CONDITIONAL AND UNCONDITIONAL STABILITY PROPERTIES OF TIME DEPENDENT SETS *

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Abstract. Let $\dot{x} = f(t,x)$ be a smooth differential equation in $\mathbf{R} \times \mathbf{R}^n$ and M be an s–compact invariant set in $\mathbf{R} \times \mathbf{R}^n$. Assume the existence of a smooth invariant set Φ in $\mathbf{R} \times \mathbf{R}^n$ containing M such that M is uniformly asymptotically stable with respect to the perturbations lying on Φ. We analyze the influence of the stability properties of Φ "near M" on the unconditional stability properties of M. A comparison with some classical results concerning the autonomous or the periodic case is given.

1. Introduction. Denote by $\|\cdot\|$ a norm in $\mathbf{R}^n$ and by ρ the induced distance. Let C be a nonempy set in $\mathbf{R}^n$ and for $a > 0$ let $B^n(C,a) = \{x \in \mathbf{R}^n : \rho(x,C) < a\}$, $B^n[C,a] = \{x \in \mathbf{R}^n : \rho(x,C) \leq a\}$, $S^n(C,a) = \{x \in \mathbf{R}^n : \rho(x,C) = a\}$. Consider a set A in $\mathbf{R} \times \mathbf{R}^n$. We say that A is s–nonempty if for any $t \in \mathbf{R}$ the section $A(t) = \{x \in \mathbf{R}^n : (t,x) \in A\}$ is nonempty. We say that A has a s–bounded diameter if A is s–nonempty and there exists a constant $\lambda > 0$ such that $diam A(t) < \lambda$ for all $t \in \mathbf{R}$. If A is s–nonempty and there exists a compact set Q in $\mathbf{R}^n$ such that $A(t) \subseteq Q$ for all $t \in \mathbf{R}$, then A is said to be s–bounded. In this case the intersection of all these sets Q will be denoted by $Q^\star(A)$. If A is s–bounded and each $A(t)$ is compact, we say that A is s–compact. When the mapping $t \to A(t)$ is ω–periodic for some $\omega > 0$ or in particular t–independent, we say that A is ω–periodic or t–independent respectively.

Consider the system of differential equations

$$\dot{x} = f(t,x), \qquad \dot{(\,)} = \frac{d}{dt} \tag{1.1}$$

where $f \in C(\mathbf{R}\times\mathbf{R}^n, \mathbf{R}^n)$ and f satisfies conditions ensuring the uniqueness of the solutions. Moreover f is supposed such that (1.1) admits an invariant s–compact set M in $\mathbf{R} \times \mathbf{R}^n$. For any $(t_0,x_0) \in \mathbf{R} \times \mathbf{R}^n$ let us denote by $x(t,t_0,x_0)$ the solution through (t_0,x_0) and by $J^+(t_0,x_0)$, $J^-(t_0,x_0)$ its maximal interval of existence in the future and in the past respectively.

Let A be any s–nonempty invariant set in $\mathbf{R}\times\mathbf{R}^n$. The stability concepts of A are known and derived from the usual concepts concerning the stability of a single trajectory. For instance A is said to be: (i) stable if for any t_0 in $\mathbf{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(t_0,\varepsilon) > 0$ such that $\rho(x_0, A(t_0)) < \delta$ implies $\rho(x(t,t_0,x_0), A(t)) < \varepsilon$ for any $t \in J^+(t_0,x_0)$; (ii) attracting if for any t_0 in $\mathbf{R}$ there exists $\mu = \mu(t_0) > 0$ for which $\rho(x_0, A(t_0)) < \mu$ implies $J^+(t_0,x_0) = [t_0, +\infty)$ and $\rho(x(t,t_0,x_0), A(t)) \to 0$ as $t \to +\infty$. We only notice that as

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in the case of a single trajectory if A has a s–bounded diameter (in particular if A is s–compact), the stability of A is equivalent to the following condition: for any t_0 in $\mathbf{R}$ and $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\rho(x_0, A(t_0)) < \delta$ implies $J^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), A(t)) < \varepsilon$ for any $t \geq t_0$. When A is t–independent, $A(t) \equiv N$, it is customary to replace A by N and then look at the stability properties of A as the stability properties of a set in $\mathbf{R}^n$.

Suppose now the existence of an invariant set Φ in $\mathbf{R} \times \mathbf{R}^n$ containing M and satisfying the two conditions: (**a**) Φ is the kernel of a function $G \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^p)$, $p \geq 1$; (**b**) M is uniformly asymptotically stable on Φ, that is for perturbations $(t_0, x_0) \in \Phi$. Without any restriction we can clearly assume $\Phi = ker F$, with $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$. The set of these functions F will be denoted by $(h)_\Phi$. If $F \in (h)_\Phi$ may be chosen to be a first integral of (1.1), then for this F we will write $F \in (H)_\Phi$.

The scope of the present paper is an analysis of the unconditional stability of M. The results show that the stability properties of M appear to be correlated to the stability properties of Φ "near M" in an appropriate sense (see Definition 2.1). Precisely we find that for f smooth the stability (the asymptotic stability) of Φ near M implies the stability (the asymptotic stability) of M in each of the following cases: (u) $(H)_\Phi$ is nonempty; (v) Φ is t–independent and satisfies some regularity conditions; (w) Φ is a manifold $z = g(t, y)$ with $(y, z) = x$, and g smooth (Section 3).

The case in which f and M are both ω–periodic in t for some constant $\omega > 0$ will be specified as the periodic case (as the autonomous case if f and M are both t–independent). In the periodic case the above results are invertible. In other words in the periodic case if Φ satisfies (u) or (v) or (w), then Φ and M have the same stability properties: M is stable (asymptotically stable) if and only if Φ is stable (asymptotically stable) near M (Section 4).

It is useful to compare these latter results with some classical results (Liapunov [4], Pliss [5], Kelley [3]). In [3] $M = \mathbf{R} \times c$, where c is an equilibrium, or the orbit of a periodic solution, or a periodic surface. Moreover (1.1) is autonomous and (suitably modified outside of a neighborhood of c) admits in $\mathbf{R}^n$ an invariant center manifold Ψ containing c and exponentially asymptotically stable near c. In terms of Ψ the results in [3] may be stated as follows: the (unconditional) stability properties of c are completely determined by the stability properties of c on Ψ; precisely if c is stable (asymptotically stable, unstable) on Ψ, then c is stable (asymptotically stable, unstable). If c is asymptotically stable on Ψ, the asymptotic stability of c follows immediately from our results with $\Phi = \mathbf{R} \times \Psi$ and $M = \mathbf{R} \times c$ (Section 4). Similarly it may be treated the known result (see for instance Chow and Hale [2]) concerning the asymptotic stability problem of a ω–periodic solution $x(t)$ to a nonautonomous ω–periodic differential system. In this case Φ and M are ω–periodic subsets of $\mathbf{R} \times \mathbf{R}^n$ and $M = \{(t, x) : t \in \mathbf{R}, x = x(t)\}$. It has to be noticed that for the asymptotic stability of c, the exponential character in [3] of the asymptotic stability of Ψ near c does not play any role. This has been the motivation to analyze in the general case the influence that the stability properties of Φ near M have on the corresponding unconditional stability properties of M. However, we do not have discussed the extendibility to our general setup of the result in [3] relative to the case that c is nonasymptotically stable on Ψ. We only remark that for this extension the assumption that Φ is exponentially asymptotically stable near M cannot be in general avoided (Section 4).

In Sections 3 and 4 some simple applications of our results are given. More significant applications, especially to the bifurcation theory from equilibrium into s–compact sets, will be provided in forthcoming papers.

2. Preliminaries. We begin by giving some definitions concerning a function $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ such that the set $A = kerF$ is invariant and contains M.

Definition 2.1 *For any $\gamma > 0$ let $I[M, \gamma] = \{(t, x) : t \in \mathbf{R}, x \in B^n[M(t), \gamma]\}$. Then we will say that A has a stability property near M if there exists $\gamma > 0$ such that the property is satisfied with respect to the perturbations $(t_0, x_0) \in I[M, \gamma]$. For instance A is said to be: (i) stable near M if there exists $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ and $\varepsilon > 0$ one may find $\delta = \delta(t_0, \varepsilon) > 0$ with the property that $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, A(t_0)) < \delta$ imply $\rho(x(t, t_0, x_0), A(t)) < \varepsilon$ for any $t \in J^+(t_0, x_0)$; (ii) attracting near M if there exists $\gamma > 0$ such that for any $t_0 \in \mathbf{R}$ one may find $\mu = \mu(t_0) > 0$ for which $x_0 \in B^n[M(t_0), \gamma]$ and $\rho(x_0, A(t_0)) < \mu$ imply $J^+(t_0, x_0) = [t_0, +\infty)$ and $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$. Similarly one may define the other stability properties of A near M.*

Remark 2.1 *Since M is contained in A, and then $\rho(x_0, A(t_0)) \leq \rho(x_0, M(t_0))$ for any $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$, the uniform attractivity of A near M may be defined as follows: There exists $\sigma > 0$ such that $t_0 \in \mathbf{R}$ and $x_0 \in B^n[M(t_0), \sigma]$ implies that $x(t, t_0, x_0)$ exists for all $t \geq t_0$ and satisfies $\rho(x(t, t_0, x_0), A(t)) \to 0$ as $t \to +\infty$, uniformly in (t_0, x_0).*

Definition 2.2 *The function F is said to be A–positive definite near M if for some $\gamma > 0$ and for any $t_0 \in \mathbf{R}$, $\alpha > 0$ there exists $\beta = \beta(t_0, \alpha) > 0$ such that if $t \in (t_0, +\infty)$, $x \in B^n[Q^\star(M), \gamma]$, and $\rho(x, A(t)) \geq \alpha$, then $F(t, x) \geq \beta$.*

We observe that because of the continuity of F we have that if the above $\beta(t_0, \alpha)$ exists for a fixed t_0, it exists for any t_0 and for $t_0' > t_0$ one may assume $\beta(t_0', \alpha) = \beta(t_0, \alpha)$. If $A(t) \equiv M(t) \equiv N$, we have $Q^\star(M) = N$ and this definition reduces to the usual concept of positive definitiveness of F with respect to N. We also need a weaker definitiveness property which involves the solutions of (1.1). For $\gamma > 0$, $t_0 \in \mathbf{R}$, consider the following set

$$\Pi(t_0, \gamma) = \{(t, x) : t \geq t_0,\, x \in B^n[M(t), \gamma],\, t_0 \in J^-(t, x),\, x(t_0, t, x) \in B^n[M(t_0), \gamma]\}.$$

Definition 2.3 *The function F is said to be weakly A–positive definite near M (with respect to (1.1)) if for some $\gamma > 0$ and for any $t_0 \in \mathbf{R}$, $\alpha > 0$ there exists $\beta = \beta(t_0, \alpha) > 0$ such that if $(t, x) \in \Pi(t_0, \gamma)$ and $\rho(x, A(t)) \geq \alpha$ then $F(t, x) \geq \beta$.*

Remark 2.2 *If M is t–independent, $M(t) \equiv N$, then we have*

$$\Pi(t_0, \gamma) = \{(t, x) : x \in B^n[N, \gamma],\, t \in P(t_0, x, \gamma)\},$$

with

$$P(t_0, x, \gamma) = \{t \geq t_0 : t_0 \in J^-(t, x),\, x(t_0, t, x) \in B^n[N, \gamma]\}.$$

When F is a first integral, the weak A–positive definitiveness of F is connected to the stability of A near M. Precisely the following result holds:

Lemma 2.1 *Assume that F is a first integral of (1.1). Then:*

(i) *If A is stable near M, then F is weakly A–positive definite near M.*

(ii) *Assume that $A(t) \equiv M(t) \equiv N$. Then M is stable if and only if there exists $\gamma > 0$ such that F is weakly A–positive definite near M.*

Proof. (i) For some fixed $\gamma > 0$ and for any $t_0 \in \mathbf{R}$, $\alpha > 0$ there exists $\eta = \eta(t_0, \alpha) > 0$ such that if x_0 is in $B^n[M(t_0), \gamma]$ and $\rho(x_0, A(t_0)) < \eta$ then $\rho(x(t, t_0, x_0), A(t)) < \alpha$ for $t \geq t_0$. For fixed t_0 let us consider the function $F(t_0, \cdot)$. One has $F(t_0, x_0) > 0$ for any $x_0 \notin A(t_0)$ and $F(t_0, x_0) = 0$ for $x_0 \in A(t_0)$. By setting

$$\beta(t_0, \alpha) = \min \{F(t_0, x_0) : x_0 \in B^n[M(t_0), \gamma],\ \rho(x_0, A(t_0)) \geq \eta(t_0, \alpha)\},$$

we easily obtain

$$x_0 \in B^n[M(t_0), \gamma],\ F(t_0, x_0) < \beta(t_0, \alpha) \text{ imply } \rho(x(t, t_0, x_0), A(t)) < \alpha \quad \forall t \in J^+(t_0, x_0). \tag{2.1}$$

Given any $(t, x) \in \Pi(t_0, \gamma)$, let $x_0 = x(t_0, t, x)$. By definition $x_0 \in B^n[M(t_0), \gamma]$. Hence from (2.1) it follows

$$\rho(x, A(t)) \geq \alpha \Longrightarrow F(t_0, x(t_0, t, x)) \geq \beta(t_0, \alpha).$$

In conclusion, since $F(t, x) = F(t_0, x(t_0, t, x))$ we have that if

$$(t, x) \in \Pi(t_0, \gamma) \text{ and } \rho(x, A(t)) \geq \alpha,$$

then $F(t, x) \geq \beta(t_0, \alpha)$. The proof of ($i$) is complete.
(ii) Since now $M = A$, necessity follows from (i). Then it remains only to prove sufficiency. This is obtained by the arguments which are used when F is positive definite in the usual sense. Indeed, choose any t_0 in $\mathbf{R}$ and $\varepsilon \in (0, \gamma)$. Taking into account Definition 2.3 and Remark 2.2, for any $x \in S^n(N, \varepsilon)$ and $t \in P(t_0, x, \varepsilon)$ one has $F(t, x) \geq \beta(t_0, \varepsilon) > 0$. Let $\delta = \delta(t_0, \varepsilon) \in (0, \varepsilon)$ be such that if $x \in B^n(N, \delta)$ then $F(t_0, x) < \beta(t_0, \varepsilon)$. Suppose the existence of x_0 in $B^n(N, \delta)$ and $t^\star > t_0$ such that $\rho(x(t, t_0, x_0), N) < \varepsilon$ for $t \in [t_0, t^\star)$ and $\rho(x(t^\star, t_0, x_0), N) = \varepsilon$. Then necessarily one has $t^\star \in P(t_0, x(t^\star, t_0, x_0), \varepsilon)$ and

$$\beta(t_0, \varepsilon) > F(t_0, x_0) = F(t^\star, x(t^\star, t_0, x_0)) \geq \beta(t_0, \varepsilon),$$

a contradiction. This completes the proof of (ii). ■

3. Stability results. In the sequel we will denote by $\mathcal{L}(x)$ the class of functions $g : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $(t, x) \to g(t, x)$, which are locally Lipschitzian with respect to x. We write $g \in \mathcal{L}_u(x)$ if for every compact set $K \subset \mathbf{R}^n$ there exists a constant $L(K) > 0$ such that $\|g(t, x) - g(t, y)\| \leq L(K)\|x - y\|$ for all x, y in K and t in $\mathbf{R}$, and write $g \in \mathcal{L}_{ub}(x)$ if in addition for every compact set $K \subset \mathbf{R}^n$ g is bounded in $\mathbf{R} \times K$. Trivially $g \in \mathcal{L}_u(x)$ implies $g \in \mathcal{L}_{ub}(x)$ if there exists at least one $x \in \mathbf{R}^n$ such that the fonction $g(\cdot, x)$ is bounded. We assume from now on the existence of an invariant set Φ containing M which is the kernel of a function $G \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^p)$, $p \geq 1$, and satisfies the condition that M is uniformly asymptotically stable for perturbations $(x_0, y_0) \in \Phi$. As in Section 1, we will denote by $(h)_\Phi$ the set of functions $F \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ such that $\Phi = ker F$, and write $F \in (H)_\Phi$ if $F \in (h)_\Phi$ and F is a first integral. Often a further condition on Φ will be assumed. This will be one of the following:

(u) The set $(H)_\Phi$ is nonempty;

(v) $\Phi = \mathbf{R} \times ker\varphi$, where $\varphi \in C^1(\mathbf{R}^n, \mathbf{R}^q)$, $1 \leq q \leq n$, and $rank[\partial\varphi/\partial x] = q$ for any $x \in ker\varphi$;

(w) $\Phi = \{(t, y, z) : z = g(t, y)\}$, with $(y, z) = x$, $y \in \mathbf{R}^m$, $z \in \mathbf{R}^{n-m}$, and $g \in C^1$, $g \in \mathcal{L}'_{ub}(y)$.

Here by $g \in \mathcal{L}'_{ub}(y)$ we want to mean that g belongs to $\mathcal{L}_{ub}(y)$ together with its partial derivatives.

Lemma 3.1 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_u(x)$. Assume the existence of a function $F \in (H)_\Phi$ which is weakly Φ–positive definite near M. Then M is stable.*

Proof. The proof is very similar to that given in [6] in the case $f(t, 0) \equiv 0$ and $M = \mathbf{R} \times \{0\}$. Choose $\gamma > 0$ as in Definition 2.3, with $A = \Phi$. For any $\varepsilon > 0$ let $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ be such that if $t_0 \in \mathbf{R}$, $y_0 \in \Phi(t_0)$, and $\rho(y_0, M(t_0)) < \delta$, then $\rho(x(t, t_0, y_0), M(t)) < \varepsilon$ for all $t \geq t_0$. Let $\sigma \in (0, \delta(\gamma))$ be chosen with the condition that for any $\nu > 0$ there exists a number $T = T(\nu) > 0$ such that if $y_0 \in \Phi(t_0)$ and $\rho(y_0, M(t_0)) < \sigma$, then $\rho(x(t, t_0, y_0), M(t)) < \nu$ for all $t \geq t_0 + T$.

Given any $\varepsilon \in (0, \sigma)$, let $\delta_1 = (1/2)\delta(\varepsilon/2)$, $\tau = T((1/4)\delta(\varepsilon))$ and $\bar{\delta} = (1/4)\delta(\varepsilon)exp(-k\tau)$ where $k = L(B^n[Q^\star(M), \gamma])$. By Definition 2.3 there exists $\beta = \beta(t_0, \bar{\delta})$ such that one has:

$$(t, x) \in \Pi(t_0, \gamma) \text{ and } F(t, x) < \beta \text{ imply } \rho(x, \Phi(t)) < \bar{\delta}. \tag{3.1}$$

Fix t_0 in $\mathbf{R}$ and assume $x_0 \in B^n(M(t_0), \delta_1)$, $F(t_0, x_0) < \beta$. Since $\delta_1 < \gamma$ and then trivially $(t_0, x_0) \in \Pi(t_0, \gamma)$, from (3.1) it follows $\rho(x_0, \Phi(t_0)) < \bar{\delta}$. Hence there exists $y_0 \in \Phi(t_0)$ with $\|x_0 - y_0\| < \bar{\delta}$. Thus we obtain

$$\rho(y_0, M(t_0)) \leq \|x_0 - y_0\| + \rho(x_0, M(t_0)) \leq \frac{3}{2}\delta_1 < \delta(\frac{\varepsilon}{2}),$$

and then in $[t_0, t_0 + \tau]$ we have

$$\rho(x(t, t_0, y_0), M(t)) < \frac{\varepsilon}{2} \quad \text{with} \quad \rho(x(t_0 + \tau, t_0, y_0), M(t_0 + \tau)) < \frac{\delta_1}{2}.$$

It follows

$$\begin{aligned} &\rho(x(t, t_0, x_0), M(t)) \leq \|x(t, t_0, x_0) - x(t, t_0, y_0)\| + \rho(x(t, t_0, y_0), M(t)) \\ &\leq \|x_0 - y_0\| exp(k\tau) + \rho(x(t, t_0, y_0), M(t)) < \frac{\delta_1}{2} + \frac{\varepsilon}{2} < \varepsilon \end{aligned} \tag{3.2}$$

for all t in $[t_0, t_0 + \tau]$ and

$$\rho(x(t_0 + \tau, t_0, x_0), M(t_0 + \tau)) < \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1. \tag{3.3}$$

Setting now $t_1 = t_0 + \tau$ and $x_1 = x(t_1, t_0, x_0)$, and taking into account that F is a first integral, we then recognize that $x_1 \in B^n(M(t_1), \delta_1)$ and $F(t_1, x_1) < \beta$. Since $(t_1, x_1) \in \Pi(t_0, \gamma)$, by virtue of (3.1) we still have $\rho(x_1, \Phi(t_1)) < \bar{\delta}$. Therefore the result expressed by (3.2), (3.3) holds with (t_0, x_0) replaced by (t_1, x_1), and so on. In other words for each $\varepsilon \in (0, \sigma)$ and $t_0 \in \mathbf{R}$ there exist two positive numbers δ_1 and β such that if $x_0 \in B^n(M(t_0), \delta_1)$ and $F(t_0, x_0) < \beta$, then

$$\rho(x(t, t_0, x_0), M(t)) < \varepsilon \quad \forall t \geq t_0. \tag{3.4}$$

Let now $\lambda = \lambda(t_0, \varepsilon) \in (0, \delta_1)$ be such that $F(t_0, x) < \beta$ for any $x \in B^n(M(t_0), \lambda)$. Then (3.4) holds for each x_0 in $B^n(M(t_0), \lambda)$ and this proves that M is stable. ∎

Theorem 3.1 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_u(x)$. Assume (u). Then, if Φ is stable near M, M is stable.*

Proof. Let $F \in (H)_\Phi$. By Lemma 2.1 we see that F is weakly positive definite near M. Hence the result follows from Lemma 3.1. ■

Theorem 3.2 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_{ub}(x)$. Assume (v). Then, if Φ is stable near M, M is stable.*

Proof. The set Φ is t–independent and its invariance is equivalent to the invariance of $ker\varphi$. Let $\mathcal{B}$, $\mathcal{B}'$ be two bounded open sets in $\mathbf{R}^n$ with $cl\mathcal{B} \subset \mathcal{B}'$ and $Q^\star(M) \subset \mathcal{B}$. Consider the system

$$\dot{x} = f(t,x)\alpha(x), \tag{3.5}$$

where $\alpha \in C^\infty(\mathbf{R}^n, [0,1])$ is such that $\alpha(x) = 1$ for $x \in \mathcal{B}$ and $\alpha(x) = 0$ for $x \notin \mathcal{B}'$. The r.h.s. of (3.5) is in $\mathcal{L}_{ub}(x)$ and then in $\mathcal{L}_u(x)$. Because of the local character of our stability problem, system (3.5) may replace the original system (1.1). We denote by $(x_{(3.5)}(t, t_0, x_0))$ the solution of (3.5) through (t_0, x_0). This solution clearly exists in all $\mathbf{R}$. The proof is divided into two steps.

(a) Let us prove that $\Phi = \mathbf{R} \times ker\varphi$ is invariant even for (3.5). Along the solutions of (3.5) we have

$$[\frac{d\varphi}{dt}]_{(3.5)} = \alpha(x)\langle\frac{\partial\varphi}{\partial x}(x), f(t,x)\rangle = 0 \quad \text{for any} \quad t \in \mathbf{R}, x \in ker\varphi, \tag{3.6}$$

because $ker\varphi$ is invariant under (1.1). To complete the proof of the above invariance, set

$$u = \varphi(x) \tag{3.7}$$

and consider any $(t_0, x_0) \in \Phi$. Equation (3.7) is satisfied for $x = x_0$ and $u = 0$. Moreover the determinant of at least one of the $q \times q$ matrices contained in $[\partial\varphi/\partial x](x_0)$ is different from zero. Suppose for instance that this matrix is that contained in the first q columns of $[\partial\varphi/\partial x](x_0)$ and set $x = (y, z)$, $x_0 = (y_0, z_0)$, with $y = (x_1, x_2, ..., x_q)$, $z = (x_{q+1}, x_{q+2}, ..., x_n)$. Then (3.7) defines in a neighborhood $\mathcal{N}$ of $z = z_0, u = 0$, an implicit function $y = y(z, u)$, $y(z_0, 0) = y_0$. Hence in $\mathcal{N}$ equation (3.5) in terms of z, u may be written as

$$\begin{aligned} \dot{z} &= Z(t,z,u) \\ \dot{u} &= U(t,z,u), \end{aligned} \tag{3.8}$$

where $U(t, z, 0) \equiv 0$ by virtue of (3.6), (3.7). Let $(z(t), u(t))$ be the solution of (3.8) such that $z(t_0) = z_0$, $u(t_0) = 0$. As well as this solution exists in $\mathcal{N}$, one has $u(t) \equiv 0$. Indeed $(3.8)_2$ is satisfied by assuming $u(t) \equiv 0$, while $(3.8)_1$ with $u = 0$ admits one and only one solution such that $z(t_0) = z_0$. Hence, since (t_0, x_0) is any point on Φ, the invariance of Φ under (3.5) is clearly proved.

(b) Since any solution of (3.5) exists for all t in $\mathbf{R}$, we may define a function $G \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^+)$ by assuming

$$G(t,x) \equiv \|\varphi(x_{(3.5)}(0,t,x))\|.$$

Let us prove that $kerG = \Phi$. Indeed $(t, x) \in kerG$ implies $(0, x_0) \in \Phi$, with $x_0 = x_{(3.5)}(0, t, x)$. The invariance of Φ under (3.5) then implies $(t, x) \in \Phi$. Similarly one can prove that $(t, x) \in \Phi$ implies $(t, x) \in kerG$. Since G is a first integral for (3.5), for this equation we have $G \in (H)_\Phi$. Moreover Φ is stable near M even for (3.5). Hence from Theorem 3.1 it follows that M is stable for (3.5) and then stable for the original equation (1.1). The proof is complete. ■

Theorem 3.3 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_{ub}(x)$. Assume (w). Then, if Φ is stable near M, M is stable.*

Proof. Letting $u = z - g(t, y)$, system (1.1) in terms of the variables y, u becomes

$$\dot{y} = Y(t, y, u)$$

(3.9)

$$\dot{u} = U(t, y, u),$$

where Y, U are continuous functions such that $Y, U \in \mathcal{L}_{ub}(y, u)$ and $U(t, y, 0) \equiv 0$, while Φ becomes the set $\tilde{\Phi} = \{(t, y, u) : u = 0\}$ and M becomes a set $\tilde{M}$. The problem of stability of M for (1.1) is equivalent to the problem of stability of $\tilde{M}$ for (3.9). Setting $\varphi(y, u) \equiv u$ we have $\tilde{\Phi} = \mathbf{R} \times ker\varphi$ and clearly φ satisfies for (3.9) the conditions in Theorem 3.2 with $q = n - m$. Since $\tilde{\Phi}$ is uniformly stable near $\tilde{M}$, the result follows from Theorem 3.2. The proof is complete. ■

Thus we find in each of the above theorems that the stability of Φ near M implies the stability of M provided Φ satisfies (u) or (v) or (w). Conversely, the *uniform* stability of M implies in general (that is without any of the conditions (u) or (v) or (w)) the *uniform* stability of Φ near M. Precisely the following theorem holds:

Theorem 3.4 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_u(x)$. If M is uniformly stable, then Φ is uniformly stable near M.*

Proof. For any $\varepsilon > 0$ let $\delta = \delta(\varepsilon) > 0$ be the number associated with ε in the definition of the uniform stability of M. Let $\sigma > 0$ be such that $\rho(x(t, t_0, y_0), M(t)) \to 0$ as $t \to +\infty$, uniformly in $\{(t_0, y_0) : t_0 \in \mathbf{R}, y_0 \in \Phi(t_0) \cap B^n[M(t_0), \sigma]\}$. Thus if $t_0 \in \mathbf{R}$ and $y_0 \in \Phi(t_0) \cap B^n[M(t_0), \delta(\sigma)]$, then: (i) $\rho(x(t, t_0, y_0), M(t)) < \sigma \quad \forall t \geq t_0$; (ii) for any $\nu > 0$, there exists $T = T(\nu) > 0$ such that $\rho(x(t, t_0, y_0), M(t)) < \nu \quad \forall t \geq t_0 + T$. Let $\gamma \in (0, \delta(\sigma)/2)$. Fixing now $\varepsilon \in (0, \gamma)$ and $\nu \in (0, \delta)$, choose a number $\eta = \eta(\varepsilon) > 0$ with the condition

$$0 < \eta < \frac{\delta - \nu}{exp(kT)}, \quad k = L(B^n[Q^\star(M), \sigma]).$$

Let $t_0 \in \mathbf{R}$. Assume $x_0 \in B^n[M(t_0), \gamma]$ and $y_0 \in \Phi(t_0)$ such that $\rho(x_0, \Phi(t_0)) < \eta$ and $\|x_0 - y_0\| < \eta$. Since

$$\|x(t, t_0, x_0) - x(t, t_0, y_0)\| < \eta exp(kT) < \delta - \nu < \varepsilon \quad \forall t \in [t_0, t_0 + T], \tag{3.10}$$

and Φ is an invariant set, one has

$$\rho(x(t, t_0, x_0), \Phi(t)) < \varepsilon \qquad \forall t \in [t_0, t_0 + T]. \tag{3.11}$$

We also have $\rho(y_0, M(t_0)) \leq \|x_0 - y_0\| + \rho(x_0, M(t_0)) < \eta + \gamma < 2\gamma < \delta(\sigma)$ from which it follows by virtue of (ii)

$$\rho(x(t, t_0, y_0), M(t)) < \nu \quad \forall t \leq t_0 + T. \tag{3.12}$$

Consequently by virtue of (3.10), (3.12),

$$\begin{aligned}&\rho(x(t_0+T,t_0,x_0),M(t_0+T))\\ &\leq \|x(t_0+T,t_0,x_0)-x(t_0+T,t_0,y_0)\|+\rho(x(t_0+T,t_0,y_0),M(t_0+T))<\delta(\varepsilon).\end{aligned}$$

Thus $\rho(x(t,t_0,x_0),M(t))<\varepsilon$ for all $t\geq t_0+T$. Hence, since $M(t)\subseteq\Phi(t)$ for every t, the inequality (3.11) is satisfied even for $t>t_0+T$. In conclusion for each $\varepsilon\in(0,\gamma)$ there exists $\eta>0$ such that

$$(t_0,x_0)\in\mathbf{R}\times B^n[M(t_0),\gamma] \text{ and } \rho(x_0,\Phi(t_0))<\eta \quad\text{imply}\quad \rho(x(t,t_0,x_0),\Phi(t))<\varepsilon \quad \forall t\geq t_0.$$

The proof is complete. ■

We wish now to analyze the connections between the asymptotic stability of Φ near M and the unconditional asymptotic stability of M. We begin by a statement which is the corresponding of Theorem 3.1 for asymptotic stability.

Theorem 3.5 *Suppose that f of (1.1) is continuous and $f\in\mathcal{L}_u(x)$. Assume (u). Then, if Φ is asymptotically stable near M, M is asymptotically stable.*

Proof. Since $(H)_\Phi$ is nonempty and Φ is stable near M, M is stable by virtue of Theorem 3.1. It remains only to prove that M is attracting. Denote by $\delta(\varepsilon)>0$ the number associated with (ε) in the definition of the uniform stability of M on Φ. Since M is uniformly attracting on Φ, for some fixed $\sigma>0$ and for every $\nu>0$ there exists $T=T(\nu)>0$ such that $\rho(x(t,t_0,y_0),M(t))<\frac{\nu}{2}$ for all t_0 in $\mathbf{R}$, $y_0\in B^n[M(t_0),\delta(\sigma)]\cap\Phi(t_0)$, and $t\geq t_0+T$. For every $\nu\in(0,\delta(\sigma))$ choose $\nu'=\nu'(\nu)$ with the condition

$$0<\nu'<\frac{\nu}{2exp(kT)},\quad k=L(B^n[Q^\star(M),\sigma]). \tag{3.13}$$

Let $\gamma\in(0,\delta(\sigma)/2)$ and $\mu(t_0)\in(0,\gamma)$ be chosen as in Definition 2.1(ii) for $A=\Phi$. Then for every $\nu\in(0,\delta(\sigma))$ there exists $T'=T'(t_0,x_0,\nu)>0$ such that $x_0\in B^n[M(t_0),\gamma]$ and $\rho(x_0,\Phi(t_0))<\mu(t_0)$ imply

$$J^+(t_0,x_0)=[t_0,+\infty) \quad\text{and}\quad \rho(x(t,t_0,x_0),\Phi(t))<\nu' \ \ \forall t\geq t_0+T'. \tag{3.14}$$

Given any $x_0\in B^n(M(t_0),\mu(t_0))$, we will now prove that

$$\rho(x(\tau_0+T,\tau_0,\xi_0),M(\tau_0+T))<\nu, \tag{3.15}$$

where $\tau_0\geq t_0+T'$, $\xi_0=x(\tau_0,t_0,x_0)$. Since $M(t_0)\subseteq\Phi(t_0)$ one has $\rho(x_0,\Phi(t_0))<\mu(t_0)$ and then, by virtue of (3.14), $\rho(\xi_0,\Phi(\tau_0))<\nu'$. Hence there exists $y_0\in\Phi(\tau_0)$ such that $\|\xi_0-y_0\|<\nu'$. Since

$$\rho(y_0,M(\tau_0))\leq\|\xi_0-y_0\|+\rho(\xi_0,M(\tau_0))<\nu'+\gamma<\frac{\delta(\sigma)}{2}+\frac{\delta(\sigma)}{2}=\delta(\sigma)$$

and

$$\begin{aligned}\rho(x(\tau_0+T,\tau_0,\xi_0),M(\tau_0+T)) &\leq \|x(\tau_0+T,\tau_0,\xi_0)-x(\tau_0+T,\tau_0,y_0)\|\\ &+ \rho(x(\tau_0+T,\tau_0,y_0),M(\tau_0+T)),\end{aligned}$$

it follows

$$\rho(x(\tau_0+T,\tau_0,\xi_0),M(\tau_0+T))<\nu' exp(kT)+\frac{\nu}{2}<\frac{\nu}{2}+\frac{\nu}{2}=\nu.$$

Therefore (3.15) is proved. Because of the arbitrarity of $\tau_0\geq t_0+T'$ we then obtain

$$\rho(x(t,t_0,x_0),M(t))<\nu \quad \forall t_0\in\mathbf{R},\ \forall x_0\in B^n(M(t_0),\mu(t_0)),\ \forall t\geq t_0+T(\nu)+T'(t_0,x_0,\nu).$$

This completes the proof. ■

By using Theorem 3.5 and the same arguments employed in the proofs of Theorems 3.2, 3.3 we easily find corresponding statements for asymptotic stability:

Theorem 3.6 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_{ub}(x)$. Assume (v) or (w). Then, if Φ is asymptotically stable near M, M is asymptotically stable.*

For asymptotic stability the converse Theorem 3.4 may be formulated as follows:

Theorem 3.7 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}_u(x)$. If M is uniformly asymptotically stable, then Φ is uniformly asymptotically stable near M.*

Proof. By virtue of Theorem 3.4 the set Φ is uniformly stable near M. Let $\varepsilon > 0$ and denote by $\delta(\varepsilon)$, $\delta'(\varepsilon)$ the positive numbers respectively associated with this property and with the uniform stability of M. Let $\gamma > 0$ and $\sigma \in [0, min(\delta(\gamma), \delta'(\gamma))]$ be such that $t_0 \in \mathbf{R}$ and $x_0 \in B^n[M(t_0), \sigma]$ imply that $\rho(x(t, t_0, x_0), M(t)) \to 0$ as $t \to +\infty$, uniformly in (t_0, x_0). Since $M(t) \subseteq \Phi(t)$ for any t in $\mathbf{R}$, then we have

$$\rho(x(t, t_0, x_0), \Phi(t)) \to 0 \quad \text{as} \quad t \to +\infty,$$

uniformly in $t_0 \in \mathbf{R}$, $x_0 \in B^n[M(t_0), \sigma]$. Thus Φ is uniformly attractive near M (see Remark 2.1). The proof is complete. ■

Remark 3.1 *Obviously all the above results are still true if for the differential system and in any definition or assumption the t–axis is replaced by an interval $(\tau, +\infty)$, $\tau \in \mathbf{R}$.*

We conclude the section with a simple example which depicts the situation in Remark 3.1. Consider in $(\tau, +\infty) \times \mathbf{R}^2$, $\tau \in \mathbf{R}$, the system:

$$\begin{aligned} \dot{y} &= -2y - [y - exp(-2t)]y^2 + yz^2 \\ \dot{z} &= -z^3(1 + y^2). \end{aligned} \tag{3.16}$$

Since t is varying in an interval bounded from below, the r.h.s. of (3.16) is in $\mathcal{L}_{ub}(y, z)$. The set $\Phi = \{(t, y, z) : t \in (\tau, +\infty), y \in \mathbf{R}, z = 0\}$ is invariant and the y–part of the solutions lying on Φ are the solutions of the equation

$$\dot{y} = -2y - [y - exp(-2t)]y^2. \tag{3.17}$$

Consider the two solutions $y = 0$ and $y = exp(-2t)$ of equation (3.17). These solutions are both uniformly asymptotically stable. The property is evident for $y = 0$. To prove the statement for $y = exp(-2t)$, set $v = y - exp(-2t)$. From (3.17) it follows

$$\dot{v} = -[2 + exp(-4t)]v - 2exp(-2t)v^2 - v^3, \tag{3.18}$$

and then the assert. Relatively to system (3.16), let

$$\begin{aligned} M &= \{(t, y, z) : t \in (\tau, +\infty), y = 0, z = 0\}, \\ M' &= \{(t, y, z) : t \in (\tau, +\infty), y = exp(-2t), z = 0\}, \\ M^\star &= \{(t, y, z) : t \in (\tau, +\infty), y \in [0, exp(-2t)], z = 0\}. \end{aligned}$$

These three sets are all s–compact invariant subsets of Φ (the s–compactness of M' and $M^\star$ is ensured by the restriction assumed on t). Furthermore these sets are uniformly asymptotically stable on Φ. Consider any $(y_0, z_0) \in \mathbf{R}^2$ and $t_0 \in (\tau, +\infty)$ and let (y(t),z(t)) be the corresponding solution. By using $(3.16)_2$ ve find that $|z(t)| \leq |z_0|$ as well as the solution (y(t),z(t)) exists. By Theorem 3.3 and Remark 3.1, the sets M, M' and $M^\star$ are stable. Because these sets are all s–compact, by assuming (y_0, z_0) sufficiently close to M or to M' or to $M^\star$, the solution (y(t),z(t)) exists for all $t \geq t_0$. Hence, by using again $(3.16)_2$, we recognize that Φ is asymptotically stable near each one of these sets. Therefore by Theorem 3.6 and Remark 3.1 we see that these sets are all (unconditionally) asymptotically stable.

4. The periodic case. Let us assume that f of (1.1) is continuous and $f \in \mathcal{L}(x)$. The case in which f and M are both ω–periodic in t for the same constant $\omega > 0$ will be specified as the periodic case. In particular f or M or both may be t–independent. In the periodic case one has: (i) $f \in \mathcal{L}_{ub}(x)$; (ii) the stability and the asymptotic stability of M when occurring are always uniform. Consequently under the conditions (u) or (v) or (w) even the stability or the asymptotic stability of Φ near M when occuring are uniform. Indeed if any of these conditions is satisfied and Φ is stable near M, M is uniformly stable and then Φ is uniformly stable near M by virtue of Theorem 3.4. Similarly one may proceed for asymptotic stability by the aid of Theorem 3.7. By Theorems 3.4, 3.7, it follows that in the periodic case Theorems 3.1, 3.2, 3.3 and Theorems 3.5, 3.6 are invertible. In other words the following theorem holds:

Theorem 4.1 *Suppose that f of (1.1) is continuous and $f \in \mathcal{L}(x)$. Moreover assume that f and M are both ω–periodic in t for the same constant $\omega > 0$. Then, under the conditions (u) or (v) or (w), M is stable (asymptotically stable) if and only if Φ is stable (asymptotically stable) near M.*

The results in [3] for the part concerning the problem of asymptotic stability may be obtained by using Theorem 4.1. For simplicity we consider the case that 0 is an equilibrium and $M = \mathbf{R} \times \{0\}$. Consider the autonomous system

$$\begin{aligned} \dot{y} &= Ay + u(y, z), \\ \dot{z} &= Bz + v(y, z), \end{aligned} \tag{4.1}$$

$y \in \mathbf{R}^m$, $z \in \mathbf{R}^{n-m}$. Here A and B are square matrices, the eigenvalues of A have zero real parts and the eigenvalues of B have negative real parts. Finally u and v are C^2 functions which vanish together with their derivatives at the origin. It is known (see for instance [1], [2]) the existence of a differential system $\mathcal{S}$ associated to (4.1) having the same regularity of (4.1) and such that: (1) $\mathcal{S}$ coincides with (4.1) for $\|y\| < \delta$, $\delta > 0$ small; (2) $\mathcal{S}$ admits an invariant manifold in $\mathbf{R} \times \mathbf{R}^n$, $\Phi = \{(t, y, z) : t \in \mathbf{R}, y \in \mathbf{R}^m, z = g(y)\}$ with $g \in C^2$, $g(0) = 0$. Moreover Φ is exponentially asymptotically stable for $\mathcal{S}$ near $M = \mathbf{R} \times \{0\}$. The set $\Phi^\star = \{(t, y, z) : t \in \mathbf{R}, \|y\| < \delta, z = g(y)\}$ is locally invariant for (4.1). Clearly the unconditional stability properties of M and the stability properties of M on $\Phi^\star$ are preserved when the original system (4.1) is replaced by $\mathcal{S}$ and $\Phi^\star$ is replaced by Φ. Thus the result in [3] relative to the asymptotic stability of equilibrium, and expressed in terms of $\Phi^\star$ and system (4.1), may be stated in terms of the invariant manifold Φ and system $\mathcal{S}$ by saying that for $\mathcal{S}$ the asymptotic stability of M on Φ implies the asymptotic stability of M. Therefore the result is an immediate consequence of Theorem 4.1.

Similarly it may be treated the asymptotic stability problem of a nonautonomous ω–periodic solution $x(t)$ to a ω–periodic differential system. In this case Φ and M are ω–periodic subsets of $\mathbf{R} \times \mathbf{R}^n$ and $M = \{(t, x) : t \in \mathbf{R}, x = x(t)\}$.

We emphasize that in our general results the asymptotic stability of Φ is not supposed to be of exponential type. However we do not have treated here the case that the stability of M on Φ is nonasymptotic. About this case we only observe that the assumption that Φ is exponentially asymptotically stable cannot be in general avoided. This is trivially shown by the following counterexample. Consider the system

$$\begin{aligned} \dot{y} &= yz^2 \\ \dot{z} &= -z^3, \end{aligned} \tag{4.2}$$

with $y, z \in \mathbf{R}$. Clearly any solution of (4.2) exists for all $t \in \mathbf{R}$. Hence, by using $(4.2)_2$, we see that the set $\Phi = \{(t, y, z) : t \in \mathbf{R}, y \in \mathbf{R}, z = 0\}$ is an asymptotically (but nonexponentially) stable invariant manifold in $\mathbf{R} \times \mathbf{R}^2$. With respect to the solutions lying on Φ the origin is stable but nonasymptotically. We prove that the origin is unstable. Indeed $(4.2)_1$ by means of $(4.2)_2$ may be written as

$$\dot{y} = \frac{yz_0^2}{1 + 2z_0^2(t - t_0)}$$

from which it follows

$$y(t, t_0, y_0, z_0) = y_0[1 + 2z_0^2(t - t_0)]^{\frac{1}{2}}.$$

Thus $y(t, t_0, y_0, z_0) \to +\infty$ as $t \to +\infty$ for any choice of $y_0 \neq 0, z_0 \neq 0$. Hence our assert follows.

We conclude by a simple example concerning the case that M does not consist of a single trajectory. Consider the following system:

$$\begin{aligned} \dot{y} &= ycost + 3yexp(sint) - 3y^2 + yz \\ \dot{z} &= -\lambda z^3(1 + y^2), \end{aligned} \tag{4.3}$$

were $t, y, z, \lambda \in \mathbf{R}$ and λ is a constant. The set $\Phi = \{(t, y, z) : t \in \mathbf{R}, y \in \mathbf{R}, z = 0\}$ is invariant. The y–part of the solutions lying on Φ are the solutions of the equation

$$\dot{y} = [cost + 3exp(sint)]y - 3y^2. \tag{4.4}$$

Equation (4.4) admits the equilibrium position $y = 0$ and the periodic solution $y = exp(sint)$. Clearly: (i) $y = 0$ is asymptotically stable on the left and unstable on the right; (ii) the solution $y = exp(sint)$ is asymptotically stable. To prove (ii) it is sufficient to show that setting $v = y - exp(sint)$ from (4.4) it follows

$$\dot{v} = -[3exp(sint) - cost]v - 3v^2.$$

Relatively to system (4.3) , consider the two sets

$$M^\star = \{(t, y, z) : t \in \mathbf{R}, y = exp(sint), z = 0\},$$
$$M = \{(t, y, z) : t \in \mathbf{R}, y \in [0, exp(sint)], z = 0\}.$$

These two sets are both s–compact invariant subsets of Φ and asymptotically stable on Φ. Moreover, by using the same arguments employed for the example in Section 3, we see that Φ is asymptotically stable (stable, unstable) near $M^\star$ and near M if $\lambda > 0$ ($\lambda = 0$, $\lambda < 0$). Then by using Theorem 4.1 we see that $M^\star$ and M are both (unconditionally) asymptotically stable for $\lambda > 0$, stable for $\lambda = 0$, and unstable for $\lambda < 0$.

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