A NOTE ON GOLDEN BISECTION NUMBER AND TWO-PLAYER INFINITE GAME

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ABSTRACT. The golden bisection number $g = \frac{1}{2}\left(\sqrt{5}-1\right)$ is famous as a mark of beauty in the history of art. It is also famous in mathematics, such as continued fraction and Fibonacci sequence of integers, *etc.* We find that in the two-player infinite games a lot of interesting results, where g and some pair of numbers lying near to g play an important role. We show four examples.

1 Introduction. The numbers $g = \dfrac{1}{2}\left(\sqrt{5}-1\right) \approx 0.61803$ and $g/\bar{g} = g^{-1} = \dfrac{1}{2}\left(\sqrt{5}+1\right)$ are called golden bisection number and golden ratio, respectively, and g is a mark of beauty in the history of art, from the ancient era of Greece. It is also famous in mathematics, such as the infinite continued fraction $g = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$ and Fibonacci sequence of integers, where the consecutive two integers grow in the limit by the ratio $g:1$, *etc.* The object of this note is to show that g has an interesting connection with two-player infinite games, and also that, by extending the notion of g, we are led to, as might be called, the golden trisection numbers. We will explain this fact by giving four simple examples, in Sections $2 \sim 5$. Our conclusion is given in Section 6.

2 Hi-Lo Stud Poker. For the two-player poker discussed in Sections 2 and 3, see Ref.[2; Chapter 6]. Suppose that two players I and II receive hands x and y, respectively, which are r.v.s distributed *i.i.d.* with $U_{[0,1]}$. Players have to choose one of the two alternatives Hi and Lo. Choices are made simultaneosly and independently of his opponent's choice. Then players make show-down, and one with the higher (lower) hand than the opponent's, is the winner, if players' choices come out to be Hi-Hi (Lo-Lo). If players choose Hi-Lo or Lo-Hi, the game is draw.

Let the payoff be given by

$$\begin{array}{cc} & \begin{array}{cc} \text{Hi} & \text{Lo} \end{array} \\ \begin{array}{c} \text{Hi} \\ \text{Lo} \end{array} & \left[\begin{array}{cc} g^{-1}sgn(x-y) & 0 \\ 0 & sgn(y-x) \end{array}\right]. \end{array}$$

Note that $g^{-1} = \dfrac{1}{2}\left(\sqrt{5}+1\right) \approx 1.61803$ is the golden ratio and the Lo-Lo element is $sgn(y-x)$. It is easily shown that the value of the game is zero and players have the common optimal strategy;

Choose Hi (Lo), if his hand is $> (<) g$.

Next we consider the bilateral-move version of this poker. The game is described by

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Players' hand	First move	Second move	Payoff to I
I ; x	Hi		
	Lo		
II ; y		Hi	$g^{-1}sgn(x-y)$
		Lo	0
		Hi	0
		Lo	$sgn(y-x)$

Player II, at his move, knows which choice was made by I in the previous move, and therefore he can utilize the information in deciding his choice.

It is shown that the value of the game is $-\frac{1}{4}g \approx -0.15415$ and an optimal play proceeds as;

$$\text{I chooses } \begin{cases} \text{Hi}, \\ (\text{Hi}, \text{Lo}; \bar{g}, g), \quad \text{if} \\ \text{Lo}, \end{cases} \begin{cases} x > b_1 \equiv \frac{1}{2}(1+g) \approx 0.80902, \\ b_1 > x > b_0, \\ x < b_0 \equiv \frac{1}{2}g \approx 0.30902, \end{cases}$$

$$\text{If I chooses } \left\{ \begin{matrix} \text{Hi} \\ \text{Lo} \end{matrix} \right\}, \text{then II chooses Hi (Lo), if } y > (<) \left\{ \begin{matrix} b_1 \\ b_0 \end{matrix} \right\}.$$

where $(\text{Hi}, \text{Lo}; \bar{g}, g)$ means the mixed strategy that chooses Hi and Lo with probabilities $\bar{g}$ and g, respectively.

Since the first mover I stands unfavorable over the second mover II, he randomizes his choice in order to improve his disadvantage. The value of the game is negative. See Ref.[4; Theorems 1 and 2].

3 High-Hand-Wins Draw Poker. Players have to choose one of the two alternatives K (*i.e.* keep his hand) and E (*i.e.* exchange his hand with a newly drawn hand). The payoff matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{K} & \text{E} \end{array} \\ \begin{array}{c} \text{K} \\ \text{E} \end{array} & \left[\begin{array}{cc} sgn(x-y) & sgn(x-u) \\ sgn(z-y) & sgn(z-u) \end{array} \right]. \end{array}$$

where $z(u)$ is the newly drawn hand for I (II). All r.v.s x, y, z and u are *i.i.d.* distributed as $U_{[0,1]}$. It is easily shown that the value of the game is zero and players have the common optimal strategy :

Choose K (E), if his hand $> (<) g$.

Consider the bilateral-move version described by :

Players' hand	First move	Second move	Payoff to I
I ; x	K		
	E		
II ; y		K	$sgn(x-y)$
		E	$sgn(x-u)$
		K	$sgn(z-y)$
		E	$sgn(z-u)$

It is proven, in Ref.[3 : Theorem 2] that the optimal play proceeds as follows :

I chooses K (E), if $x > (<) b_0$:

If I chooses $\left\{ \begin{array}{c} K \\ E \end{array} \right\}$, then II chooses K (E), if $y > (<) \left\{ \begin{array}{c} b_1 \\ 1/2 \end{array} \right\}$,

where $b_0 \approx 0.56736$ is a unique root in $[0,1]$ of the cubic equation $x^3 + x - 3/4 = 0$, and $b_1 = \frac{1}{2}(1 + b_0^2) \approx 0.66095$.

The value of the game is

$$-\left(b_1^2 - \frac{3}{4} b_0\right) \approx -0.01133.$$

Although player I, the first mover, stands unfavorable, he can give no information to II by choosing E, and he doesn't need to randomize his choice. His situation is improved, but the game value is still negative.

4 Noisy Duel. For a wide review of duels, see Ref.[2 : Chapter 5]. Two duelists I and II, starting at time $t = 0$ (at a distance 2 apart) walk toward each other at a constant unit speed with no opportunity to retreat. I (II) has $m(n)$ noisy bullets. Let $P_i(t)$ $(i = 1, 2)$ be the probability for player i of hitting his opponent, if he fires at time $t \in [0, 1]$. $P_i(t)$ is assumed to be continuous and increasing, satisfying $P_i(0) = 0$ and $P_i(1) = 1$. Payoff is $1(-1)$ for the sole survivor (his opponent), and zero otherwise.

It is proven in Ref.[1], that for each player, there exist a set $\{t_{mn}\}$ and a unique set $\{v_{mn}\}$ satisfying

$$\begin{aligned} v_{mn} &= P_1(t_{mn}) + \overline{P_1}(t_{mn}) v_{m-1,n} \\ &= -P_2(t_{mn}) + \overline{P_2}(t_{mn}) v_{m.n-1} \quad (v_{m0} = 1, v_{0n} = -1, \forall m, n) \end{aligned}$$

If player I (II) has $m(n)$ noisy bullets, then for each player, there exists an ε-optimal strategy such that he doesn't fire his first shot at time $t < t_{mn}$. The game has the value v_{mn}. (See Ref.[1]).

Consider the case where $P_1(t) = t$ and $P_2(t) = t^2$. Then we easily find that

(a) $t_{11} = \frac{1}{2}(\sqrt{5} - 1) \equiv g \approx 0.61803$, and $v_{11} = 2g - 1 \approx 0.2361$

(b) $t_{12} \approx 0.47726$ is a unique root in $[0,1]$ of the equation $x^2 + (1 + g)x - 1 = 0$, and $v_{12} = 2t_{12} - 1 \approx -0.0454$

(c) $t_{21} \approx 0.45589$ is a unique root in $[0,1]$ of the equation $x^2 + \bar{g}x - \bar{g} = 0$, and $v_{21} = 1 - 2t_{21}^2 \approx 0.5843$

(d) $t_{22} \approx 0.38167$ is a unique root in $[0,1]$ of the equation $(1 - t_{21}^2)x^2 + (1 - t_{12})x - (1 - t_{12} - t_{21}^2) = 0$, and $v_{22} = t_{22} + \overline{t_{22}} v_{12} \approx 0.3536$.

Note that the optimal play proceeds as follows : Player I fires at time $t_{22} \approx 0.382$, and if he misses, then II fires at time $t_{12} \approx 0.477$. If he misses, then I and II simultaneously fire at time $t_{11} \approx 0.618$. The game value is $v_{22} \approx 0.354$.

5 Two-Player Game of "Keep-or-Exchange". Consider the two players I and II (sometimes they are denoted by 1 and 2). I(II) observes the sequence of random variables $X_1, X_2, \cdots, X_n (Y_1, Y_2, \cdots, Y_n)$ one-bye-one sequentially. We assume that X_i's and Y_i' s are i.i.d., each with uniform distribution in $[0,1]$. I(II) chooses his (or her) decreasing sequence of decision levels

$$\begin{array}{rcl} 1 & \equiv & a_0^{(n)} > a_1^{(n)} > a_2^{(n)} > \cdots > a_{n-1}^{(n)} > a_n^{(n)} \equiv 0 \\ (1 & \equiv & b_0^{(n)} > b_1^{(n)} > b_2^{(n)} > \cdots > b_{n-1}^{(n)} > b_n^{(n)} \equiv 0) \end{array}$$

so that

$$\begin{array}{rcl} \text{I accepts (rejects) } X_i & = & x, \text{ if } x > (<) a_i^{(n)} \\ \text{II accepts (rejects) } Y_i & = & y, \text{ if } y > (<) b_i^{(n)} \end{array}$$

Define each palyer's score by the r.v. he chooses to accept it. The player with the higher score than his opponent is the *winner*. Each player aims to maximize the probability of his winning. The game is called "Keep-or-Exchange". Here, Keep is, in other words, "Accept" or "Stop". Exchange is "Reject" or "Continue".

When $n = 2$, the game is essentially identical to one considered in Section 3. So, the game has a unique eq.point $(a_1^{(2)}, b_1^{(2)}) = (g, g)$, and the eq.values are $\frac{1}{2}, \frac{1}{2}$.

When $n = 3$, the game has the common eq.strategy $a_i^{(3)} = b_i^{(3)}$, where, by omitting sup-script (3), $(1 >) b_1 > b_2 (> 0)$ is a unique root of the simultaneous equations

$$\begin{array}{l} b_1^2 + (1 + b_2)^{-1} b_1 - 1 = 0 \\ b_2^2 + b_2 - (b_1^{-1} - 1 + b_1) = 0. \end{array}$$

Computation gives the unique root is $b_1^0 \approx 0.743, b_2^0 \approx 0.657$.
The eq.values of the game are $\frac{1}{2}, \frac{1}{2}$ (see Ref.[5 ; Theorem 2]).

6 Conclusion. The set of two numbers $b_1 = \frac{1}{2}(1+g) \approx 0.809$, and $b_0 = \frac{1}{2} g \approx 0.309$ in Section 2 ; $b_1 \approx 0.661$ and $b_0 \approx 0.567$ in Section 3 ; and $t_{12} \approx 0.477$ and $t_{22} \approx 0.382$ in Section 4 ; and $b_1^0 \approx 0.734$ and $b_2^0 \approx 0.657$ in Section 5 are considered as "golden trisection numbers" (as they might be). We pass over an unrest that whether people feel beautiful those "trisection ratios", for example, $(1 - b_1) : (b_1 - b_0) : b_0 \approx 0.191 : 0.5 : 0.567$.

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