AN ANALYSIS ON THE INTERNAL STRUCTURE OF THE CELEBRATED FURUTA INEQUALITY VIA OPERATOR MEAN

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Abstract. For $A,\ B > 0$, the chaotic order $A \gg B$ is defined by $\log A \geq \log B$. It is known $A \gg B$ if and only if $A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I$ for all $r \geq 0$ and $p \geq 0$. We show this by Uchiyama's technique without using Furuta inequality. Furthermore we give an interpretation for the genelarized Furuta inequality, that is, if $A,\ B > 0$ and $A \gg (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$ for $p > t \geq 0$, then

$$A^{-r+t} \sharp_{\frac{\delta+r}{\beta-t+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \natural_{\frac{\delta}{p-t}} B^p$$

holds for $r \geq 0$ and $0 \leq \delta \leq \beta - t$. If $t = 0$ and $\beta = p$, then this shows the chaotic case of Furuta inequality and the case $t = 1$ corresponds to the Ando-Hiai inequality.

1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. invertible) operator.
The Furuta inequality is given as follows:

Furuta inequality: *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(F) $$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(F) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p + r.$$

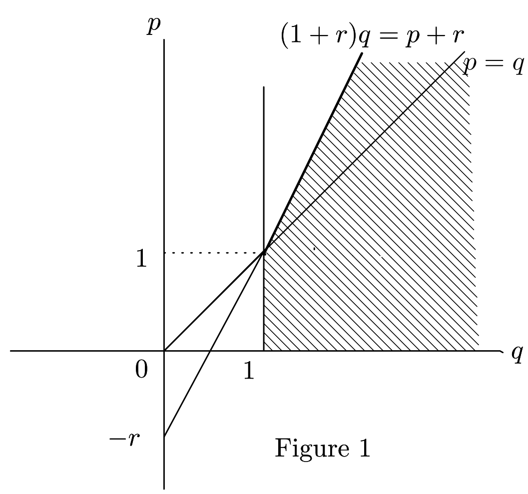

Figure 1

This yields the Löwner-Heinz inequality;

(LH) $$A \geq B \geq 0 \ \text{ implies } \ A^{\alpha} \geq B^{\alpha} \ \text{ for any } \ \alpha \in [0,\ 1].$$

We had reformed (F) in terms of the α-power mean (or generalized geometric operator mean) of A and B which is introduced by Kubo-Ando [18] as follows:

$$A \sharp_{\alpha} B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\alpha} A^{\frac{1}{2}} \quad \text{for } \ 0 \leq \alpha \leq 1.$$

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Incidentally we use also the notation $\natural$ to distinguish from the operator mean $\sharp$;

$$A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}} \quad \text{for} \quad r \notin [0,\ 1].$$

By using the α-power mean, Furuta inequality [7] (cf.[8]) is given as follows [13] (cf.[2]):

(F) $$A \geq B \geq 0 \ \text{ implies } \ A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq A \ \text{ for } \ p \geq 1 \ \text{ and } \ r \geq 0.$$

Based on this reformulation, we had proposed a satellite form of (F) [13] (cf.[14]);

(SF) $$A \geq B \geq 0 \ \text{ implies } \ A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \ \text{ for } \ p \geq 1 \ \text{ and } \ r \geq 0.$$

Moreover, to investigate our satellite form, we had introduced the chaotic order $A \gg B$ for $A,\ B > 0$ by $\log A \geq \log B$ [4]. It is well-known $A \geq B$ implies $\log A \geq \log B$ but the coverse is false in general.

Under the assumption $A \gg B$, we have obtained a similar result to (F) or (SF) [16], [17], that is,

(CF) $$A \gg B \ \text{ implies } \ A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \ \text{ for } \ p \geq 1 \ \text{ and } \ r \geq 0.$$

A characterization of the chaotic order is given in [3], [16], [17] as follows:

(C) $$A \gg B \ \text{ if and only if } \ A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \ \text{ for all } \ p,\ r \geq 0.$$

Especially, Uchiyama [19] has shown a skillful proof in which he used (F).

The purpose of this note is to clarify some advantage of the viewpoint from the chaotic order. As an example, we investigate that the point of the Furuta inequality is the inequality (C) under the assumption $A \geq B > 0$. That is,

(C_0) $$A \geq B > 0 \text{ implies } \ A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \ \text{ for all } \ p,\ r \geq 0.$$

As a consequence, we give a proof of (C) depending on Uchiyama's idea without using (F), and moreover we have the Furuta inequality itself. In addition, we note that (C_0) follows from the Ando-Hiai inequality.

Another is a proposal of a multi-variant of the Ando-Hiai inequality. Precisely, if $A,\ B > 0$ and for $p > t \geq 0$

$$A \gg (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}},$$

then

$$A^{-r+t} \sharp_{\frac{\delta+r}{\beta-t+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \natural_{\frac{\delta}{p-t}} B^p$$

holds for $r \geq 0$ and $0 \leq \delta \leq \beta - t$. If $t = 0$ and $\beta = p$, then this shows the chaotic case of Furuta inequality and the case $t = 1$ corresponds to the Ando-Hiai inequality.

2. Chaotic Furuta inequality.

In this section, we attempt a chaotical approach to the Furuta inequality.

Lemma 1. *If $A \geq B \geq 0$ and $p \geq 0$, then*

$$A^{-n} \sharp_{\frac{n}{p+n}} B^p \leq I$$

holds for $n = 1, 2, \ldots$ and $p \geq 0$.

Proof. We prove this by induction. Since $A^{-1} \natural_{\frac{1}{p+1}} B^p \le B^{-1} \natural_{\frac{1}{p+1}} B^p = I$, we have the case $n = 1$. If this holds for n, that is, $A^{-n} \natural_{\frac{n}{p+n}} B^p \le I$ or $(A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{\frac{n}{p+n}} \le A^n$, then it implies $(A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{\frac{1}{p+n}} \le A$ by (LH), then

$$\begin{aligned} & A^{-n-1} \natural_{\frac{n+1}{p+n+1}} B^p = A^{-\frac{n}{2}}(A^{-1} \natural_{\frac{n+1}{p+n+1}} A^{\frac{n}{2}} B^p A^{\frac{n}{2}})A^{-\frac{n}{2}} \\ \le \; & A^{-\frac{n}{2}}((A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{-\frac{1}{p+n}} \natural_{\frac{n+1}{p+n+1}} A^{\frac{n}{2}} B^p A^{\frac{n}{2}})A^{-\frac{n}{2}} = A^{-n} \natural_{\frac{n}{p+n}} B^p \le I. \end{aligned}$$

Theorem 2. *If $A \ge B \ge 0$, then*

$$\text{(C)} \qquad A^{-r} \natural_{\frac{r}{p+r}} B^p \le I$$

holds for all $r \ge 0$ and $p \ge 0$.

Proof. The case $0 \le r \le 1$, $A^{-r} \natural_{\frac{r}{p+r}} B^p \le B^{-r} \natural_{\frac{r}{p+r}} B^p = I$, is assured by (LH). If $r = n + \epsilon$ for positive integer n and $0 \le \epsilon < 1$, then

$$(A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{\frac{\epsilon}{p+n}} \le A^{\epsilon}$$

by Lemma 1 and (LH). Hence we have

$$\begin{aligned} & A^{-r} \natural_{\frac{r}{p+r}} B^p = A^{-(n+\epsilon)} \natural_{\frac{n+\epsilon}{p+n+\epsilon}} B^p \\ = \; & A^{-\frac{n}{2}}(A^{-\epsilon} \natural_{\frac{n+\epsilon}{p+n+\epsilon}} A^{\frac{n}{2}} B^p A^{\frac{n}{2}})A^{-\frac{n}{2}} \\ \le \; & A^{-\frac{n}{2}}((A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{-\frac{\epsilon}{p+n}} \natural_{\frac{n+\epsilon}{p+n+\epsilon}} A^{\frac{n}{2}} B^p A^{\frac{n}{2}})A^{-\frac{n}{2}} \\ = \; & A^{-\frac{n}{2}}(A^{\frac{n}{2}} B^p A^{\frac{n}{2}})^{\frac{n}{p+n}} A^{-\frac{n}{2}} = A^{-n} \natural_{\frac{n}{p+n}} B^p \le I. \end{aligned}$$

Theorem 2 is a key of the Furuta inequality (F). That is, the way from Theorem 2 to (F) is quite easy as follows:

Corollary 3.(Furuta Inequality) *If $A \ge B \ge 0$, then*

$$\text{(F)} \qquad A^{-r} \natural_{\frac{1+r}{p+r}} B^p \le B \le A$$

holds for all $r \ge 0$ and $p \ge 1$.

Proof. We here use well-known formulas on $\natural_{\alpha}$;

$$A \natural_{\alpha} B = B \natural_{1-\alpha} A \quad \text{and} \quad A \natural_{\alpha\beta} B = A \natural_{\alpha} (A \natural_{\beta} B).$$

Thus it follows from Theorem 2 that

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p = B^p \natural_{\frac{p-1}{p+r}} A^{-r} = B^p \natural_{\frac{p-1}{p}} (B^p \natural_{\frac{p}{p+r}} A^{-r}) \le B^p \natural_{\frac{p-1}{p}} I = B \le A.$$

We now note that Theorem 2 is motivated by (C), the characterization of the chaotic order in [16], [17]. Uchiyama [19] gave it an elegant proof by using (F) with his skillful technique. We give here a proof with no use of (F) but along to Furuta's methods in [10] which is a version of Uchiyama's technique [19]. Our basement is of course Theorem 2. We recall the following essential formula;

$$\lim_{n\to\infty}(I + \frac{1}{n}\log X)^n = X \quad for \quad any \quad X > 0.$$

Theorem 4. *For $A,\ B>0$, $A\gg B$ if and only if $A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p\le I$ holds for all $r\ge 0$ and $p\ge 0$.*

Proof. Suppose $A\gg B$, then $A_1=I+\frac{1}{n}\log A\ge I+\frac{1}{n}\log B=B_1>0$ holds for sufficiently large natural nummber n. Applying Theorem 2 to A_1 and B_1, we have

$$I\ge A_1^{-nr}\ \natural_{\frac{nr}{np+nr}}\ B_1^{np}=(I+\frac{1}{n}\log A)^{-nr}\ \natural_{\frac{r}{p+r}}\ (I+\frac{1}{n}\log B)^{np}.$$

Since

$$(I+\frac{1}{n}\log A)^{-nr}\longrightarrow A^{-r}\ \text{ and }\ (I+\frac{1}{n}\log B)^{np}\longrightarrow B^p\ \ (\text{as n}\longrightarrow\infty)$$

and $A,\ B$ are invertible, we have $A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p\le I$. The converse is easily seen because $(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}\le A^r$ implies $(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{p+r}}\ll A$ for all $r\ge 0$ and $p\ge 0$. $A\gg B$ is the case $r=0$.

The following result is understood as a corollary of Theorem 4, which was shown in [16] or [17] as generalizations of (C).

Corollary 5. *If $A,\ B>0$ and $A\gg B$, then*

(a) $$A^{-r}\ \natural_{\frac{\delta+r}{p+r}}\ B^p\le B^\delta\quad for\ \ all\ \ 0\le\delta\le p\ \ and\ \ r\ge 0,$$

(b) $$A^{-r}\ \natural_{\frac{\delta+r}{p+r}}\ B^p\le A^\delta\quad for\ \ all\ \ -r\le\delta\le 0\ \ and\ \ p\ge 0.$$

Proof. It is very similar to that of Corollary 3:

(a) $$A^{-r}\ \natural_{\frac{\delta+r}{p+r}}\ B^p=B^p\ \natural_{\frac{p-\delta}{p+r}}\ A^{-r}=B^p\ \natural_{\frac{p-\delta}{p}}\ (B^p\ \natural_{\frac{p}{p+r}}\ A^{-r})\le B^p\ \natural_{\frac{p-\delta}{p}}\ I=B^\delta$$

and

(b) $$A^{-r}\ \natural_{\frac{\delta+r}{p+r}}\ B^p=A^{-r}\ \natural_{\frac{\delta+r}{r}}\ (A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p)\le A^{-r}\ \natural_{\frac{\delta+r}{r}}\ I=A^\delta.$$

Concluding this section, we point out the monotonicity of an operator function $A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p$ for $p\ge 0$ and $r\ge 0$.

Theorem 6. *If $A,\ B>0$ and $A\gg B$, then $A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p$ is decreasing for $p\ge 0$ and $r\ge 0$.*

Proof. Let $p_1\ge p$ and $r_1\ge r$, then

$$\begin{aligned}A^{-r_1}\ \natural_{\frac{r_1}{p_1+r_1}}\ B^{p_1}&=A^{-r_1}\ \natural_{\frac{r_1}{p+r_1}}\ (A^{-r_1}\ \natural_{\frac{p+r_1}{p_1+r_1}}\ B^{p_1})\le A^{-r_1}\ \natural_{\frac{r_1}{p+r_1}}\ B^p\\&=B^p\ \natural_{\frac{p}{p+r_1}}\ A^{-r_1}=B^p\ \natural_{\frac{p}{p+r}}\ (B^p\ \natural_{\frac{p+r}{p+r_1}}\ A^{-r_1})\\&=B^p\ \natural_{\frac{p}{p+r}}\ (A^{-r_1}\ \natural_{\frac{-r+r_1}{p+r_1}}\ B^p)\le B^p\ \natural_{\frac{p}{p+r}}\ A^{-r}=A^{-r}\ \natural_{\frac{r}{p+r}}\ B^p\end{aligned}$$

The first inequality follows from (a) Corollary 4 and the second one is led by (b).

3. Ando-Hiai's inequality. Ando and Hiai showed the next inequality in [1]. If $A\ \natural_\alpha\ B\le I$ for $A,\ B>0$, then $A^r\ \natural_\alpha\ B^r\le 1$ holds for $r\ge 1$. From this, they obtained the following inequality (AH). It is equivalent to the main result of log majorization and can be given as the following form:

(AH) $A^{-1} \sharp_{\frac{1}{p}} A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}} \le I$ implies $A^{-r} \sharp_{\frac{1}{p}} (A^{-\frac{1}{2}}B^pA^{-\frac{1}{2}})^r \le I$ *for* $p \ge 1$ *and* $r \ge 1$.

Furuta [9] caught (AH) in a framework of (F)(cf.[11]). He gave a generalization of (F) as follows:

If $A \ge B \ge 0$ *and* A *is invertible, then for each* $1 \le p$ *and* $0 \le t \le 1$,

$$A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s \le A^{1-t} \tag{GF}$$

holds for $t \le r$ *and* $1 \le s$.

The importance of (GF) is interpolating (F) and (AH). When $s = r$ and $t = 1$, (GF) is just (AH) and the case where $s = 1$ and $t = 0$ reduces (GF) to (F).

(GF) can be rewritten by the formula of operator mean and replacing s by $\frac{\beta-t}{p-t}$ for $\beta \ge p$ as follows [5]:

$$A^{-r+t} \sharp_{\frac{1-t+r}{\beta-t+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le B \le A. \tag{GF$'$}$$

Several satellite forms of (GF) had shown in [6], [15], etc..

We propose a generalized form of (F) from the viewpoint of (AH). The following order (1) is observed in [20] by Yanagida, Ito and Ymazaki.

Theorem 7. *If* $A,\ B > 0$ *and for* $p > t \ge 0$

$$A \gg (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1}{p-t}}, \tag{1}$$

then

$$A^{-r+t} \sharp_{\frac{\delta+r}{\beta-t+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \le A^t \natural_{\frac{\delta}{p-t}} B^p \tag{2}$$

holds for $r \ge 0$ *and* $0 \le \delta \le \beta - t$.

Proof. This is easily obtained by applying Corollary 4 to A and $(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{1}{p-t}}$, that is,

$$A^{-r} \sharp_{\frac{\delta+r}{q+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{q}{p-t}} \le (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\frac{\delta}{p-t}}$$

holds for $0 \le \delta \le \beta$, which is equivalent to

$$A^{-r+t} \sharp_{\frac{\delta+r}{q+r}} (A^t \natural_{\frac{q}{p-t}} B^p) \le A^t \natural_{\frac{\delta}{p-t}} B^p.$$

So we have the conclusion replacing q with $\beta - t$.

We note that the formula (2) is divided into two cases: (i) If $0 \le \beta \le p$, then

$$A^{-r+t} \sharp_{\frac{\delta+r}{\beta-t+r}} (A^t \sharp_{\frac{\beta-t}{p-t}} B^p) \le A^t \sharp_{\frac{\delta}{p-t}} B^p.$$

(ii) If $\beta \geq p$ and $p - t \geq \delta$, then

$$A^{-r+t} \sharp_{\frac{\delta+r}{\beta-t+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^t \sharp_{\frac{\delta}{p-t}} B^p.$$

In the latter, we put $\delta = 1 - t$ for $t \in [0, 1]$. Then we have the following.

Corollary 8. *If* (1) *in* Theorem 7 *holds and* $\beta \geq p \geq 1 \geq t \geq 0$, *then*

$$(2') \qquad A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \leq (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1-t}{p-t}}.$$

Remark 1. We here remark that Theorem 7 is a multi-variant of (SF) and (AH): If $t = 0$, then (1) is $A \gg B$, and $\beta = p$, then $(2')$ is just (SF).
If $t = 1$, then (1) implies that $A^{-1} \sharp_{\frac{1}{p}} A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}} \leq I$ by Theorem 4. Thus (AH) ensures $A^{-r} \sharp_{\frac{1}{p}} (A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}})^r \leq I$ for $p,\ r \geq 1$, which is obtained as the case $\beta = r(p-1) + 1$ in $(2')$ of Corollary 8.

Remark 2. Finally we have to say that Theorem 2 is implied by the Ando-Hiai inequality, i.e., If $A \sharp_\alpha B \leq I$ for $A,\ B > 0$, then $A^r \sharp_\alpha B^r \leq 1$ holds for $r \geq 1$. We now suppose that $A \geq B > 0$. As in the proof of Theorem 2, it follows that $A^{-r} \sharp_{\frac{r}{q+r}} B^q \leq I$ for $0 < r \leq 1$ and $q > 0$. So we may assume that $p,\ r > 1$ are given. Since $A^{-1} \sharp_{\frac{1}{q+1}} B^q \leq I$ for $q = \frac{p}{r}$, it follows that

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p = (A^{-1})^r \sharp_{\frac{1}{\frac{p}{r}+1}} (B^q)^r \leq 1,$$

as desired.

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