A NOTE ON AN OPERATOR TRANSFORM $S(T)$

S.M. PATEL, K. TANAHASHI[1] AND A. UCHIYAMA

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Abstract. In the present article, we introduce a new operator transform of a bounded linear operator on a complex Hilbert space, the definition of which is parallel to that of the Aluthge transform. Also we study the relationship between this new transform and several classes of non-hyponormal operators.

1. Introduction

Let $B(\mathcal{H})$ be the Banach algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. For $T \in B(\mathcal{H})$, we shall use the notations $\sigma(T), W(T), r(T)$ and $w(T)$ to denote the spectrum, the numerical range, the spectral radius, and the numerical radius of T. An operator $T \in B(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$, where $p > 0$; log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$; class $A(s,t)$ operator if $(|T^*|^t|T|^{2s}|T^*|^t)^{t/(s+t)} \geq |T^*|^{2t}$ where $s,t > 0$; convexoid if conv $\sigma(T)$ (convex hull of $\sigma(T)$) coincides with the closure of $W(T)$, and normaloid if $r(T) = \|T\|$. It is known that classes of p-hyponormal operators and log-hyponormal operators are subclasses of class $A(s,t)$ operators, and if T is a class $A(s,t)$ operator with $s \leq s', t \leq t'$, then T is a class $A(s',t')$ operator (see [6], [10], [14], [15], [18], [19]). Also a class $A(s,t)$ operator is normaloid([7]). In [1], Aluthge studied p-hyponormal operators by elegantly using the operator transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ of $T \in B(\mathcal{H})$, where $T = U|T|$ is the polar decomposition. Named after Aluthge, the transform $\tilde{T}$ is known as the Aluthge transform in the literature. A further extension of $\tilde{T}$ called the generalized Aluthge transform is defined as $T(s,t) = |T|^sU|T|^t$. Both the transforms have been proved to be powerful tools in introducing and exploring the properties of several classes of non-hyponormal operators ([2], [5], [6], [7], [12], [16], [17], [18]). By interchanging U with $|T|^{1/2}$ in the Aluthge transform, we define below a new transform.

Definition. Let $T \in B(\mathcal{H})$ with the polar decomposition $T = U|T|$. Then the transform $S(T)$ of T is defined as

$$S(T) = U|T|^{1/2}U.$$

In Section 2, we establish some basic properties of $S(T)$. Section 3 is devoted to obtaining some conditions on $S(T)$ implying the normality of T. In Section 4, we focus on conditions on $S(T)$ under which T is k-hyponormal or a selfadjoint partial isometry or a projection operator. Section 5 deals with the polar decomposition of $S(T)$.

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2. BASIC PROPERTIES

First we list some elementary properties of the transform $S(T)$.

Theorem 2.1. *For an operator $T \in B(\mathcal{H})$, the following assertions hold.*
(i) $\|S(T)\|^2 \leq \|T\|$.
(ii) $\ker S(T) = \ker U^2$.
(iii) $S(T^*) = |T|^{1/2}U^{*2}$.
(iv) $\ker S(T) \subset \ker S(T)^* \cap \ker S(T^*)$ *if* $\ker T \subset \ker T^*$.
(v) $\ker S(T)^* \cap \ker S(T^*) \subset \ker S(T)$ *if* $\ker T^* \subset \ker T$.
(vi) $\sigma(S(T^*)) = \sigma(S(T)^*)$.

Proof. Assertions (i) and (ii) are obvious. Assertion (iii) follows from the fact that $T^* = U^*|T^*|$ is the polar decomposition of T^*.

(iv) Suppose $S(T)x = 0$. Then $|T|^{1/2}Ux = 0$ implying $Ux \in \ker U \subset \ker U^*$. Hence $Ux = 0$ and so $U^*x = 0$ by the kernel condition. Hence $S(T)^*x = 0$ and $S(T^*)x = 0$. This proves (iv).

(v) If $S(T)^*x = 0$, then $|T|^{1/2}U^*x \subset \ker U^* = \ker T^* \subset \ker T = \ker U$. This gives $U|T|^{1/2}U^*x = 0$ or $|T|^{1/2}U^*x = 0$. Hence $U^*x = 0$ and $Ux = 0$ as $\ker U^* \subset \ker U$. From the hypothesis, we have $x \in \ker T = \ker U$. Hence $S(T)x = 0$. If $S(T^*)x = 0$, then $U^{*2}x = 0$. Hence $U^*x \in \ker U^* \subset \ker U$ by the kernel condition. Hence $U^*x = 0$ and $T^*x = 0$. Again $S(T)x = 0$. This proves (v).

(vi) Note that $\sigma(S(T)) \setminus \{0\} = \sigma(S(T^*)^*) \setminus \{0\}$. Also $S(T)$ is invertible if and only if U and $|T|$ are invertible if and only if $S(T^*)$ is invertible. Therefore $\sigma(S(T)) = \sigma(S(T^*)^*)$ or $\sigma(S(T)^*) = \sigma(S(T^*))$.

□

Theorem 2.2. *Let T be a p-hyponormal operator with $0 < p \leq 1$.*
(i) *If $0 < p \leq 1/2$, then $S(T)$ is $2p$-hyponormal.*
(ii) *If $1/2 < p \leq 1$, then $S(T)$ is hyponormal.*

Proof.
(i) Note that $S(T)^*S(T) = U^*|T|U$ and

$$S(T)S(T)^* = U|T|^{1/2}UU^*|T|^{1/2}U^* \leq U|T|U^*.$$

Since T is p-hyponormal and $U|T|^qU^* = |T^*|^q$ for $0 < q$,

$$\begin{aligned}(S(T)^*S(T))^{2p} &= (U^*|T|U)^{2p} \\ &\geq U^*|T|^{2p}U \quad \text{(by Hensen's inequality[8])} \\ &\geq |T|^{2p} \geq U|T|^{2p}U^* = (U|T|U^*)^{2p} \\ &\geq (S(T)S(T)^*)^{2p} \text{ (by Lower-Heinz's inequality[9], [11]).}\end{aligned}$$

(ii) If $1/2 < p \leq 1$, then T is semi-hyponormal. Hence, by (i), it follows that $S(T)$ is hyponormal. □

Remark. The proof of Theorem 2.2 indicates that for a p-hyponormal T with $0 < p \leq 1/2$, the following inequalities hold:

$$|S(T)|^{4p} \geq |T|^{2p} \geq |S(T)^*|^{4p}.$$

A fairly natural question presents itself: Does this inequality implies T is p-hyponormal? In case T satisfies the kernel condition $\ker T^* \subset \ker T$ then the question has an affirmative answer. Because $\ker T^* \subset \ker T$ implies $U^*U \leq UU^*$, hence

$$\begin{aligned}|T|^{2p} \geq |S(T)^*|^{4p} &= (U|T|^{1/2}UU^*|T|^{1/2}U^*)^{2p}\\ &= (U|T|U^*)^{2p} = |T^*|^{2p}.\end{aligned}$$

For $p = 1/2$, it is not difficult to verify that operators satisfying above inequality are w-hyponormal operators. However, the question is still remains unanswered.

Theorem 2.3. *If T is a log-hyponormal operator, then so is $S(T)$.*

Proof. Since T is invertible, $|S(T)^*|^2 = U|T|U^*$ and $|S(T)|^2 = U^*|T|U$. Therefore

$$\begin{aligned}2\log|S(T)| &= \log(U^*|T|U) = U^*(\log|T|)U\\ &\geq U^*(\log|T^*|)U = \log|T| \geq \log|T^*|\\ &= \log(U|T|U^*) = U(\log|T|)U^* = 2\log|S(T)^*|.\end{aligned}$$

This proves the result. □

Next, we relate the approximate point spectra of an operator T and $S(T)$ when T is either p-hyponormal or log-hyponormal.we first prove a couple of theorems that shall be needed.

Theorem 2.4. *Let $T = U|T|$ be p-hyponormal with $0 < p \leq 1$. Let $X = U^2|T|^{1/2}$. Then $X = U^2|T|^{1/2}$ is the polar decomposition of X and the following assertions hold.*
(i) *If $0 < p \leq 1/2$, then X is $2p$-hyponormal.*
(ii) *If $1/2 < p \leq 1$, then X is hyponormal.*

Proof. Since T is p-hyponormal, $\ker T \subset \ker T^*$. Hence $U^*U^2 = U$ and $U^2U^{*2}U^2 = U^2U^*U = U^2$. Also, $\ker U^2 = \ker U = \ker|T|^{1/2}$. This implies $X = U^2|T|^{1/2}$ is the polar decomposition of X.

If $0 < p < 1/2$, then

$$\begin{aligned}(X^*X)^{2p} &= |T|^{2p} \geq |T^*|^{2p}\\ &= U|T|^{2p}U^* \geq U^2|T|^{2p}U^{*2}.\end{aligned}$$

Since

$$(U^2|T|^{2p}U^{*2})(U^2|T|^{2p}U^{*2}) = U^2|T|^{4p}U^{*2},$$

we have $f(U^2|T|^{2p}U^{*2}) = U^2f(|T|)U^{*2}$ for any polynomial $f(x)$ with $f(0) = 0$. Hence

$$(X^*X)^{2p} \geq U^2|T|^{2p}U^{*2} = (U^2|T|U^{*2})^{2p} = (XX^*)^{2p}.$$

If $1/2 \leq p \leq 1$, then T is semi-hyponormal. Hence, by (i), it follows that X is hyponormal. □

Theorem 2.5. *Let $T = U|T|$ be log-hyponormal. Then $X = U^2|T|^{1/2}$ is the polar decomposition of X and X is log-hyponormal.*

Proof. That X is invertible and $X = U^2|T|^{1/2}$ is the polar decomposition should be fairly apparent. To show that X is log-hyponormal, observe first that $|X| = |T|^{1/2}$ and $|X^*| = U|T^*|^{1/2}U^*$. Since T is log-hyponormal, we find

$$\begin{aligned}\log|X| &= \frac{1}{2}\log|T| \geq \frac{1}{2}\log|T^*| = \frac{1}{2}U(\log|T|)U^*\\ &\geq U(\log|T^*|)U^* = \log(U|T^*|^{1/2}U^*) = \log|X^*|.\end{aligned}$$

Hence X is log-hyponormal. □

The approximate point spectrum of T is definded by

$$\sigma_a(T) = \{z \in \mathbb{C} \mid \exists \text{unit vectors } x_n, (T-z)x_n \to 0\}.$$

It is known ([3]) that if T is p-hyponormal, then

$$\sigma_a(T) = \sigma_{na}(T) = \{z \in \mathbb{C} \mid \exists \text{unit vectors } x_n, (T-z)x_n, (T-z)^* x_n \to 0\}.$$

Theorem 2.6. *Let $T = U|T|$ be either p-hyponormal or log-hyponormal, then*

$$\sigma_a(U^2|T|^{1/2}) = \{r^{1/2}e^{2i\theta} | re^{i\theta} \in \sigma_a(T)\} = \sigma_a(S(T)).$$

Proof. Let T be p-hyponormal and $0 \neq re^{i\theta} \in \sigma_a(T)$. Then there exist unit vectors x_n such that

$$(|T| - r)x_n \to 0, (U - e^{i\theta})x_n \to 0.$$

Hence $(U^2|T|^{1/2} - r^{1/2}e^{2i\theta})x_n \to 0$. If $0 \in \sigma_a(T)$, then there exist unit vectors x_n such that $|T|x_n \to 0$. Hence $U^2|T|^{1/2}x_n \to 0$.

Conversely, let $0 \neq \rho e^{2i\phi} \in \sigma_a(U^2|T|^{1/2})$. Since $U^2|T|^{1/2}$ is the polar decomposition of $2p$-hyponormal operator by Theorem 2.4, there exist unit vectors x_n such that

$$(U^2|T|^{1/2} - \rho e^{2i\phi})x_n \to 0, (U^2|T|^{1/2} - \rho e^{2i\phi})^* x_n \to 0.$$

Hence

$$(|T|^{1/2} - \rho)x_n \to 0, (U^2|T|^{1/2}U^{*2} - \rho)x_n \to 0$$

and

$$(U^2 - e^{2i\phi})x_n = (U + e^{i\phi})(U - e^{i\phi})x_n \to 0.$$

If there exists a subsequence x_{n_k} such that $(U - e^{i\phi})x_{n_k} \to 0$, then $(T - \rho^2 e^{i\phi})x_{n_k} \to 0$. Hence $\rho^2 e^{i\phi} \in \sigma_a(T)$.

Suppose there is no such subsequence. For a sequence u_n of unit vectors and $|z| = 1$, it is known that $(U - zI)u_n \to 0$ if and only if $(U - zI)^* u_n \to 0$. Hence we may assume that $\|(U - e^{i\phi})^* x_n\| \geq \varepsilon$ for some $\varepsilon > 0$. We show $(|T^*|^{1/2} - \rho)x_n \to 0$. Since $(|T|^{1/2} - \rho)x_n \to 0, (U^2|T|^{1/2}U^{*2} - \rho)x_n \to 0$, we have

$$(|T|^p - \rho^{2p})x_n \to 0, (U^2|T|^pU^{*2} - \rho^{2p})x_n \to 0.$$

T is p-hyponormal, hence

$$|T|^{2p} \geq U|T|^{2p}U^* \geq U^2|T|^{2p}U^{2*}$$

and

$$|T|^p \geq U|T|^pU^* \geq U^2|T|^pU^{2*}.$$

Then

$$\|U|T|^pU^*x_n\| \to \rho^{2p}, \langle U|T|^pU^*x_n, x_n\rangle \to \rho^{2p}.$$

Therefore

$$\begin{aligned} &\|(|T^*|^p - \rho^{2p})x_n\|^2 \\ &\quad = \|U|T|^pU^*x_n\|^2 - 2\rho^{2p}\langle U|T|^pU^*x_n, x_n\rangle + \rho^{4p} \to 0. \end{aligned}$$

Hence $(|T^*|^p - \rho^{2p})x_n \to 0$ and $(|T^*|^{1/2} - \rho)x_n \to 0$.

Set

$$y_n = (U - e^{i\phi})^* x_n / \|(U - e^{-i\phi})^* x_n\|.$$

Since $(U^2 - e^{2i\phi})x_n \to 0$, we have $(U + e^{i\phi})^* y_n \to 0$ and $(U + e^{i\phi})y_n \to 0$. Now

$$(|T^*|^{1/2} - \rho)x_n = (U|T|^{1/2}U^* - \rho)x_n \to 0$$

implies $(|T|^{1/2} - \rho)U^*x_n \to 0$. Consequently,

$$\begin{aligned}&(|T|^{1/2} - \rho)(U - e^{i\phi})^* x_n\\ &\quad = (|T|^{1/2} - \rho)U^*x_n - e^{-i\phi}(|T|^{1/2} - \rho)x_n \to 0\end{aligned}$$

and $(|T|^{1/2} - \rho)y_n \to 0$. Thus $(T+\rho^2 e^{i\phi})y_n = (T-\rho^2 e^{i(\phi+\pi)})y_n \to 0$ and $\rho^2 e^{i(\phi+\pi)} \in \sigma_a(T)$.

If $0 \in \sigma_a(U^2|T|^{1/2})$, then there exist unit vectors x_n such that $|T|^{1/2}x_n \to 0$ or $Tx_n \to 0$.

Now assume that T is log-hyponormal. Then the similar reasoning will lead to the desired conclusion. □

3. NORMALITY

In [12], the first author proved that a p-hyponormal operator is normal if its Aluthge transform is normal. More generally the result is found to be true for w-hyponormal operators by [2], those are class $A(1/2, 1/2)$ operators by [6]. As a further extension, it has been shown that a class $A(s,t)$ operator is normal provided its generalized Aluthge transform $T(s,t)$ is normal. That this result holds if we assume the normality of $S(T)$ instead of the normality of $T(s,t)$ will follow as a corollary to the following theorem.

Theorem 3.1. *Let $T(s,s) = |T|^s U|T|^s$. If $S(T)$ is normal and $\ker S(T) = \ker T$, then $T(s,s)$ is normal.*

Proof. First we show that

$$\ker T^* \subset \ker T. \tag{3.1}$$

Since $S(T)$ is normal, we have

$$U^*|T|U = U|T|^{1/2}UU^*|T|^{1/2}U^*. \tag{3.2}$$

Suppose $T^*x = 0$. Then $U^*x = 0$. And therefore (3.2) implies $U^*|T|Ux = (|T|^{1/2}U)^*|T|^{1/2}Ux = 0$ or $|T|^{1/2}Ux = 0$. This in turn gives $S(T)x = 0$ and so by the kernel condition, $Tx = 0$, which establishes (3.1). Note that by (3.1), $\ker UU^* \subset \ker U^*U$. Hence $UU^*|T|^{1/2} = |T|^{1/2}$. Then (3.2) reduces to

$$U^*|T|U = U|T|U^*. \tag{3.3}$$

If $Tx = 0$, then $Ux = 0$ and so $U^*x = 0$ or $T^*x = 0$ by (3.3). Thus by (3.1), $\ker T^* = \ker U^* = \ker U = \ker T$. Clearly U is normal. Then (3.3) implies $U^*|T|^sU = U|T|^sU^*$. Now the normality of $T(s,s)$ is immediate. □

Some consequences of Theorem 3.1 are of particular interest and list them below as corollaries.

Corollary 3.2. *Let T be a class $A(s,t)$ operator. If $S(T)$ is normal, then T is normal.*

Proof. We may assume $t \le s$. Then T is of class $A(s,s)$. First, we show that $T(s,s)$ is normal. This will follow from Theorem 3.1 once we show that $\ker S(T) = \ker T$. Suppose $S(T)x = 0$. Then $|T|^{1/2}Ux = 0$. Choose $z \in [\mathrm{ran}|T|^s]$ and $y \in \ker |T|^s$ such that $x = z + y$, where $[\mathrm{ran}|T|^s]$ denotes the closure of $\mathrm{ran}|T|^s$. Then

$$|T|^sUz = 0. \tag{3.4}$$

Select a sequence x_n of vectors from $\mathcal{H}$ such that $|T|^s x_n \to z$. By (3.4), $T(s,s)x_n \to 0$. Since T is of class $A(s,s)$, $|T|^s x_n \to 0$ and hence $z = 0$. Thus $x = y \in \ker |T|^s = \ker T$. This shows that $\ker S(T) \subset \ker T$. Since the reverse inclusion is obvious, we have $\ker S(T) = \ker T$ and hence $T(s,s)$ is normal. By [13, Corollary 2.2], we conclude that T is normal. □

Corollary 3.3. *If T is a class $A(s,t)$ operator and if $S(T)$ is a positive operator, then T is selfadjoint.*

Proof. By Corollary 3.2, T is normal. Therefore if $\lambda \in \sigma(T)$, then $|\lambda|^{1/2}e^{2i\theta} \in \sigma(S(T)) \subset \{x \in \mathbb{R} : x \geq 0\}$. This shows that $\sigma(T) \subset \mathbb{R}$. Hence T is selfadjoint. □

Example 1. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Then T is normal and $S(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Hence if we relax the condition " $S(T)$ is a positive operator" by assuming "$S(T)$ is a selfadjoint operator ", then the result is invalid.

Corollary 3.4.

(i) *If $S(T^*)$ is normal and $\ker S(T) = \ker T$, then $T(s,s)$ is normal.*

(ii) *If T is a class $A(s,t)$ operator for which $S(T^*)$ is normal, then T is normal.*

Proof.

(i) First we show that $\ker S(T^*) = \ker T^*$. Suppose $T^*x = 0$. Then it is obvious that $S(T^*)x = 0$. Let $S(T^*)x = 0$. By the normality of $S(T^*)$, we have $0 = S(T^*)^*x = U^2|T|^{1/2}x$. This means that $U|T|^{1/2}x \in \ker T = \ker S(T)$. Hence $S(T)|T|^{1/2}x = U|T|^{1/2}U|T|^{1/2}x = 0$ Then $|T|^{1/2}x \in \ker S(T) = \ker T = \ker |T|$, and hence we obtain $|T|^{3/2}x = 0$ or $Tx = 0$. Thus we have

$$\ker T^* \subset \ker S(T^*) \subset \ker T. \tag{3.5}$$

On the other hand, if $x \in \ker T$, then $S(T^*)^*U|T^*|^{1/2}Ux = 0$. Since $S(T^*)$ is normal,

$$0 = S(T^*)x = |T|^{1/2}U^{*2}x$$

or $U^{*2}x = 0$. From (3.5), it will follow that $U^*x \in \ker U^* \subset \ker T = \ker U$ or $U^*x = 0$. This proves $\ker T \subset \ker U^* = \ker T^*$. Combining this inclusion with (3.5) gives $\ker S(T^*) = \ker T^*$. By Theorem 3.1, $T^*(s,s)$ is normal. Since $T^*(s,s) = UT(s,s)^*U^*$, the normality of $T(s,s)$ is immediate.

(ii) The assertion follows from (i) as T being a class $A(s,t)$ operator, $\ker S(T) = \ker T$ (refer the proof of Corollary 3.2).

□

Theorem 3.5. *If $\ker T^* = \ker T$, then the following assertions hold.*

(i) $W(S(T^*)) = W(S(T)^*)$.

(ii) $\|S(T)\|^2 = \|T\| = \|S(T^*)\|^2$.

Proof.

(i) The kernel condition implies that $|T|^{1/2}UU^* = |T|^{1/2}$. Let x be a unit vector. Then

$$\begin{aligned} \langle S(T^*)x, x\rangle &= \langle |T|^{1/2}U^{*2}x, x\rangle = \langle x, U^2|T|^{1/2}UU^*x\rangle \\ &= \langle U^*x, U|T|^{1/2}UU^*x\rangle \\ &= \langle U^*x/\|U^*x\|, S(T)(U^*x)/\|U^*x\|\rangle \|U^*x\|^2. \end{aligned}$$

Thus

$$\langle S(T^*)x, x\rangle = \langle S(T)^*U^*x/\|U^*x\|, U^*x/\|U^*x\|\rangle \|U^*x\|^2. \tag{3.6}$$

If $0 \in W(S(T)^*)$, then the right hand side (3.6) belongs to $W(S(T)^*)$ as $W(S(T)^*)$ is convex and $\|U^*x\| \leq 1$. Suppose $0 \in W(S(T)^*)$. Then $S(T)$ and hence T is injective. Therefore the kernel condition shows that T^* is injective. Thus $UU^* = I$. Again the right hand side is in $W(S(T)^*)$, proving $W(S(T^*)) \subset W(S(T)^*)$. Replacing T by T^*, we obtain the reverse inclusion.

(ii) Note

$$\|S(T)\|^2 = \|U|T|^{1/2}U\|^2 = \||T|^{1/2}U\|^2 = \|U^*|T|^{1/2}\|^2$$
$$= \||T|^{1/2}UU^*|T|^{1/2}\| = \||T|\| = \|T\|.$$

Then, by the kernel condition,

$$\|S(T^*)\| = \||T|^{1/2}U^{*2}\| = \|UU^*|T|^{1/2}U^{*2}\|$$
$$= \|US(T)^*U^*\| \leq \|S(T)\|.$$

Replacing T by T^*, we get the reverse inequality. □

Example 2. Let T be the unilateral weighted shift operator on $\mathcal{H} = l^2$ with weights $\{25, 1, 1, 1, 1, \cdots\}$. Then for a vector $(x_0, x_1, x_2, \cdots) \in \mathcal{H}$, a computation shows that

$$S(T)(x_0, x_1, x_2, \cdots) = (0, 0, x_0, x_1, x_2, \cdots)$$

and

$$S(T^*)(x_0, x_1, x_2, \cdots) = (5x_2, x_3, x_4, \cdots).$$

Note that $\ker T = \{0\} \subset \ker T^*$. Let $x = (1/\sqrt{2}, 0, 1/\sqrt{2}, 0, 0, \cdots)$. Then $\langle S(T^*)x, x\rangle = 5/2$, hence we find $5/2 \in W(S(T^*))$. However, as $w(S(T)^*) \leq \|S(T)\| = 1$, it follows that $5/2 \notin W(S(T)^*)$. Hence Theorem 3.5 does not holds if the underlying kernel condition is replaced by the weaker conditions like " $\ker T^* \subset \ker T$" and " $\ker T \subset \ker T^*$".

Corollary 3.6. *Suppose* $\ker T^* = \ker T$. *Then*
(i) $S(T)$ *is convexoid if and only if* $S(T^*)$ *is convexoid.*
(ii) $S(T)$ *is normaloid if and only if* $S(T^*)$ *is normaloid.*

Proof. Note that $\sigma(S(T)) \setminus \{0\} = \sigma(S(T^*)^*) \setminus \{0\}$. Also $S(T)$ is invertible if and only if U and $|T|$ are invertible if and only if $S(T^*)$ is invertible. Therefore $\sigma(S(T)) = \sigma(S(T^*)^*)$ or $\sigma(S(T)^*) = \sigma(S(T^*))$. Now the result follows from Theorem 3.5. □

Theorem 3.7. *If* $S(T)^2 = T$ *and* $\ker T \subset \ker T^*$, *then* T *is normal.*

Proof. The condition $S(T)^2 = T$ means $U|T|^{1/2}U^2|T|^{1/2}U = U|T|$. Hence $|T|^{1/2}U^2|T|^{1/2}U = |T|$ and $U^3|T|^{1/2}U = U|T|^{1/2}$ as $\ker U = \ker |T|^{1/2}$. Note that $U^*U^2 = U$ as $\ker T \subset \ker T^*$. Therefore

$$|T|^{1/2} = U^*U^3|T|^{1/2}U = U^2|T|^{1/2}U \tag{3.7}$$

or $U^*|T|^{1/2}U^{*2} = |T|^{1/2}$ implying $\ker U^* \subset \ker U$. This together with the kernel condition gives $\ker U = \ker U^*$ and $U^*U = UU^*$. Hence

$$|T^*|^{1/2} = U|T|^{1/2}U^* = U^*U^2|T|^{1/2}U^* = U^*|T|^{1/2}U^{*2} = |T|^{1/2}.$$

Thus T is normal. □

Corollary 3.8. *If* $S(T^*)^2 = T^*$ *and* $\ker T \subset \ker T^*$, *then* T *is normal.*

Proof. The condition $S(T^*)^2 = T^*$ means $|T|^{1/2}U^{*2}|T|^{1/2}U^{*2} = |T|U^*$. Since $\ker |T|^{1/2} = \ker U$, we find $UU^{*2}|T|^{1/2}U^{*2} = U|T|^{1/2}U^*$. Hence $U^{*2}|T|^{1/2}U^{*2} = U^*UU^{*2}|T|^{1/2}U^{*2} = |T|^{1/2}U^*$ or $U|T|^{1/2} = U^2|T|^{1/2}U^2$. Since $\ker U \subset \ker U^*$, one can see that $U^*U^2 = U$ and so the last equation reduces to

$$|T|^{1/2} = U^*U|T|^{1/2} = U^*U^2|T|^{1/2}U^2 = U|T|^{1/2}U^2. \tag{3.8}$$

Now multiplying (3.8) on the left by UU^*, we find $UU^*|T|^{1/2} = U|T|^{1/2}U^2 = |T|^{1/2}$ or $|T|^{1/2} = |T|^{1/2}UU^*$. Especially, $\ker T^* \subset \ker T$ and hence by the hypothesis, $\ker T = \ker T^*$. Clearly, then U is normal. Now (3.8) along with the normality of U yields

$U^*|T|^{1/2}U^* = |T|^{1/2}U^2U^* = |T|^{1/2}U$. Hence $U^*|T|^{1/2} = U|T|^{1/2}U$ and $U^*|T|^{1/2}U = U|T|^{1/2}U^2 = |T|^{1/2}$. This implies $U|T|^{1/2} = UU^*|T|^{1/2}U = |T|^{1/2}U$. But then $S(T^*) = |T|^{1/2}U^{*2} = U^*|T|^{1/2}U^* = S(T)^*$. Then, by our hypothesis, it follows that $S(T)^2 = T$. Now the result follows from Theorem 3.7. □

Theorem 3.9. *$T \in B(\mathcal{H})$ is normal if any one of the following conditions holds.*

(i) $|S(T)^*|^2 = |T|$.
(ii) $|S(T^*)|^2 = |T|$, *where T is a class $A(s,t)$ operator.*
(iii) $|S(T)|^2 = |T|$, *where T is a class $A(s,t)$ operator.*

Proof.
(i) Note that $|S(T)^*|^2 = |T|$ implies

$$|T| = U|T|^{1/2}UU^*|T|^{1/2}U^* \le U|T|U^*.$$

Then clearly $\ker U^* \subset \ker U$ or $U^*U \le UU^*$. This in turn shows that $|T| = |S(T)^*|^2 = U|T|U^* = |T^*|$. Hence T is normal.

(ii) The condition $|S(T^*)|^2 = |T|$ implies $U^2|T|U^{*2} = |T|$. Hence $\ker T^* \subset \ker T$. On the other hand if $Tx = 0$, then it follows from the equation $U^2|T|U^{*2} = |T|$ that $|T|^{1/2}U^{*2}x = 0$ or $U^{*2}x = 0$. Then $U^*x \in \ker T^* \subset \ker T$. Hence $TU^*x = 0$ or $U|T|U^*x = 0$, which is the same as $T^*x = 0$. Thus $\ker T = \ker T^*$ or U is normal. Therefore the equation $U^2|T|U^{*2} = |T|$ implies $U|T|U^* = U^*U^2|T|U^{*2}U = U^*|T|U$. Now it is easy to show that $S(T)$ is normal. By Corollary 3.2, we conclude that T is normal.

(iii) Notice that the underlying condition is equivalent to $U|T|U^* = U^*|T|U$. Therefore if $Ux = 0$, then $U|T|U^*x = 0$ or $T^*x = 0$, giving $\ker U \subset \ker U^*$. On the other hand if $U^*x = 0$, then $U^*|T|Ux = 0$ implying $|T|^{1/2}Ux = 0$ or equivalently, $U^2x = 0$. Since $\ker U \subset \ker U^*$, we find $U^*Ux = 0$ or $Ux = 0$. Therefore $\ker U = \ker U^*$, which shows that U is normal. Hence

$$S(T)^*S(T) = U^*|T|U = U|T|U^* = S(T)S(T)^*.$$

Then $S(T)$ is normal and T is normal by Corollary 3.2. □

4. PARTIAL ISOMETRY, PROJECTION

Theorem 4.1. *If T is normaloid, then $S(T)$ is normaloid and $\|T\| = \|S(T)\|^2$.*

Proof. First we observe that $\|S(T)\|^2 \le \||T|\| = \|T\|$ for any operator T. Since T is normaloid, $\|T\| = |z|$ for some $z \in \sigma(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T-z)x_n \to 0$ and $(T-z)^*x_n \to 0$. If $z = |z|e^{i\theta}$, then $(|T|^{1/2} - |z|^{1/2})x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$. Consequently, $(S(T) - |z|^{1/2}e^{2i\theta})x_n \to 0$ and therefore $|z|^{1/2} \le r(S(T)) \le \|S(T)\|$. Hence

$$r(T) = \|T\| \le r(S(T))^2 \le \|S(T)\|^2 \le \|T\|.$$

Thus

$$r(T) = \|T\| = r(S(T))^2 = \|S(T)\|^2.$$

□

Example 3. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Clearly $T^2 = I$. If $T = U|T|$, then $U|T|U = |T|^{-1}$. Define $A = U|T|^2$. Then $S(A) = U|A|^{1/2}U = U|T|U = |T|^{-1}$. This shows that $S(A)$ is a positive invertible operator. Clearly $\|S(A)\|^2 = \|A\|$. We assert A is not normaloid. Suppose to the contrary that A is normaloid. Since a normaloid operator on a two-dimensional space is normal, it will follow that U commutes with $|T|^2$ and hence with $|T|$. But then T will be normal, which is not true. Hence the converse of Therorem 4.1 is not true.

Definition. Let $T \in B(\mathcal{H})$ with the polar decomposition $T = U|T|$. For each positive integer n, the n-th Aluthge transform $\tilde{T}(n)$ of T is defined as the Aluthge transform of $\tilde{T}(n-1)$ and $\tilde{T}(1) = \tilde{T} = |T|^{1/2}U|T|^{1/2}$. It is known that $r(T) = \lim \|\tilde{T}(n)\|$ ([17]).

Theorem 4.2. *Let T be a p-hyponormal operator. If $S(T)$ is a partial isometry, then the following assertions hold.*

(i) *T^n is a k-hyponormal operator for each positive integer n and each $k > 0$.*

(ii) *The n-th Aluthge transform $\tilde{T}(n)$ has the polar decomposition given by $\tilde{T}(n) = U|T|^{1/r(n)}$ where $r(n) = 2^n$ and $\tilde{T}(n)$ converges strongly to U.*

Proof.

(i) We may assume that $s \geq t$. Now T being a class $A(s,t)$ operator, it must be normaloid by [7]. By Theorem 4.1, $\|T\| = \|S(T)\|^2$. Since $S(T)$ is a partial isometry, it is a contraction. Therefore

$$\|T\| \leq 1. \tag{4.1}$$

Next we show that $\ker U = \ker |T|^{1/2}U$. Suppose $|T|^{1/2}Ux = 0$. Then $Ux \in \ker |T|^{1/2} = \ker T$. Since T is p-hyponormal, $\ker T \subset \ker T^*$. Hence $Ux \in kerT^* = \ker U^*$. Then $U^*Ux = 0$ or $Ux = 0$. Hence $\ker |T|^{1/2}U = \ker |T|$. Since $S(T)$ is a partial isometry, $S(T)^*S(T) = (|T|^{1/2}U)^*(|T|^{1/2}U)$ is a projection or $|T|^{1/2}U$ is a partial isometry. This in combination with the relation $\ker |T|^{1/2}U = \ker |T| = \ker U$ implies $|T|^{1/2}U$ and U are isometries on $[\mathrm{ran}|T|]$. Therefore we have

$$\|x\| = \||T|^{1/2}Ux\| = \|Ux\| \tag{4.2}$$

for $x \in [\mathrm{ran}|T|]$, and then $|T|^{1/2}Ux = Ux$ as $0 \leq |T|^{1/2} \leq 1$(see [12]). Hence

$$|T|^{1/2}U = U \tag{4.3}$$

and

$$|T|UU^* = |T|^{1/2}UU^* = UU^*.$$

Then $|T|^{2m}UU^* = UU^*$ for every positive integer m. Hence the inequality $UU^* \leq I$ implies

$$|T|^{2m} \geq |T|^m UU^* |T|^m = UU^*. \tag{4.4}$$

An application of (4.3) will show that

$$T^n = U(|T|U)^{n-1}|T| = U^n|T|.$$

Since by (4.4), $\ker T \subset \ker T^*$. Then it follows that U is quasinormal and so, in particular, $U^{*n}U^n = U^*U$. Clearly this shows that U^n is a partial isometry with $\ker U^n = \ker U = \ker |T|$. Therefore $|T^n| = |T|$ and $T^n = U^n|T|$ is the polar decomposition of T^n. Now by (4.4), we have

$$|T^n|^{2m} = |T|^{2m} \geq UU^* \geq U^nU^{*n}$$

and hence (4.1) will imply $|T^n|^{2m} \geq U^n|T|^{2m}U^{*n} = |T^{*n}|^{2m}$ or T^n is m-hyponormal for every positive integers m and n. Invoking the Lowner-Heinz Inequality, we conclude that T^n is k-hyponormal for every positive integer n and any positive real number k.

(ii) First, we note that (4.1) and (4.2) imply

$$\|x\| = \|Ux\| = \||T|^{1/2}Ux\| \leq \||T|^{1/4}Ux\| \leq$$
$$\cdots \leq \||T|^{1/r(n)}Ux\| \leq \||T|^{1/r(n+1)}Ux\| \leq \|Ux\|,$$

and hence $|T|^{1/r(n)}Ux = Ux$ for all $x \in [\text{ran}|T|^{1/r(n)}] = [\text{ran}|T|]$. Euivalently, we have $|T|^{1/r(n)}U|T|^{1/r(n)} = U|T|^{1/r(n)}$ for each positive integer n. Clearly $U|T|^{1/r(n)}$ is the polar decomposition of $|T|^{1/r(n)}U|T|^{1/r(n)}$. As a consequence of this, we find $\tilde{T}(1) = \tilde{T} = U|T|^{1/2}$ and hence $\tilde{T}(2) = |T|^{1/4}U|T|^{1/4} = U|T|^{1/4}$. An induction argument shows that $\tilde{T}(n) = |T|^{1/r(n)}U|T|^{1/r(n)} = U|T|^{1/r(n)}$. Since $|T|^{1/r(n)} \to U^*U$ strongly, it follows that $\tilde{T}(n) \to U^*UUU^*U = U$ strongly. □

Example 4. Let T to be a unilateral weighted shift with weights $\{1/2, 1, 1, 1, \cdots\}$. Then one can check that T is a non-quasinormal hyponormal operator for which $S(T)$ is an isometry. Hence if we put a stronger condition on $S(T)$ by assuming it to be an isometry, we may not get a stronger conclusion like T is quasinormal.

T is said to be paranormal if

$$\|Tx\|^2 \leq \|T^2x\|\|x\|$$

for all $x \in \mathcal{H}$. It is known that $A(1,1)$ operators are paranormal by [6]. In the next theorem, we extend the above result for paranormal operators with the closed range.

Theorem 4.3. *Let T be a paranormal operator with closed range. If $S(T)$ is a partial isometry, then the following assertions hold.*

(i) *T^n is a k-hyponormal operator for each positive integer n and each $k > 0$.*

(ii) *The n-th Aluthge transform $\tilde{T}(n)$ has the polar decomposition given by $\tilde{T}(n) = U|T|^{1/r(n)}$ where $r(n) = 2^n$ and $\tilde{T}(n)$ converges strongly to U.*

Proof. Since a paranormal operator is normaloid, as argued in Theorem 4.2, we get

$$\|T\| \leq 1. \tag{4.5}$$

Next, we show that

$$\ker |T|^{1/2}U = \ker |T|. \tag{4.6}$$

Suppose $|T|^{1/2}Ux = 0$. Then $U|T|Ux = 0$. Since ran T and therefore ran $|T|$ is closed, $x = y + z$ with some $y \in \ker |T|$ and $z \in$ ran $|T|$. Let $z = |T|u$ for some vector $u \in \mathcal{H}$. Then $0 = U|T|Ux = U|T|U|T|u$ or $T^2u = 0$. The paranormality of T implies $Tu = 0$ or $0 = |T|u = z$. This leads to $x = y \in \ker |T|$ proving (4.6).

Hence we have $|T|^{1/2}U = U$ or $U^*|T|^{1/2} = U^*$ as shown in the proof of Theorem 4.2. In paticular, $\ker T \subset \ker T^*$ as $U^* = U^*|T|$. Now using the same line of argument used in Theorem 4.2, we arrive at the desired conclusion. □

Theorem 4.4. *If $S(T)$ is an idempotent operator and $\ker T \subset \ker T^*$, then T is a selfadjoint partial isometry.*

Proof. The hypothesis $S(T)^2 = S(T)$ means $U|T|^{1/2}U^2|T|^{1/2}U = U|T|^{1/2}U$ which gives $|T|^{1/2}U^2|T|^{1/2}U = |T|^{1/2}U$ and hence $U^3|T|^{1/2}U = U^2$. Then applying the kernel condition, we obtain $U|T|^{1/2}U = U^*U$ or $U^*|T|^{1/2}U^* = U^*U$. Now it is obvious that $\ker U^* \subset \ker U$. This along with our hypothesis implies $\ker U = \ker U^*$. In particular, U is normal. Since $U|T|^{1/2}U = U^*U$, the normality of U gives $|T|^{1/2} = U^*U|T|^{1/2}UU^* = U^{*2}UU^* = U^{*2}$ or $|T|^{1/2} = U^2$. Therefore $U^*U = U|T|^{1/2}U = U^4$ and so $|T| = U^*U$. Hence $T = U$. In order to complete the proof, it is enough to show that $U^* = U$. Now $U^2 = |T|^{1/2}$ combined with $|T| = U^*U$ imply $U^2 = U^*U$ and hence $U^* = (U^*U)U^* = U^2U^* = UU^*U = U$ as U is normal. This finishes the proof. □

Example 5. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $S(T) = I$. Hence conditions in Theorem 4.4 do not guarantee that T is a projection.

Corollary 4.5. *Suppose $S(T)$ is a projection. If $\ker S(T) = \ker T$, then T is a selfadjoint partial isometry.*

Proof. First we observe that the condition $\ker S(T) = \ker T$ implies the condition $\ker U = \ker U^2$. Since $S(T)$ is normal, we have

$$U^*|T|U = U|T|^{1/2}UU^*|T|^{1/2}U^* \leq U|T|U^*. \tag{4.7}$$

Then $U^*x = 0$ implies $U^*|T|U = (|T|^{1/2}U)^*|T|^{1/2}Ux = 0$ which is the same as $U^2x = 0$. Hence $Ux = 0$ showing $\ker U^* \subset \ker U$. Obviously then $U^*U \leq UU^*$. Then from (4.7), we derive

$$U|T|U^* \leq U^*|T|U. \tag{4.8}$$

From (4.7) and (4.8), we have $U^*|T|U = U|T|U^*$. In particular, $\ker T = \ker U \subset \ker U^* = \ker T^*$. Consequently, the corollary follows from the Theorem 4.4. □

Corollary 4.6. *If $S(T)$ is idempotent and if $\ker T^* \subset \ker T$, then T is a selfadjoint partial isometry.*

Proof. By our hypothesis on $S(T)$, $U^3|T|^{1/2}U = U^2$ or $U^*|T|^{1/2}U^{*3} = U^{*2}$. Therefore, since $\ker T^* \subset \ker T$, we obtain

$$|T|^{1/2}U^{*3} = UU^*|T|^{1/2}U^{*3} = UU^{*2} = U^*. \tag{4.9}$$

Also by Theorem 2.1, $\ker S(T)^* \subset \ker S(T)$ implying $S(T)$ to be a projection. As seen in the proof of Theorem 3.5, the underlying kernel condition indicates

$$1 \geq \|S(T)\| \geq \|S(T^*)\| = \||T|^{1/2}U^{*2}\|$$

and therefore

$$\begin{aligned} &\|(U^2|T|^{1/2}U^* - U^*)x\|^2 \\ &= \|U^2|T|^{1/2}U^*x\|^2 - \langle U^2|T|^{1/2}U^*x, U^*x\rangle - \langle U^*x, U^2|T|^{1/2}U^*x\rangle + \|U^*x\|^2 \\ &\leq \|U^*x\|^2 - \langle UU^*x, x\rangle - \langle x, UU^*x\rangle + \|U^*x\|^2 = 0. \end{aligned}$$

Hence $U^2|T|^{1/2}U^* = U^*$ or equivalently, $U|T|^{1/2}U^{*2} = U$ and hence $\ker T = \ker U \subset \ker U^* = \ker T^*$. Invoking Theorem 4.4, we arrive at the desired conclusion. □

Corollary 4.7. *Suppose the following conditions hold for $T \in B(\mathcal{H})$.*
(i) *$S(T)$ is idempotent.*
(ii) *$S(T^*)$ is a contraction.*
(iii) *$\ker S(T) = \ker T$.*
Then T is a selfadjoint partial isometry.

Proof. As seen earlier, the condition (i) yields $U^3|T|^{1/2}U = U^2$. Then applying (iii) which is equivalent to $\ker U = \ker U^2$ gives $U^2|T|^{1/2}U = U$. Since $\|S(T^*)\| = \||T|U^{*2}\| \leq 1$ by (ii), we have

$$\begin{aligned} &\|(|T|^{1/2}U^{*2}U - U)x\|^2 \\ &= \||T|^{1/2}U^{*2}Ux\|^2 - \langle |T|^{1/2}U^{*2}Ux, Ux\rangle - \langle Ux, |T|^{1/2}U^{*2}Ux\rangle + \|Ux\|^2 \\ &\leq \|Ux\|^2 - \langle U^*Ux, x\rangle - \langle x, U^*Ux\rangle + \|Ux\|^2 = 0. \end{aligned}$$

Hence $|T|^{1/2}U^{*2}U = U$ and $|T|^{1/2}U^{*2} = UU^*$. Then $UU^* = U^2|T|^{1/2}$, and so $\ker T \subset \ker T^*$. Hence the result is immediate from Theorem 4.4. □

Theorem 4.8. *If $S(T) = T$ and $\ker T \subset \ker T^*$, then T is a projection.*

Proof. Since $U|T|^{1/2}U = U|T|$, $|T|^{1/2}U = |T|$. Then $|T|^{1/2}U|T|^{1/2} = |T|^{3/2}$ and $\langle Ux, x\rangle \geq 0$ for all $x \in \operatorname{ran} |T|^{1/2}$. Also, $\langle Uy, y\rangle = 0$ for all $y \in \ker |T|^{1/2}$. For any $z \in \mathcal{H}$, there exist $x \in [\operatorname{ran} |T|^{1/2}]$ and $y \in \ker |T|^{1/2}$ such that $z = x + y$. Therefore

$$\langle Uz, z\rangle = \langle Ux, x + y\rangle = \langle Ux, x\rangle + \langle x, U^*y\rangle = \langle Ux, x\rangle \geq 0$$

as $\ker T \subset \ker T^*$. This shows that U is positive. Since U is also a partial isometry, $U^*U = U^2$ is a projection. Then $\sigma(U) \subset \{0, 1\}$ as U is positive. Hence U is a projection and $U^*U = U$. Then $T = U|T| = U^*U|T| = |T|$ and $|T| = |T|^{1/2}U = |T|^{1/2}U^*U = |T|^{1/2}$. Therefore $|T|$, and hence T is a projection. □

Example 6. Let $T = \begin{pmatrix} \frac{1}{2\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 \end{pmatrix}$. Then the polar decomposition $T = U|T|$ is given by $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$ and $|T| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$. Then $S(T) = U|T|^{\frac{1}{2}}U = T$, but T is not a projection. This example shows that Theorem 4.8 does not hold if the underlying kernel condition $\ker T \subset \ker T^*$ is replaced even by the weaker condition like $\ker U = \ker U^2$.

Corollary 4.9. *Let $S(T) = T$. If either* (i) *T is convexoid or* (ii) *U is convexoid, then T is a projection.*

Proof.

(i) The condition $S(T) = T$ implies $|T|^{1/2}U = |T|$. Clearly $\tilde{T} = |T|^{1/2}U|T|^{1/2} = |T|^{3/2}$ and hence $\sigma(\tilde{T}) = \sigma(T) \subset \{x : x \geq 0\}$. Since T is convexoid, it follows that T is a positive operator. In particular, $\ker T = \ker T^*$. Now the result follows from Theorem 4.8.

(ii) Again as $|T|^{1/2}U = |T|$, we have $UU^*|T|^{1/2} = U|T|$ or $|T|^{1/2}UU^* = |T|U^*$ implying $U^2U^* = U|T|^{1/2}U^*$. Since

$$\sigma(U) = \sigma(UU^*U) = \sigma(UUU^*)$$

by [3, lemma], we have

$$\sigma(U) = \sigma(U|T|^{1/2}U^*) \subset \{t : t \geq 0\}.$$

Since U is convexoid, it follows that U is a positive operator. Consequently, $\ker T = \ker T^*$. Now the result is clear from Theorem 4.8.

□

Corollary 4.10. *If $S(T) = T$ and T is a class $A(s,t)$ operator, then T is a projection.*

Proof. We may assume $1/2 < s$ by [9]. Note that the condition $S(T) = T$ implies $|T|^{1/2}U = |T|$. Then $T(s,t) = |T|^{s-1/2}(|T|^{1/2}U)|T|^t = |T|^{s+t+1/2}$. Hence T is normal by [12] and so the result follows from Theorem 4.8. □

Example 7. Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. Then $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $|T|^{1/2} = 2^{-\frac{3}{4}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $S(T) = 2^{-\frac{3}{4}}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $S(T)$ is not idempotent. Hence $S(T)$ may not be even idempotent if T is idempotent. It is well known that $\tilde{T}$ is a projection whenever T is idempotent. However, in case T and $S(T)$ are idempotent operators, then both T and $S(T)$ turn out to be projections.

Theorem 4.11. *If T and $S(T)$ are idempotent operators, then T is projection and $T = S(T)$.*

Proof. Since $U|T| = T = T^2 = U|T|U|T|$, we have $|T|U|T| = |T| = |T|U^*|T|$. Hence $U = UU^*|T|$ and $U^*U = U^*UU^*|T| = U^*|T|$. Then $|T|U = U^*U$ and

$$\ker U = \ker U^2. \tag{4.10}$$

Now the condition that $S(T)$ is idempotent yields $U^3|T|^{1/2}U = U^2$. Then

$$U^2|T|^{1/2}U = U. \tag{4.11}$$

by (4.10). Multiplying (4.11) on the right by U, we obtain $U^2|T|^{1/2}U^2 = U^2$. By (4.10), it follows that $U|T|^{1/2}U^2 = U$. Hence $|T|^{1/2}U^2 = U^*U$ and then $|T|U^2 = |T|^{1/2}U^*U = |T|^{1/2}$. Since $|T|U = U^*U$, we get $|T|^{1/2} = (|T|U)U = U^*U^2$ and so $S(T) = U(U^*U^2)U = U^3$. Since U^3 is a contraction and $S(T)$ is idempotent, U^3 is a projection. Since $\ker U = \ker U^2 = \ker U^3$ by (4.10), $U^3 = U^*U$ or $U^{*3} = U^*U$. Now it is obvious from the last equation that $\ker T^* = \ker U^* \subset \ker U = \ker T$. Hence T is a projection as T is idempotent. Clearly $S(T) = T$ which finishes the proof. □

5. POLAR DECOMPOSITION

Theorem 5.1. *If the operator $|T|^{1/2}U$ has the polar decomposition given by $|T|^{1/2}U = W||T|^{1/2}U|$, then $S(T) = UW|S(T)|$ is the polar decomposition of $S(T)$.*

Proof. Clearly $S(T) = UW|S(T)|$. To complete the proof, we must show that UW is a partial isometry with $\ker UW = \ker|S(T)|$.

We show UW is a partial isometry. Suppose $Ux = 0$ for some $x \in \mathcal{H}$. Then $U^*|T|^{1/2}x = 0$. By our hypothesis, $W^*x = 0$. Thus $\ker U^*U \subset \ker WW^*$ or $U^*UWW^* = WW^*$. Therefore

$$UW(UW)^*UW = UWW^*U^*UW = UWW^*W = UW.$$

Next we show $\ker UW = \ker S(T)$. Let $UWx = 0$. Then $Wx \in \ker U$ and $U^*|T|^{1/2}Wx = 0$. Then $W^*Wx = 0$ and $S(T)x = 0$. On the other hand if $S(T)x = 0$, then $|T|^{1/2}Ux = U^*U|T|^{1/2}Ux = 0$. This implies that $Wx = 0$ and therefore $UWx = 0$. □

Theorem 5.2. *If $S(T) = W|S(T)|$ is the polar decomposition, then the operator $|T|^{1/2}U$ has the polar decomposition given by $U^*W||T|^{1/2}U|$.*

Proof. By our hypothesis,

$$|T|^{1/2}U = U^*W|S(T)| = U^*W||T|^{1/2}U|.$$

We show $\ker U^*W = \ker||T|^{1/2}U|$. If $U^*Wx = 0$, then we find $Wx \in \ker U^*$. Since $\ker U^* \subset \ker W^*$, $W^*Wx = 0$. Then it follows that $W^*Wx = 0$ and hence $x \in \ker|S(T)| = \ker||T|^{1/2}U|$. Conversely if $||T|^{1/2}U|x = 0$, then $Wx = 0$ and so $x \in \ker U^*W$.

Next we show U^*W is a partial isometry. Since $\ker UU^* \subset \ker WW^*$, we have $WW^*UU^* = WW^*$. Therefore

$$U^*W(U^*W)^*U^*W = U^*(WW^*UU^*)W = U^*WW^*W = U^*W.$$

□

Theorem 5.3. *Let T be a binormal operator, i.e., $|T|$ commutes with $|T^*|$. If $S(T) = W|S(T)|$ is the polar decomposition, then the Aluthge transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ has the polar decomposition given by $\tilde{T} = U^*W|\tilde{T}|$.*

Proof. By Theorem 5.2, $|T|^{1/2}U = U^*W|S(T)|$ is the polar decomposition. Then

$$\tilde{T} = U^*W|S(T)||T|^{1/2}. \tag{5.1}$$

Since $|T|$ commutes with $|T^*| = U|T|U^*$, we have

$$\begin{aligned}|S(T)|^2|T|^{1/2} &= (U^*|T|U)|T|^{1/2} = U^*(|T|U|T|^{1/2}U^*)U \\ &= U^*(U|T|^{1/2}U^*|T|)U \\ &= |T|^{1/2}U^*|T|U = |T|^{1/2}|S(T)|^2.\end{aligned}$$

Thus $|S(T)|$ commutes with $|T|^{1/2}$. As a consequence of this fact, we find $\tilde{T}^*\tilde{T} = |T|^{1/2}U^*|T|U|T|^{1/2} = |S(T)|^2|T|$ or $|\tilde{T}| = |S(T)||T|^{1/2}$. Therefore (5.1) implies

$$\tilde{T} = U^*W|\tilde{T}|. \tag{5.2}$$

Note that U^*W is a partial isometry. In order to complete the proof, we must show that $\ker U^*W = \ker \tilde{T}$. Suppose $\tilde{T}x = 0$. Then $|S(T)|^2|T|^{1/2}x = U^*|T|U|T|^{1/2}x = 0$. Since $|S(T)|$ commutes with $|T|^{1/2}, |T|^{1/2}U^*|T|Ux = 0$. Then $UU^*|T|Ux = 0$ or $U^*|T|Ux = 0$. Hence $|T|^{1/2}Ux = 0$ and $S(T)x = 0$. Consequently we have $x \in \ker U^*W$. On the other hand if $U^*Wx = 0$, then $|\tilde{T}|x = |T|^{1/2}|S(T)|x = 0$. Thus $x \in \ker \tilde{T}$. □

Theorem 5.4. *If $S(T) = W|S(T)|$ and $\tilde{T} = U^*W|\tilde{T}|$ are polar decompositions, then $|S(T)|, |T|$ and $|\tilde{T}|$ are commuting. Moreover if $\ker T^* \subset \ker T$, then T is binormal.*

Proof. By Theorem 5.2, $|T|^{1/2}U$ has the polar decomposition $U^*W|S(T)|$. From this we derive that $\tilde{T} = U^*W|S(T)||T|^{1/2}$. Therefore

$$U^*W|\tilde{T} = U^*W|S(T)||T|^{1/2}.$$

Since $(U^*W)^*U^*W|S(T)| = |S(T)|$ and $(U^*W)^*U^*W|\tilde{T}| = |\tilde{T}|$, we obtain $|\tilde{T}| = |S(T)||T|^{1/2}$. Thus $|S(T)|, |T|$ and $|\tilde{T}|$ are commuting. Suppose $\ker T^* \subset \ker T$. Then $UU^*U^* = U^*$. Since $|S(T)|^2|T|^{1/2} = U^*(|T|U|T|^{1/2}U^*)U$ and $|T|^{1/2}|S(T)|^2 = U^*(U|T|^{1/2}U^*|T|)U$, we find $U^*(|T|U|T|^{1/2}U^*)U = U^*(U|T|^{1/2}U^*|T|)U$. Now the kernel condition implies that T is binormal. □

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S.M. Patel
Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar 388 120, Gujarat, India
E-mail address: smpatel-32@yahoo.com

Kôtarô Tanahashi
Department of Mathematics
Tohoku Pharmaceutical University
Sendai 981-8558, Japan
E-mail address: tanahasi@tohoku-pharm.ac.jp

Atsushi Uchiyama
Sendai National College of Technology
Sendai 989-3128, Japan
E-mail address: uchiyama@cc.sendai-ct.ac.jp