COMPARISON AMONG SOME OPTIMAL POLICIES WITH UNCONSTRAINED ADDITIONAL ORDERS

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Abstract. This paper proposes three policies we can consider with unconstrained additional orders: (i) a policy in which additional orders are placed if the stock level is negative at the fixed investigative time; (ii) a policy without additional orders; (iii) a policy in which additional orders are always placed at the investigative time. We will refer to the optimality in each of these policies and clarify the order among the optimal quantities related to initial orderings and additional ones.

1 Introduction In a stochastic inventory model shortage of products happens if initial order quantity is less than total demands. To stave off these losses, we can consider additional ordering within the quantity of shortages. In inventory studies [3], [4], and [6] having already investigated, additional orders take place so as to replenish a part of or all shortages. Removing a constrained condition to order within the quantity of shortages, we can hope for cutting down total costs when the cost charged by holding stock is cheaper than one charged by being sold out.

In this article we suggest a new inventory model with unconstrained quantity of additional orders and explore the optimal solution for minimizing the expected total cost. Section 2 describes a stochastic inventory model permitted unconstrained additional orders. Section 3 explains three policies which we can consider with respect to additional orders and gives their expected total cost functions. We are interested in investigating the order among the optimal quantities related to initial orderings and additional ones. Section 4 shows our main results.

2 Model

2.1 Model Description We examine a stochastic inventory model with shortages and unconstrained additional ordering for a single planning period and a single commodity. Our model has the following assumptions: Let t be the length of a planning period. Orderings take place at the beginning of a period and, if it needs, at fixed investigative time t_0, $0 \leq t_0 \leq t$. We assume that the initial stock level is equal to zero without loss of generality. The ordered commodities are received with zero lead-time. Then the stock level after the initial ordering reaches S units to be depleted to meet demands for customers. The stock gradually decreases by meeting their demands for a planning period and its level is investigated at time t_0. If the initial order quantity is less than the cumulative quantity of demands in $[0, t_0)$, the quantity s may be additionally ordered because of shortages. The quantity of an additional order, s, is exhausted to be backlogged for shortages in the interval $[0, t_0)$ and, if it is received more than the quantity of shortages, to meet future demands. The number of times of additional orderings is restricted to once only. There is no stock carried over to the next period.

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The following are the relevant costs. Let c_1 and c_2 be the unit ordering cost charged in the initial orders and the one charged in the additional orders, respectively. The commodities are sold at the selling price r per unit. The holding cost per unit time is h for each unit on-hand inventory carried over. On the other hand, the penalty cost of p per unit per unit time is incurred for shortage.

The quantity of total demands during the planning period is a random variable, which is denoted by B. The value b is given as one of values in random variable B. The cumulative demand at any time T is given as a function, denoted by $g(T/t)b$, of the quantity of total demands b, where $g(x)$ is a continuous increasing function of x defined on $0 \leq x \leq 1$ such that $g(0) = 0$ and $g(1) = 1$. Let $G(x) = \int_0^x g(y)dy$.

We give the following assumptions for parameters in relation to costs and quantity.

$$\begin{cases} S \geq 0, \quad s \geq 0, \quad b \geq 0 \\ h > 0, \quad p > 0, \quad 0 < c_1 \leq c_2 \leq r \\ (1 - t_0/t)p + r - c_2 \geq 0 \end{cases}$$

The inequalities in the first and second rows are given under natural assumptions and the last implies that the additional ordering is more efficient than no ordering.

2.2 Objective When a manager has to choose his policy from the following policies at the beginning of a planning period, he needs to compare the expected total costs for three policies:

- Policy (S) in which additional orders are placed by s if the stock level is negative at time t_0
- Policy (N) without additional orders
- Policy (A) in which additional orders are always placed by s at time t_0

The objective of this paper is to investigate properties for the optimal ordering quantity and the optimal additional ordering quantity under the minimization of the expected total costs, and to examine the relation of the optimal orders in these policies.

3 Problem Formulation This section identifies the properties of three policies. For the purpose, we begin with calculating total cost functions in some cases and their expectations.

3.1 Policy (S) Policy (S) can be expressed by the model described in Section 2.1. In this model we observe five inventory situations for the different values of demand b as follows.

Case (I): If the quantity of total demands is less than or equal to the initial order quantity, that is, demand b satisfies $0 \leq b \leq S$, S units ordered at time 0 meet all demands in a planning period. The stock level is always nonnegative at any time T, $0 \leq T \leq t$. Therefore this situation does not have additional orders.

Case (II): We next consider the situation in which the quantity of total demands b is given by $S < b \leq S/g(t_0/t)$. The stock level keeps positive at investigate time t_0, and the additional ordering does not take place. Because of excess demands, however, the stock level drops down into the situation of shortages between times t_0 and t.

Case (III): When the quantity of total demands b satisfies $b > (S + s)/g(t_0/t)$, shortages at investigative time t_0 cause to s units of additional orders. The commodities replenished by additional ordering backlog a part of excess demands having appeared before time t_0. Because the additional quantity s is fewer rather than cumulative quantity of shortages at time t_0, the stock level after that time always remains negative.

Case (IV): Given the quantity of total demands b satisfying $\max\{S/g(t_0/t), S+s\} < b \leq (S+s)/g(t_0/t)$, shortages occur by time t_0 and s units of additional orders are made at that time. These additional commodities backlog all of excess demands having appeared before time t_0. Moreover, the stock level goes up to be positive after receiving additional commodities. The stock level, however, drops down into the situation of shortages by the end of the planning period because of sequential consumption again.

Case (V): At last, we consider the situation in which the quantity of total demands b is given by $S/g(t_0/t) < b \leq S+s$. Shortages by investigative time t_0 lead to s units of additional orders at that time. As the additional order quantity s is much enough, the additional commodities backlog all of excess demands having appeared before time t_0 and meet all demands from time t_0 to the end of the planning period. Therefore, the stock level always keeps to be nonnegative after additional ordering. Of course, this case exsits only if it holds that $S/g(t_0/t) < S+s$.

In these cases the stock levels at any time T, denoted by $Q(T)$, can be expressed by

$$Q(T) = S - g(T/t)b, \quad 0 \leq T \leq t \quad \text{for Cases (I), (II)} \tag{1}$$

and

$$Q(T) = \begin{cases} S - g(T/t)b, & 0 \leq T < t_0 \\ S + s - g(T/t)b, & t_0 \leq T \leq t \end{cases} \quad \text{for Cases (III), (IV), and (V).} \tag{2}$$

Letting $C(S,s;b)$ and $C_i(S,s;b), i = 1,\dots,5$ be the total cost function for a given b and ones corresponding to each of inventory situations described above, respectively, we obtain the following equation:

$$C(S,s;b) = \begin{cases} C_1(S,s;b), & \text{for } \ 0 \leq b \leq S \\ C_2(S,s;b), & \text{for } \ S < b \leq S/g(t_0/t) \\ C_3(S,s;b), & \text{for } \ b > (S+s)/g(t_0/t) \\ C_4(S,s;b), & \text{for } \ \max\{S/g(t_0/t), S+s\} < b \leq (S+s)/g(t_0/t) \\ C_5(S,s;b), & \text{for } \ S/g(t_0/t) < b \leq S+s, \end{cases}$$

which is given by

$$\begin{aligned}
C_1(S,s;b) &= [c_1 + h]S - [hG(1) + r]b, \\
C_2(S,s;b) &= [c_1 - p - r]S + (h+p)\{Sg^{-1}(S/b) - bG(g^{-1}(S/b))\} + pbG(1), \\
C_3(S,s;b) &= [c_1 - p - r]S + (h+p)\{Sg^{-1}(S/b) - bG(g^{-1}(S/b))\} + pbG(1) \\
&\quad +[c_2 - r - p(1 - t_0/t)]s, \\
C_4(S,s;b) &= [c_1 - p - r]S + (h+p)[S\{g^{-1}(S/b) + g^{-1}((S+s)/b) - t_0/t\} \\
&\quad +sg^{-1}((S+s)/b) - b\{G(g^{-1}(S/b)) + G(g^{-1}((S+s)/b)) - G(t_0/t)\}] \\
&\quad +pbG(1) - [ht_0/t + p + r - c_2]s, \\
C_5(S,s;b) &= [c_1 - pt_0/t + h(1 - t_0/t)]S + [h(1 - t_0/t) + c_2]s - [hG(1) + r]b \\
&\quad +(h+p)[Sg^{-1}(S/b) + b\{G(t_0/t) - G(g^{-1}(S/b))\}].
\end{aligned}$$

See Appendix for these details on calculations.

Letting $E[C(S,s;B)]$ denote the expectation of the total cost $C(S,s;b)$ with respect to b, we can represent it as

$$E[C(S,s;B)] = \begin{cases} E_1[C(S,s;B)], & S + s \leq S/g(t_0/t) \\ E_2[C(S,s;B)], & S + s > S/g(t_0/t), \end{cases}$$

which is calculated by

$$\begin{aligned} E_1[C(S,s;B)] &= \int_0^S C_1(S,s;b)\phi(b)db + \int_S^{S/g(t_0/t)} C_2(S,s;b)\phi(b)db \\ &\quad + \int_{S/g(t_0/t)}^{(S+s)/g(t_0/t)} C_4(S,s;b)\phi(b)db + \int_{(S+s)/g(t_0/t)}^{\infty} C_3(S,s;b)\phi(b)db, \\ E_2[C(S,s;B)] &= \int_0^S C_1(S,s;b)\phi(b)db + \int_S^{S/g(t_0/t)} C_2(S,s;b)\phi(b)db \\ &\quad + \int_{S/g(t_0/t)}^{S+s} C_5(S,s;b)\phi(b)db + \int_{S+s}^{(S+s)/g(t_0/t)} C_4(S,s;b)\phi(b)db \\ &\quad + \int_{(S+s)/g(t_0/t)}^{\infty} C_3(S,s;b)\phi(b)db. \end{aligned}$$

Then the following result is obtained.

Proposition 1. In Policy (S) there exists the unique optimal solution $s = s^*$ satisfying $\frac{\partial}{\partial s}E[C(S,s;B)] = 0$ among $(0,\infty)$ and the optimal solution $S = S^*$ exists among $[0,\infty)$.

Proof. For a fixed S, $E_1[C(S,s;B)]$ and $E_2[C(S,s;B)]$ are continuous convex functions with respect to s. They satisfy

$$\frac{\partial}{\partial s}E_1[C(S,s;B)]\bigg|_{s=0} < 0 \quad \text{and} \quad \lim_{s\to+\infty}\frac{\partial}{\partial s}E_2[C(S,s;B)] > 0.$$

Also, $E_1[C(S,s;B)]$ and $E_2[C(S,s;B)]$ are continuous functions of S for a fixed s, however their derivatives are not continuous on the line $S + s = s/g(t_0/t)$. Furthermore, the value of $E_2[C(S,s;B)]$ is bounded at $S = 0$ and it tends to $+\infty$ as $S \to +\infty$. These statements give the result of Proposition 1.

3.2 Policy (N) We next consider another policy, say Policy (N), without additional ordering at time t_0. That is, a single ordering just takes place at the beginning of a planning period. Since Policy (N) is constructed by Cases (I) and (II) having described in the preceding subsection, the expected total cost in this policy, denoted by $E_3[C(S,s;B)]$, is calculated as

$$\begin{aligned} E[C(S,s;B)] &= E_3[C(S,s;B)] \\ &= \int_0^S C_1(S,s;b)\phi(b)db + \int_S^{\infty} C_2(S,s;b)\phi(b)db. \end{aligned}$$

The convexity of function $E_3[C(S,s;B)]$ gives the following proposition.

Proposition 2. In Policy (N) there uniquely exists the optimal solution $S = S^0$ satisfying $\frac{d}{dS}E[C(S,s;B)] = 0$ among $(0,\infty)$.

3.3 Policy (A) At last, we discuss the other policy, say Policy (A), in which additional ordering always takes place whether the stock level is nonpositive or not at time t_0. Thus, two orderings certainly take place: the initial ordering at time 0 and the additional ordering at investigative time t_0. In Policy (A), there appear the following situations instead of Cases (I), (II) of Policy (S).

Case (I'): Consider the situation in which the quantity of total demands b satisfies $0 \le b \le \min\{S/g(t_0/t), S+s\}$. Although the stock level keeps to be nonnegative at time t_0, the additional ordering takes place at that time. The total of quantity of initial ordering and additional ordering, $S+s$, meets all of demands for a planning period. The stock level always keeps to be nonnegative.

Case (II'): We next consider the situation in which the quantity of total demands b satisfies $S+s<b\leq S/g(t_0/t)$. The additional ordering takes place at time t_0. Since the quantity $S+s$ is not as much as the total demands, the stock level drops down into the situation of shortages before the end of a planning period. Of course, this case exists only if it holds that $S+s<S/g(t_0/t)$.

For Cases (I') and (II'), the stock level at any time T can be expressed by Eq.(2). The total costs corresponding to these situations, $C_i(S,s;b), i=1',2'$, are given as follows:

$$\begin{aligned} C_{1'}(S,s;b) &= [c_1+h]S+[h(1-t_0/t)+c_2]s-[hG(1)+r]b, \\ C_{2'}(S,s;b) &= [c_1-p-r]S+[c_2-ht_0/t-p-r]s+(h+p)(S+s)g^{-1}((S+s)/b) \\ &\quad +\{pG(1)-(h+p)G(g^{-1}((S+s)/b))\}b. \end{aligned}$$

Then the expectation of the total cost $C(S,s;b)$ for Policy (A), $E[C(S,s;B)]$, is calculated by

$$E[C(S,s;B)]=\begin{cases} E_4[C(S,s;B)], & S+s\leq S/g(t_0/t) \\ E_5[C(S,s;B)], & S+s>S/g(t_0/t), \end{cases}$$

where

$$\begin{aligned} E_4[C(S,s;B)] &= \int_0^{S+s} C_{1'}(S,s;b)\phi(b)db+\int_{S+s}^{S/g(t_0/t)} C_{2'}(S,s;b)\phi(b)db \\ &\quad +\int_{S/g(t_0/t)}^{(S+s)/g(t_0/t)} C_4(S,s;b)\phi(b)db+\int_{(S+s)/g(t_0/t)}^{\infty} C_3(S,s;b)\phi(b)db, \\ E_5[C(S,s;B)] &= \int_0^{S/g(t_0/t)} C_{1'}(S,s;b)\phi(b)db+\int_{S/g(t_0/t)}^{S+s} C_5(S,s;b)\phi(b)db \\ &\quad +\int_{S+s}^{(S+s)/g(t_0/t)} C_4(S,s;b)\phi(b)db+\int_{(S+s)/g(t_0/t)}^{\infty} C_3(S,s;b)\phi(b)db. \end{aligned}$$

Then the following result is obtained.

Proposition 3. In Policy (A), the unique optimal solutions $S=S^1$ and $s=s^1$ exist among $[0,\infty)$, respectively.

Proof. These equations lead us to the equality $E_4[C(S,s;B)]=E_5[C(S,s;B)]$. This function is continuous and convex in S for a fixed s. Also, it is continuous convex function of s for a fixed S. Furthermore, it is bounded at $S=0$ and $s=0$, and it satisfies

$$\lim_{S\to+\infty}\frac{\partial}{\partial S}E_4[C(S,s;B)]>0, \quad \text{and} \quad \lim_{s\to+\infty}\frac{\partial}{\partial s}E_5[C(S,s;B)]>0.$$

These statements give the result of Propostion 3.

4 Relation of ordering size With this background in place, we can establish our main results. This section investigates the relation of ordering size among three types of policies. First we show the relation between the optimal additional ordering quantities s^* of Policy (S) and s^1 of Policy (A).

Theorem 1. Suppose that S units of initial ordering are made on both policies (S) and

(A). Then we obtain the following results.
(i) If s^* satisfies $S + s^* \leq S/g(t_0/t)$, then

$$\frac{\partial}{\partial s} E_4[C(S, s; B)]\bigg|_{s=s^*} \begin{cases} < 0 & \Rightarrow \quad s^* < s^1 \\ = 0 & \Rightarrow \quad s^* = s^1 \\ > 0 & \Rightarrow \quad s^* > s^1 \end{cases}$$

(ii) If s^* satisfies $S + s^* > S/g(t_0/t)$, then

$$s^1 \leq s^*.$$

Proof. From Proposition 1, Policy (S) uniquely has an optimal solution s^* on functions $E_1[C(S, s; B)]$ or $E_2[C(S, s; B)]$. We now consider two ranges with respect to s^* for a fixed S:

(i) Suppose that it follows $S + s^* \leq S/g(t_0/t)$. Then the optimal solution s^* satisfies $\frac{\partial}{\partial s} E_1[C(S, s; B)]\Big|_{s=s^*} = 0$. Substituting the value s^* for the s-partial derivative of function $E_4[C(S, s; B)]$, we obtain

$$\begin{aligned} \frac{\partial}{\partial s} E_4[C(S, s; B)]\bigg|_{s=s^*} &= [(1 - t_0/t)\, h + c_2] \int_0^{S+s^*} \phi(b)db \\ &\quad - [(1 - t_0/t)\, p + r - c_2] \int_{S+s^*}^{S/g(t_0/t)} \phi(b)db \\ &\quad + (h + p) \int_{S+s^*}^{S/g(t_0/t)} \left\{g^{-1}\left((S + s^*)/b\right) - t_0/t\right\} \phi(b)db. \end{aligned}$$

Thus from the optimality of s^1, the former of Theorem 1 is derived.

(ii) Suppose that it follows $S + s^* > S/g(t_0/t)$. Then the optimal solution s^* satisfies that the partial derivative $\frac{\partial}{\partial s} E_2[C(S, s; B)]$ is equal to zero. By substituting this value for the s-partial derivative of function $E_5[C(S, s; B)]$ and using the statement of Proposition 3, we have

$$\frac{\partial}{\partial s} E_5[C(S, s; B)]\bigg|_{s=s^*} = [(1 - t_0/t)\, h + c_2] \int_0^{S/g(t_0/t)} \phi(b)db \;\geq\; 0.$$

This completes the proof of Theorem 1.

Next we show the relation among the optimal initial order quantities S^0 of Policy (N), S^* of Policy (S), and S^1 of Policy (A).

Theorem 2. Suppose that s units of additional ordering are made on both policies (S) and (A). For a given s, we define

$$\begin{aligned} A(S^*, s) &= r \int_{S^*}^{S^*+s} \phi(b)db + (h + p) \int_{S^*}^{S^*+s} \left\{1 - g^{-1}\left(S^*/b\right)\right\} \phi(b)db \\ &\quad + (h + p) \int_{S^*+s}^{S^*/g(t_0/t)} \left\{g^{-1}\left((S^* + s)/b\right) - g^{-1}\left(S^*/b\right)\right\} \phi(b)db, \\ B(S^*, s) &= (h + p) \int_{S^*/g(t_0/t)}^{(S^*+s)/g(t_0/t)} \left\{g^{-1}\left((S^* + s)/b\right) - \frac{t_0}{t}\right\} \phi(b)db, \\ M(S^*, s) &= \left\{(h + p) \int_{t_0/t}^{g^{-1}((S^*+s)/S^* g(t_0/t))} \left\{S^* + s - g(x)S^*/g(t_0/t)\right\} dx \right. \\ &\quad \left. - [(1 - t_0/t)\, p + r - c_2]\, s\right\} \phi\left(S^*/g(t_0/t)\right)/g(t_0/t) \end{aligned}$$

Then we obtain the following results.
(I) For $S^* = 0$:

$$0 = S^* \le S^1 \le S^0$$

(II) For $S^* \neq 0$:
(1) If S^* and s satisfy $S^* + s \le S^*/g(t_0/t)$, then
(i) $M(S^*, s) \le -A(S^*, s) \quad \Rightarrow \quad S^* \le S^1 \le S^0$,
(ii) $-A(S^*, s) < M(S^*, s) < B(S^*, s) \quad \Rightarrow \quad S^1 < S^* < S^0$,
(iii) $M(S^*, s) \ge B(S^*, s) \quad \Rightarrow \quad S^1 \le S^0 \le S^*$,
(2) If S^* and s satisty $S^* + s > S^*/g(t_0/t)$, then

$$S^1 \le S^* \le S^0$$

Proof. We first show the relation between the optimal order quantities S^0 of Policy (N) and S^1 of Policy (A).

(a) Suppose that $S^0 + s \le S^0/g(t_0/t)$ for a fixed s. Then the optimal solution S^0 satisfies that the derivative $\frac{d}{dS}E_3[C(S, s; B)]$ is equal to zero. By substituting the value S^0 for the S-partial derivative of function $E_4[C(S, s; B)]$ and arranging it, we obtain

$$\frac{\partial}{\partial S}E_4[C(S, s; B)]\bigg|_{S=S^0} \ge 0.$$

(b) Suppose that $S^0 + s > S^0/g(t_0/t)$ for given s. As in above discriptions, we substitute the equation $\frac{d}{dS}E_3[C(S, s; B)] = 0$ for the S-partial derivative of function $E_4[C(S, s; B)]$. Then we have

$$\frac{\partial}{\partial S}E_4[C(S, s; B)]\bigg|_{S=S^0} \ge 0.$$

Therefore, it follows $S^1 \le S^0$ for any given s.

Next, we show the relation between S^* and S^1, that is the initial order quantity of Policy (S) and one of Policy (A).

(a) Suppose that $S^* + s \le S^*/g(t_0/t)$ for a fixed s. We assume that there exists an optimal solution S^* satisfying $\frac{\partial}{\partial S}E_1[C(S, s; B)] = 0$. Substituting this value for the S-partial derivative of function $E_4[C(S, s; B)]$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial S}E_4[C(S, s; B)]\bigg|_{S=S^*} \\ &= r\int_{S^*}^{S^*+s} \phi(b)db + (h+p)\int_{S^*}^{S^*+s} \left\{1 - g^{-1}\left(S^*/b\right)\right\}\phi(b)db \\ &\quad + (h+p)\int_{S^*+s}^{S^*/g(t_0/t)} \left\{g^{-1}\left((S^*+s)/b\right) - g^{-1}\left(S^*/b\right)\right\}\phi(b)db \\ &\quad + \left\{(h+p)\int_{t_0/t}^{g^{-1}((S^*+s)/S^* g(t_0/t))} \left\{S^* + s - g(x)S^*/g(t_0/t)\right\}dx \right. \\ &\quad \left. - \left[(1 - t_0/t)\,p + r - c_2\right] s\right\}\phi\left(S^*/g(t_0/t)\right)/g(t_0/t). \end{aligned} \tag{3}$$

Then it leads to the following results.

$$\frac{\partial}{\partial S}E_4[C(S, s; B)]\bigg|_{S=S^*} \begin{cases} < 0 & \Rightarrow \quad S^* < S^1 \\ = 0 & \Rightarrow \quad S^* = S^1 \\ > 0 & \Rightarrow \quad S^* > S^1. \end{cases}$$

If the last term of Eq.(3) is non-negative, it always holds $S^1 \le S^*$.

(b) Suppose that $S^* + s > S^*/g(t_0/t)$ for a fixed s. We assume that there exists an optimal solution S^* satisfying $\frac{\partial}{\partial S}E_2[C(S,s;B)] = 0$. Substituting this value for the S-partial derivative of function $E_4[C(S,s;B)]$, we obtain

$$\frac{\partial}{\partial S}E_4[C(S,s;B)]\bigg|_{S=S^*} \ge 0.$$

Therefore it holds $S^1 \le S^*$.

At last, we show the relation between S^* and S^0, that is the initial order quantity of Policy (S) and one of Policy (N).

(a) Suppose that $S^* + s \le S^*/g(t_0/t)$ for a fixed s. We assume that there exists an optimal solution S^* satisfying $\frac{\partial}{\partial S}E_1[C(S,s;B)] = 0$. Substituting this value for the derivative of function $E_3[C(S,s;B)]$, we obtain

$$\begin{aligned} &\frac{d}{dS}E_3[C(S,s;B)]\bigg|_{S=S^*} \\ &= -(h+p)\int_{S^*/g(t_0/t)}^{(S^*+s)/g(t_0/t)} \left\{ g^{-1}\left((S^*+s)/b\right) - \frac{t_0}{t} \right\} \phi(b)db \\ &\quad + \left\{ (h+p)\int_{t_0/t}^{g^{-1}((S^*+s)/S^* g(t_0/t))} \{S^* + s - g(x)S^*/g(t_0/t)\}\, dx \right. \\ &\quad \left. - \left[(1-t_0/t)\,p + r - c_2\right] s \right\} \phi\left(S^*/g(t_0/t)\right)/g(t_0/t). \end{aligned} \tag{4}$$

Then we obtain the following results.

$$\frac{d}{dS}E_3[C(S,s;B)]\bigg|_{S=S^*} \begin{cases} < 0 & \Rightarrow \quad S^* < S^0 \\ = 0 & \Rightarrow \quad S^* = S^0 \\ > 0 & \Rightarrow \quad S^* > S^0. \end{cases}$$

If the last term of Eq.(4) is non-positive, it follows $S^* \le S^0$.

(b) Suppose that $S^* + s > S^*/g(t_0/t)$ for a fixed s. We assume that there exists an optimal solution S^* satisfying $\frac{\partial}{\partial S}E_2[C(S,s;B)] = 0$. Substituting this value for the derivative of function $E_3[C(S,s;B)]$, we obtain

$$\frac{d}{dS}E_3[C(S,s;B)]\bigg|_{S=S^*} \le 0,$$

which implies $S^* \le S^0$.

These statements completes the proof of Theorem 2.

5 Concluding Remarks This paper suggested a stochastic inventory model with unconstrained additional orders and provided a mathematical formulation. We explained three policies we can consider for additional orders and searched on the optimality in three considerable policies. It showed that each of the optimal quantities of orderings without initial one of Policy (S) has uniqueness. We also clarified the order among the optimal quantities related to initial orderings and additional ones.

Appendix. It is indispensable for our study to calculate the expectation of the total costs. We give the total cost functions for all cases mentioned in Section 3 as follows.

Case (I): In Case (I) of Policy (S), the stock level $Q(T)$ is given by Eq.(1). Then the average quantity I_1 is calculated by

$$\begin{aligned} I_1 &= \frac{1}{t}\int_0^t \{S - g(T/t)b\}dT \\ &= S - G(1). \end{aligned}$$

Because this situation has no shortage, the average shortage quantity I_2 is equal to zero. The initial order quantity is S and the number of commodities sold is b. Hence the total cost $C_1(S, s; b)$ is given by

$$\begin{aligned} C_1(S, s; b) &= c_1 S + hI_1 + pI_2 - rb \\ &= [c_1 + h]S - [hG(1) + r]b. \end{aligned}$$

Case (II): The average quantity I_1 and the average shortage quantity I_2 are calculated by

$$\begin{aligned} I_1 &= \frac{1}{t}\int_0^{tg^{-1}(S/b)} \{S - g(T/t)b\}dT \\ &= Sg^{-1}(S/b) - bG(g^{-1}(S/b)) \end{aligned} \tag{5}$$

and

$$\begin{aligned} I_2 &= \frac{1}{t}\int_{tg^{-1}(S/b)}^t \{g(T/t)b - S\}dT \\ &= b[G(1) - G(g^{-1}(S/b))] - S[1 - g^{-1}(S/b)]. \end{aligned}$$

The initial order quantity is S and the number of commodities sold is S. Hence the total cost $C_2(S, s; b)$ is given by

$$\begin{aligned} C_2(S, s; b) &= c_1 S + hI_1 + pI_2 - rS \\ &= [c_1 - p - r]S + (h + p)\{Sg^{-1}(S/b) - bG(g^{-1}(S/b))\} + pbG(1). \end{aligned}$$

Case (III): The average quantity I_1 is given by Eq.(5). On the other hand, the average shortage quantity I_2 is calculated by

$$\begin{aligned} I_2 &= \frac{1}{t}\int_{tg^{-1}(S/b)}^{t_0} \{g(T/t)b - S\}dT + \frac{1}{t}\int_{t_0}^t \{g(T/t)b - S - s\}dT \\ &= Sg^{-1}(S/b) - G(g^{-1}(S/b))b + G(1)b - S - \left(1 - \frac{t_0}{t}\right)s. \end{aligned}$$

The initial order quantity is S and the number of commodities sold is $S + s$. Hence the total cost $C_3(S, s; b)$ is given by

$$\begin{aligned} C_3(S, s; b) &= c_1 S + hI_1 + pI_2 - r(S + s) \\ &= [c_1 - p - r]S + (h + p)\{Sg^{-1}(S/b) - bG(g^{-1}(S/b))\} + pbG(1) \\ &\quad + [c_2 - r - p(1 - t_0/t)]s. \end{aligned}$$

Case (IV): Under the stock level $Q(T)$ represented by Eq.(2), the average quantity I_1 and the average shortage quantity I_2 are calculated by

$$I_1 = \frac{1}{t}\int_0^{tg^{-1}(S/b)} \{S - g(T/t)b\}dT + \frac{1}{t}\int_{t_0}^{tg^{-1}((S+s)/b)} \{S + s - g(T/t)b\}dT$$

$$
\begin{aligned}
&= \{g^{-1}(S/b)+g^{-1}((S+s)/b)-t_0/t\}S+\{g^{-1}((S+s)/b)-t_0/t\}s \\
&\quad -\{G(g^{-1}(S/b))+G(g^{-1}((S+s)/b))-G(t_0/t)\}b
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{t}\int_{tg^{-1}(S/b)}^{t_0}\{g(T/t)b-S\}dT+\frac{1}{t}\int_{tg^{-1}((S+s)/b)}^{t}\{g(T/t)b-S-s\}dT \\
&= \{g^{-1}(S/b)+g^{-1}((S+s)/b)-1-t_0/t\}S-\{1-g^{-1}((S+s)/b)\}s \\
&\quad +\{G(1)+G(t_0/t)-G(g^{-1}((S+s)/b))-G(g^{-1}(S/b))\}b.
\end{aligned}
$$

The initial ordering and the additional ordering are S units and s units, respectively. The number of commodities sold is $S+s$. Hence the total cost $C_4(S,s;b)$ is given by

$$
\begin{aligned}
C_4(S,s;b) &= c_1S+hI_1+pI_2-r(S+s)+c_2s \\
&= [c_1-p-r]S+(h+p)[S\{g^{-1}(S/b)+g^{-1}((S+s)/b)-t_0/t\} \\
&\quad +sg^{-1}((S+s)/b)-b\{G(g^{-1}(S/b))+G(g^{-1}((S+s)/b))-G(t_0/t)\}] \\
&\quad +pbG(1)-[ht_0/t+p+r-c_2]s.
\end{aligned}
$$

Case (V): The average quantity I_1 and the average shortage quantity I_2 are calculated by

$$
\begin{aligned}
I_1 &= \frac{1}{t}\int_0^{tg^{-1}(S/b)}\{S-g(T/t)b\}dT+\frac{1}{t}\int_{t_0}^{t}\{S+s-g(T/t)b\}dT \\
&= Sg^{-1}(S/b)-bG(g^{-1}(S/b))+(1-t_0/t)(S+s)-\{G(1)-G(t_0/t)\}b
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{t}\int_{tg^{-1}(S/b)}^{t_0}\{g(T/t)b-S\}dT \\
&= \{g^{-1}(S/b)-t_0/t\}S+\{G(t_0/t)-G(g^{-1}(S/b))\}b.
\end{aligned}
$$

The initial ordering and the additional ordering are S units and s units, respectively. The number of commodities sold is b. Hence the total cost $C_5(S,s;b)$ is given by

$$
\begin{aligned}
C_5(S,s;b) &= c_1S+hI_1+pI_2+c_2s-rb \\
&= [c_1-pt_0/t+h(1-t_0/t)]S+[h(1-t_0/t)+c_2]s-[hG(1)+r]b \\
&\quad +(h+p)[Sg^{-1}(S/b)+b\{G(t_0/t)-G(g^{-1}(S/b))\}].
\end{aligned}
$$

Case (I'): In Case (I') of Policy (A), the stock level $Q(T)$ is given by Eq.(2). Then the average quantity I_1 is calculated by

$$
\begin{aligned}
I_1 &= \frac{1}{t}\int_0^{t_0}\{S-g(T/t)b\}dT+\frac{1}{t}\int_{t_0}^{t}\{S+s-g(T/t)b\}dT \\
&= S-G(1)b+(1-t_0/t)s.
\end{aligned}
$$

Because this situation has no shortage, the average shortage quantity I_2 is equal to zero. The initial ordering and the additional ordering are S units and s units, respectively. The number of commodities sold is b. Hence the total cost $C_{1'}(S,s;b)$ is given by

$$
\begin{aligned}
C_{1'}(S,s;b) &= c_1S+hI_1+pI_2-rb+c_2s \\
&= [c_1+h]S+[h(1-t_0/t)+c_2]s-[hG(1)+r]b.
\end{aligned}
$$

Case (II'): The average quantity I_1 and the average shortage quantity I_2 are calculated by

$$\begin{aligned} I_1 &= \frac{1}{t}\int_0^{t_0}\{S - g(T/t)b\}dT + \frac{1}{t}\int_{t_0}^{g^{-1}((S+s)/b)t}\{S + s - g(T/t)b\}dT \\ &= Sg^{-1}((S+s)/b) - G(g^{-1}((S+s)/b))b + \{g^{-1}((S+s)/b) - t_0/t\}s \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{t}\int_{g^{-1}((S+s)/b)t}^{t}\{g(T/t)b - S - s\}dT \\ &= (g^{-1}((S+s)/b) - 1)(S+s) + (G(1) - G(g^{-1}((S+s)/b)))b. \end{aligned}$$

The initial ordering and the additional ordering are S units and s units, respectively. The number of commodities sold is $S+s$. Hence the total cost $C_{2'}(S,s;b)$ is given by

$$\begin{aligned} C_{2'}(S,s;b) &= c_1S + hI_1 + pI_2 - r(S+s) + c_2s \\ &= [c_1 - p - r]S + [c_2 - ht_0/t - p - r]s + (h+p)(S+s)g^{-1}((S+s)/b) \\ &\quad + \{pG(1) - (h+p)G(g^{-1}((S+s)/b))\}b. \end{aligned}$$

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