. Let demand B subject to an exponential disribution with mean $1/\lambda$. That is

$$\phi(b) = \begin{cases} \lambda e^{-\lambda b} & \text{if} \quad b \geq 0, \\ 0 & \text{if} \quad b < 0. \end{cases} \tag{4.2}$$

5 Example 1. Our various inventory models are created by fixing a demand pattern function $g(x)$ and the probabilistic density function $\phi(b)$ of a demand distribution. We first consider in the case $g(x) = x$.

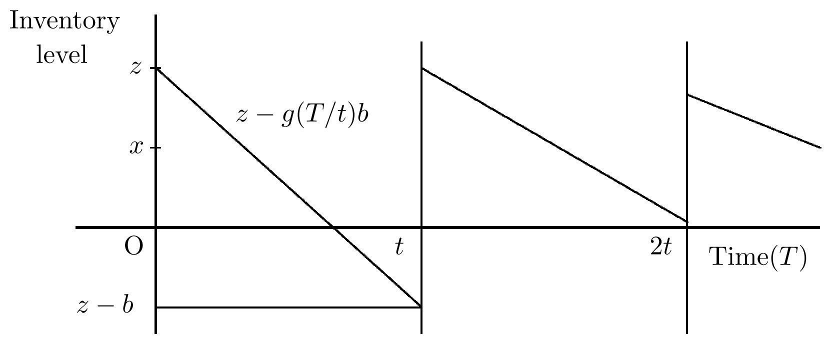

Figure 2. The case $g(x) = x$.

Then $g^{-1}(x) = x$ and $G(x) = \frac{1}{2}x^2$. It follows from (3.1) that

$$w(z) = \int_0^z \phi(b)db + z\int_z^\infty \frac{\phi(b)}{b}db \tag{5.1}$$

Since $z - g(T/t)b = -(b/t)T + z$ $(0 \leq T \leq t)$, the inventory level is shown in the Figure 2.

5.1 Demand of uniform distribution. Let demand B subject to uniform distribution of 4.1. Then it is shown that

$$w(z) = \begin{cases} \dfrac{z}{d}(1 + \log d - \log z) & \text{if} \quad 0 < z \leq d, \\ 1 \quad \text{if} \quad z > d, \end{cases} \tag{5.2}$$

and $\lim_{z\to+0} w(z) = 0$. It follows from Proposition 2.2 that

$$\begin{aligned} H'_+(0) = \lim_{z\to+0} \Bigg[(c-p) + (h+p) \Bigg\{ &\int_0^z \phi(b)db \\ &+ z\int_z^\infty \frac{\phi(b)}{b}db - \frac{z}{2}\int_z^\infty \frac{\phi(b)}{b}db - \frac{1}{2z}\int_0^z b\phi(b)db \Bigg\} \Bigg] \\ = c - p + \lim_{z\to+0} &\left[\frac{z}{d}(\log d - \log z) - \frac{z}{4d} \right] = c - p. \end{aligned}$$

In particular the function $H(z)$ implies all assumptions in this case. Since $\kappa + \alpha c/(h+p) < 1 = w(d)$, we may set $R_2 = d$ by (3.6). If $0 < z < d$, then $w'(z) = (\log d - \log z)/d > 0$. Thus it works to use algorithm 3.2. Therefore $w(\bar{x}_1) = \kappa$ and hence

$$\bar{x}_1 = w^{-1}(\kappa) \quad \text{and} \quad 0 < \bar{x}_1 < d = R_2. \tag{5.3}$$

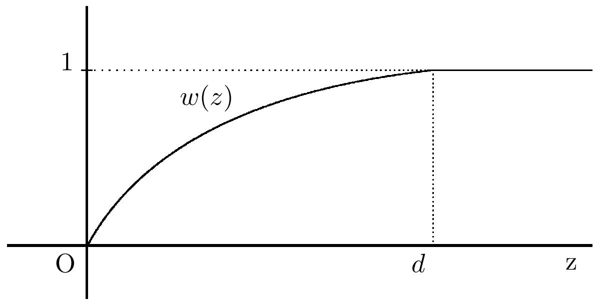

Figure 3. Function $w(z)$

It is obtained by (3.7) that if $\bar{x}_n \le z \le R_2$, then

$$w_n(z) = w(z) + \frac{\alpha}{d}\int_{\bar{x}_n}^{z}\left(w_{n-1}(b) - \kappa - \frac{\alpha c}{h+p}\right)db. \tag{5.4}$$

Numerical example. Approximate values of $\bar{x}_n$ $(1 \le n \le 4)$ are gotten by the mathematical soft Mathematica. Let $c = 100$, $p = 200$, $h = 5$ and $\alpha = 0.95$. Then $\kappa = 20/41$ and a solution of the equation $w(z) = \kappa$ is 1.795 and so $\bar{x}_1 \approx 1.795$. To get $\bar{x}_2$, $\bar{x}_3$ and $\bar{x}_4$ we use values of functions $u_n(z) := w_n(z) - \kappa - \alpha c/(h+p)$.

Table 1: Function $u_1(z) = w_1(z) - \kappa - \alpha c/(h+p)$

z	5.416	5.417	5.418	5.419
$u_1(z)$	-1.015×10^{-4}	-3.508×10^{-6}	9.446×10^{-5}	1.924×10^{-4}

The values of the function $u_1(z)$ are calculated by bisection method in Table 1 that shows $\bar{x}_2 \approx 5.418$. The approximate value $\bar{x}_2$ is obtained using the Table 2 and which shows $\bar{x}_3 \approx 6.812$.

Table 2: Function $u_2(z) = w_2(z) - \kappa - \alpha c/(h+p)$

z	6.810	6.811	6.812	6.813
$u_2(z)$	-7.388×10^{-5}	-2.360×10^{-5}	2.666×10^{-5}	7.692×10^{-5}

Repeating the same method we observe $\bar{x}_4 \approx 7.041$ by Table 3.

Table 3: Function $u_3(z) = w_3(z) - \kappa - \alpha c/(h+p)$

z	7.039	7.040	7.041	7.042
$u_2(z)$	-4.453×10^{-5}	-8.354×10^{-6}	2.781×10^{-5}	6.397×10^{-5}

5.2 Demand of exponential distribution. Let demand B subject to exponential distribution of 4.2. In this case if $z > 0$, then

$$w(z) = \int_0^z \lambda e^{-\lambda b} db + z \int_z^\infty \frac{\lambda e^{-\lambda b}}{b} db. \tag{5.5}$$

By the same way in 5.1 we see that $H'_+(0) = c - p$, $\lim_{z \to +0} w(z) = 0$. If $z > 0$, then $w'(z) = \int_z^\infty \frac{\lambda e^{-\lambda b}}{b} db > 0$. Find a number R_2 by (3.11) and $\bar{x}_n$ $(n = 1, 2, \ldots, N)$ are obtained by algorithm 3.2. It follows from (3.7) that

$$w_n(z) = w(z) + \alpha\lambda \int_{\bar{x}_n}^z \left(w_{n-1}(b) - \kappa - \frac{\alpha c}{h+p} \right) e^{-\lambda(z-b)} db \tag{5.6}$$

Numerical example. Let $c = 100,\ p = 200,\ h = 5$ and $\alpha = 0.95$ as 5.1. Let $\lambda = 1/20$. Since $w(36.010) - \kappa - \alpha c/(h+p) \approx -2.422 \times 10^{-6}$ and $w(36.011) - \kappa - \alpha c/(h+p) \approx 8.108 \times 10^{-7}$, it is adequate to set $R_2 = 36.011$. Then, in view of quite similar method as before, $\bar{x}_1 \approx 5.107, \bar{x}_2 \approx 18.937$ and $\bar{x}_3 \approx 28.073$.

6 Example 2. Let $g(x) = \sqrt{x}$. Then $g^{-1}(x) = x^2$ and it follows from (3.1) that

$$w(z) = \int_0^z \phi(b) db + z^2 \int_z^\infty \frac{\phi(b)}{b^2} db. \tag{6.1}$$

Because of $z - g(T/t)b = -(b/\sqrt{t})\sqrt{T} + z$ $(0 \le T \le t)$, the inventory level is indicated as a graph of Figure 1. Comparing two figures, Figure 1 and Figure 2, it must be that $\bar{x}_n$ of Example 1 is less than one of Example 2. It can be proved and our numeriacl examples show it.

6.1 Demand of uniform disribution. Assume that demand B subjects to uniform distribution. Then

$$w(z) = \begin{cases} \dfrac{z}{d^2}(2d - z) & \text{if} \quad 0 < z \le d, \\ 1 \quad \text{if} \quad z > d \end{cases} \tag{6.2}$$

and $H'_+(0) = c - p$. Set $R_2 = d$ by (3.6). If $0 < z < d$, then $w'(z) = 2(d-z)/d^2 > 0$. Whence we get $\bar{x}_2,\ \bar{x}_3, \cdots, \bar{x}_N$ by algorithm 3.2.

Numerical example. Let situations be the same as a numerical example in section 5.1. We are able to get approximate values of $\bar{x}_n(1 \le n \le 4)$ by solving equation (3.11) because the equation (6.1) is simple in this case. In fact we have $\bar{x}_1 \approx 2.843,\ \bar{x}_2 \approx 6.484,\ \bar{x}_3 \approx 7.648$ and $\bar{x}_4 \approx 7.790$.

6.2 Demand of exponential disribution. Let demand subject to exponential distribution. In view of Proposition 2.2 $H'_+(0) = c - p$ and if $z > 0$, then

$$w(z) = \int_0^z \lambda e^{-\lambda b} db + z^2 \int_z^\infty \frac{\lambda e^{-\lambda b}}{b^2} db. \tag{6.3}$$

The function $w(z)$ is increasing and continuous on $[0, \infty)$. Indeed we have that $H'_+(0) = c - p$, $\lim_{z \to +0} w(z) = 0$ and if $z > 0$, then $w'(z) = \int_z^\infty \left(\lambda e^{-\lambda b}/b \right) db > 0$. A number R_2 is found by 3.10 and using algorithm 3.2 numbers $\bar{x}_n$ $(n = 1,\ 2,\ \ldots, N)$ are obtained.

Numerical example. Consider the same data in a numerical example of 5.2. That is $c = 100,\ p = 200,\ h = 5,\ \alpha = 0.95$ and $\lambda = 1/20$. The algorithm 3.2 implies $R_2 = 43.412,\ \bar{x}_1 \approx 8.062,\ \bar{x}_2 \approx 23.986$ and $\bar{x}_3 \approx 34.045$.

7 Discussion. The good part of the algorithm given in this article is to restrict the range where $\bar{x}_n$ exists. However it happens to be quitely difficult to get it and the source of the trouble comes from a demand pattern function $g(x)$ and the probabilistic density function $\phi(b)$. The example 6.1 is only simple one among examples because solving the equation is praticable by the computer soft Mathematica. In other examples we are forced to use the method of bisection to get solutions of the equation

$$w_i(z) = \kappa + \frac{\alpha c}{h+p} \tag{7.1}$$

since it is too complex for Mathematica to deal it when i is more than 2. It needs to find another way to manage well and we have attempted to do a little change of the function $w_i(z)$ but we now fail in improving our method because dymanic theory is applied to the algorithm and we must use $\bar{x}_i$ to form $w_i(z)$. Since a little error of $\bar{x}_1$ transmits a large error of $\bar{x}_i$, it seems difficult to set it up in general. We only succeed in a special case.

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