UNIQUENESS OF MEROMORPHIC FUNCTIONS *

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Abstract. In this paper, we deal with the uniqueness problems on meromorphic functions concerning differential polynomials that share a small meromorphic function. Moreover, we improve some former results.

1 Introduction and Results In this paper, we assume all the functions are nonconstant meromorphic functions in the complex plane C. We shall use the standard notations of Nevanlinna theory of meromorphic functions such as $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), S(r,f), etc. \cdots$.

It is well known that if f and g share four distinct values CM, then f is a Möbius transformation of g. Recently, corresponding to one famous question of Hayman [1], many uniqueness theorems for some certain types of differential polynomials sharing one value were obtained (See [2, 3, 4, 5]).

In 2001, M. Fang and W. Hong proved:

Theorem A [3]. Let f and g be two transcendental entire functions, $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.

Afterwards, W. Lin and H. X. Yi improved Theorem A and obtained the following results:

Theorem B [4]. Let f and g be two transcendental entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f(z) \equiv g(z)$.

Theorem C [4]. Let f and g be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then either $f(z) \equiv g(z)$ or $f = \{(n+2)h(1-h^{n+1})\}/\{(n+1)(1-h^{n+2})\}$ and $g = \{(n+2)(1-h^{n+1})\}/\{(n+1)(1-h^{n+2})\}$, where h is a nonconstant meromorphic function.

Recently, W. Lin and H. X. Yi extended Theorem B and Theorem C concerning to fix-points (See [5]).

In this paper, some uniqueness problems of meromorphic functions are investigated, which are improvement and complementary results for the above theorems.

Throughout this paper, we use the following notations:

Let $E(f) = \{z | f(z) = 0\}$, where a zero of f with multiplicity m is counted m times. If $E(f-\alpha) = E(g-\alpha)$, then we say that f and g share α CM, especially, we say that $f(z)$ and $g(z)$ have the same fixed-points if $\alpha(z) = z$. Let $E_{k)}(f) = \{z|$ zeros of $f(z)$ with multiplicity at most $k\}$, where a zero with multiplicity $m(\leq k)$ is counted m times. Obviously, if $E(f) = E(g)$, then $E_{k)}(f) = E_{k)}(g)$, for $k = 1, 2, \cdots$.

Let f be a meromorphic function. We denote by $n_{k)}(r,f)$ the number of poles of f with multiplicity at most k in $|z| < r$ counting its multiplicities. We denote by $n_{(k}(r,f)$

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the number of poles of f with multiplicity at least k in $|z| < r$ counting its multiplicities. We denote by $n_2(r, f)$ the number of poles of f in $|z| < r$, where a simple pole is counted once and a multiple pole is counted two times. We denote by $\overline{n}(r, f)$ as the counting function of poles of f counted with ignoring multiplicities. $N_{k)}(r, f), N_{(k}(r, f), N_2(r, f)$, $\overline{N}_{k)}(r, f), \overline{N}_{(k}(r, f), N_{k)}(r, 1/(f-a)), N_{(k}(r, 1/(f-a)), N_2(r, 1/(f-a))$ and so on are defined in the usual way, respectively.

Let f, g and α be meromorphic functions. Let $\Psi_f(z) = f^{n+1}(z)(f^m(z) + a) + \alpha(z)$, where a is a constant. We note that

$$\Psi_f'(z) = (n + m + 1)\{f^n(z)(f^m(z) + a_1)f'(z) + \alpha_1(z)\},$$

where $a_1 = (n + 1)a/(n + m + 1)$ and $\alpha_1(z) = \alpha'(z)/(n + m + 1)$.

Theorem 1. Let f and g be two transcendental entire functions, α be a meromorphic function such that $T(r, \alpha_1) = o(T(r, f) + T(r, g))$ and $\alpha_1 \not\equiv 0, \infty$. Let $\Psi_f(z)$ be as above, and a be a nonzero constant. Suppose that m, n and k are positive integers such that $(k - 1)n > 7 + 3m + k(5 + m)$. If $E_{k)}(\Psi_f') = E_{k)}(\Psi_g')$, then $f(z) \equiv g(z)$.

Remark 1. Under the condition of Theorem 1, letting $k \to \infty$, we obtain that $f(z) \equiv g(z)$ if $E(\Psi_f') = E(\Psi_g')$ and $n > 5+m$. Obviously, Theorem 1 improves Theorem A and Theorem B.

Theorem 2. Let f and g be two transcendental meromorphic functions, α_1 be a meromorphic function such that $T(r, \alpha) = o(T(r, f)+T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that m, n and k are positive integers such that $(k-1)n > 14+3m+k(10+m)$. If $E_{k)}(\Psi_f') = E_{k)}(\Psi_g')$, then

(i) if $m \geq 2$, then $f(z) \equiv g(z)$;

(ii) if $m = 1$, either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n + 1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n + 2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.

Remark 2. Under the condition of Theorem 2, letting $k \to \infty$, we obtain that the result of Theorem 2 is still valid if $E(\Psi_f') = E(\Psi_g')$ and $n > 10+m$. Obviously, Theorem 2 improves Theorem C.

Theorem 3. Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r, \alpha) = o(T(r, f) + T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that $\Theta(\infty, f) + \Theta(\infty, g) > (2/5)\{10 + m - n + 2(n + m + 2)/(k + 1)\}$ holds for positive integers m, n, k such that $k \geq 2$ and $n \geq 10 + m$. If $E_{k)}(\Psi_f') = E_{k)}(\Psi_g')$, then

(i) if $m \geq 2$, $f(z) \equiv g(z)$;

(ii) if $m = 1$, either $f(z) \equiv g(z)$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\varpi_1, \varpi_2) = (n + 1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n + 2)(\varpi_1^{n+1} - \varpi_2^{n+1})$.

As the consequence of Theorem 2 and Theorem 3, letting $k \to \infty$, we have

Remark 3. Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r, \alpha) = o(T(r, f)+T(r, g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant, and m, n ($m \geq 2, n \geq 10 + m$) be positive integers. If $E(\Psi_f') = E(\Psi_g')$ and $\Theta(\infty, f) > 0$, then $f(z) \equiv g(z)$.

Corollary. Let f and g be two transcendental meromorphic functions, α be a meromorphic function such that $T(r,\alpha) = o(T(r,f)+T(r,g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant. Suppose that $\Theta(\infty, f) > 2/(n+1)$ holds. If $E(\Psi'_f) = E(\Psi'_g)$ holds for positive integers m and $n \geq 10 + m$, then $f(z) \equiv g(z)$.

Remark 4. In the case $m = 1$, the following example shows that the condition of $\Theta(\infty, f) > 2/(n+1)$ is necessary.
Example. Let

$$f = \frac{(n+2)h(h^{n+1}-1)}{(n+1)(h^{n+2}-1)} \qquad g = \frac{(n+2)(h^{n+1}-1)}{(n+1)(h^{n+2}-1)},$$

where $u = \exp\{(2\pi i)/(n+2)\}$ and $h = (u^2e^z - u)/(e^z - 1)$. It is easy to find $E(\Psi'_f) = E(\Psi'_g)$ and $\Theta(\infty, f) = 2/(n+1)$, but $f(z) \not\equiv g(z)$.

2 Lemmas

For proving the theorems, we need the following lemmas.

Lemma 1 [6]. Let $f(z)$ be a nonconstant meromorphic function, and

$$R(f) = \sum_{k=0}^{n} a_k f^k \Big/ \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n, m\}$.

Lemma 2. Let f and g be two nonconstant meromorphic functions, and α be a meromorphic function such that $T(r,\alpha) = o(T(r,f) + T(r,g))$ and $\alpha \not\equiv 0, \infty$. Let a be a nonzero constant, and n and m be positive integers. Set

$$F = f^n(f^m - a_1)f', \qquad G = g^n(g^m - a_1)g'.$$

If $E_{k)}(F-\alpha) = E_{k)}(G-\alpha)$ and $(n-6)k-m > 4$, then $S(r,f)$ and $S(r,g)$ are equivalent, that is, if $A(r) = S(r,f)$, then $A(r) = S(r,g)$, and also if $A(r) = S(r,g)$, then $A(r) = S(r,f)$.

Proof. By Lemma 1, we have

$$(n+m)T(r,f) = T(r, f^n(f^m + a)) + S(r,f) \leq T(r,F) + T(r,f') + S(r,f).$$

Therefore we have

$$T(r,F) \geq (n+m-2)T(r,f) + S(r,f).$$

By the second fundamental theorem, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{F-\alpha}\Big) + S(r,F) \\ &\leq \overline{N}(r,F) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}_{k)}\Big(r,\frac{1}{F-\alpha}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{F-\alpha}\Big) + S(r,f) \\ &\leq \overline{N}(r,F) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}_{k)}\Big(r,\frac{1}{G-\alpha}\Big) + \frac{1}{k+1}N\Big(r,\frac{1}{F-\alpha}\Big) + S(r,f) \\ &\leq (4+m)T(r,f) + \frac{1}{k+1}T(r,F) + T(r,G) + S(r,f) \end{aligned}$$

Noting that $T(r,G) \le T(r, g^n(g^m-1)) + T(r,g') \le (n+m+2)T(r,g) + S(r,g)$, we deduce that

$$\Big(\frac{k(n+m-2)}{k+1} - 4 - m\Big)T(r,f) \le (n+m+2)T(r,g) + S(r,f) + S(r,g).$$

We note that $k(n+m-2)/(k+1)-4-m > 0$ and the conditions for f and g are symmetric. Thus $S(r,f)$ and $S(r,g)$ are equivalent.

Lemma 3. Let F and G be two nonconstant meromorphic functions such that $E_{k)}(F-1) = E_{k)}(G-1)$, and let

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

If $H \not\equiv 0$, then

$$\begin{aligned}\frac{1}{2}\Big\{T(r,F) + T(r,G)\Big\} \quad \le \quad & N_2(r,F) + N_2\Big(r,\frac{1}{F}\Big) + N_2(r,G) + N_2\Big(r,\frac{1}{G}\Big)\\ & + \overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big) + S(r),\end{aligned}$$

where $T(r) = \max\{T(r,F), T(r,G)\Big\}$, $S(r) = o(T(r))$ $(r \to \infty, r \notin E)$ and E is a set of finite linear measure.

Proof. By the second fundamental theorem, we have

$$T(r,F) \le \overline{N}(r,F) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{F-1}\Big) - N_0\Big(r,\frac{1}{F'}\Big) + S(r,F)$$

and

$$T(r,G) \le \overline{N}(r,G) + \overline{N}\Big(r,\frac{1}{G}\Big) + \overline{N}\Big(r,\frac{1}{G-1}\Big) - N_0\Big(r,\frac{1}{G'}\Big) + S(r,G),$$

where $N_0(r, 1/F')$ is the counting function of the zeros of F' in $|z| < r$ that is not the zeros of $F-1$ and F. In the same way, we can define $N_0(r, 1/G')$. Thus we have

$$\begin{aligned}T(r,F) + T(r,G) \quad \le \quad & \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\Big(r,\frac{1}{G}\Big) + \overline{N}\Big(r,\frac{1}{F-1}\Big)\\ & + \overline{N}\Big(r,\frac{1}{G-1}\Big) - N_0\Big(r,\frac{1}{F'}\Big) - N_0\Big(r,\frac{1}{G'}\Big) + S(r). \qquad (1)\end{aligned}$$

We also have

$$\begin{aligned}\overline{N}\Big(r,\frac{1}{F-1}\Big) \quad + \quad & \overline{N}\Big(r,\frac{1}{G-1}\Big)\\ \le \quad & \overline{N}_{k)}\Big(r,\frac{1}{F-1}\Big) + \overline{N}_{k)}\Big(r,\frac{1}{G-1}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big)\\ \le \quad & \frac{1}{2}\Big\{N_{1)}\Big(r,\frac{1}{F-1}\Big) + N_{k)}\Big(r,\frac{1}{F-1}\Big) + N_{1)}\Big(r,\frac{1}{G-1}\Big) + N_{k)}\Big(r,\frac{1}{G-1}\Big)\Big\}\\ & + \overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big).\end{aligned}$$

Let z_0 be a simple pole of F. By a simple calculation , we know that z_0 is not a pole of $F''/F' - 2F'/(F-1)$. Let z_1 be a zero of $F-1$ with multiplicity t, where $1 \le t \le k$.

We know also that z_1 is not a pole of H, especially, z_1 is a simple zero of H if $k=1$. In fact, by a simple calculation, we can prove that any common simple 1-point of F and G is a zero of

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Let z_0 be a common simple 1-point of F and G. If we expand F in a neighborhood of z_0 as

$$F = 1 + A(z - z_0) + B(z - z_0)^2 + D(z - z_0)^3 + O((z - z_0)^4), \quad (A \neq 0).$$

Then we have

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{-1}{z - z_0} + O(z - z_0).$$

Similarly, we have

$$\frac{G''}{G'} - \frac{2G'}{G-1} = \frac{-1}{z - z_0} + O(z - z_0).$$

Thus we obtain that

$$H(z) = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) = O(z - z_0),$$

that is, z_0 is a zero of H.

Similarly, by a simple calculation, we can also prove that any simple pole of F is not a pole of $F''/F' - 2F'/(F-1)$, and any simple pole of G is not a pole of $G''/G' - 2G'/(G-1)$.

Thus we have

$$\begin{aligned} N_{1)}\Big(r, \frac{1}{F-1}\Big) &\leq \overline{N}\Big(r, \frac{1}{H}\Big) \leq T(r, H) + S(r, F) = N(r, H) + S(r, F) \\ &\leq \overline{N}_{(2}(r, F) + \overline{N}_{(2}(r, G) + \overline{N}_{(k+1}\Big(r, \frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r, \frac{1}{G-1}\Big) \\ &\quad + N_0\Big(r, \frac{1}{F'}\Big) + N_0\Big(r, \frac{1}{G'}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{F}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{G}\Big) + S(r, F). \end{aligned} \quad (2)$$

Similarly we have

$$\begin{aligned} N_{1)}\Big(r, \frac{1}{G-1}\Big) &\leq \overline{N}_{(2}(r, F) + \overline{N}_{(2}(r, G) + \overline{N}_{(k+1}\Big(r, \frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r, \frac{1}{G-1}\Big) \\ &\quad + N_0\Big(r, \frac{1}{F'}\Big) + N_0\Big(r, \frac{1}{G'}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{F}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{G}\Big) + S(r, G). \end{aligned} \quad (3)$$

By (1), (2) and (3), we have the desired inequality.

Lemma 4[7]. Let H be defined as in Lemma 3. If $H \equiv 0$ and

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{\overline{N}\big(r, 1/F\big) + \overline{N}\big(r, 1/G\big) + \overline{N}(r, F) + \overline{N}(r, G)}{T(r)} < 1,$$

where I is a set of infinite linear measure, then $FG \equiv 1$ or $F \equiv G$.

3 Proof of Theorems

(I) Proof of Theorem 2.

Let

$$F = \frac{f^n(f^m + a_1)f'}{\alpha_1(z)}, \qquad G = \frac{g^n(g^m + a_1)g'}{\alpha_1(z)}, \quad (4)$$

$$F_1 = \frac{1}{n+m+1} f^{n+m+1} + \frac{a_1}{n+1} f^{n+1}, \quad G_1 = \frac{1}{n+m+1} g^{n+m+1} + \frac{a_1}{n+1} g^{n+1}, \quad (5)$$

and

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Here $a_1 = (n+1)a/(n+m+1)$ and $\alpha_1 = (-\alpha')/(n+m+1)$. Then $E_{k)}(F-1) = E_{k)}(G-1)$. By Lemma 1 and Lemma 2, we have $S(r,f) = S(r,g)$ $(= S(r)$, say) and

$$T(r,F_1) = (n+m+1)T(r,f) + S(r,f),\ \ T(r,G_1) = (n+m+1)T(r,g) + S(r,g). \qquad (6)$$

Since $F_1' = \alpha_1(z)F$ and $G_1' = \alpha_1(z)G$, we deduce that

$$\begin{aligned} T(r,F_1) + T(r,G_1) &\le T(r,F) + N\Big(r,\frac{1}{F_1}\Big) - N\Big(r,\frac{1}{F}\Big) \\ &\quad + T(r,G) + N\Big(r,\frac{1}{G_1}\Big) - N\Big(r,\frac{1}{G}\Big) + S(r) \\ &= T(r,F) + (n+1)N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f^m+a}\Big) \\ &\quad - nN\Big(r,\frac{1}{f}\Big) - N\Big(r,\frac{1}{f^m+a_1}\Big) - N\Big(r,\frac{1}{f'}\Big) \\ &\quad + T(r,G) + (n+1)N\Big(r,\frac{1}{g}\Big) + N\Big(r,\frac{1}{g^m+a}\Big) \\ &\quad - nN\Big(r,\frac{1}{g}\Big) - N\Big(r,\frac{1}{g^m+a_1}\Big) - N\Big(r,\frac{1}{g'}\Big) + S(r) \\ &= T(r,F) + N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f^m+a}\Big) - N\Big(r,\frac{1}{f^m+a_1}\Big) \\ &\quad - N\Big(r,\frac{1}{f'}\Big) + T(r,G) + N\Big(r,\frac{1}{g}\Big) + N\Big(r,\frac{1}{g^m+a}\Big) \\ &\quad - N\Big(r,\frac{1}{g^m+a_1}\Big) - N\Big(r,\frac{1}{g'}\Big) + S(r). \end{aligned} \qquad (7)$$

If $H \not\equiv 0$, by Lemma 3, we have

$$\begin{aligned} T(r,F) + T(r,G) \quad \le \quad & 2\Big\{N_2(r,F) + N_2\Big(r,\frac{1}{F}\Big) + N_2(r,G) + N_2\Big(r,\frac{1}{G}\Big) \\ & + \overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big) + \overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big)\Big\} + S(r). \end{aligned} \qquad (8)$$

It follows from (4) that

$$\begin{aligned} & N_2(r,F) + N_2\Big(r,\frac{1}{F}\Big) + N_2(r,G) + N_2\Big(r,\frac{1}{G}\Big) \\ & \quad \le 2\Big\{\overline{N}(r,f) + N\Big(r,\frac{1}{f}\Big)\Big\} + N\Big(r,\frac{1}{f^m+a_1}\Big) + N\Big(r,\frac{1}{f'}\Big) \\ & \qquad + 2\Big\{\overline{N}(r,g) + N\Big(r,\frac{1}{g}\Big)\Big\} + N\Big(r,\frac{1}{g^m+a_1}\Big) + N\Big(r,\frac{1}{g'}\Big) + S(r). \end{aligned} \qquad (9)$$

Then we have from (6) $\sim$ (9)

$$\begin{aligned}
(n+m+1)\Big\{T(r,f)+T(r,g)\Big\} &= \Big\{T(r,F_1)+T(r,G_1)\Big\}+S(r)\\
&\le T(r,F)+T(r,G)+N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\\
&\quad +N\Big(r,\frac{1}{f^m+a}\Big)+N\Big(r,\frac{1}{g^m+a}\Big)-N\Big(r,\frac{1}{f^m+a_1}\Big)\\
&\quad -N\Big(r,\frac{1}{g^m+a_1}\Big)-N\Big(r,\frac{1}{f'}\Big)-N\Big(r,\frac{1}{g'}\Big)+S(r)\\
&\le 4\Big\{\overline{N}(r,f)+\overline{N}(r,g)\Big\}+5\Big\{N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big\}\\
&\quad +2\Big\{N\Big(r,\frac{1}{f^m+a_1}\Big)+N\Big(r,\frac{1}{g^m+a_1}\Big)\Big\}+2\Big\{N\Big(r,\frac{1}{f'}\Big)+N\Big(r,\frac{1}{g'}\Big)\Big\}\\
&\quad +2\Big\{\overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big)+\overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big)\Big\}+N\Big(r,\frac{1}{f^m+a}\Big)+N\Big(r,\frac{1}{g^m+a}\Big)\\
&\quad -N\Big(r,\frac{1}{f^m+a_1}\Big)-N\Big(r,\frac{1}{g^m+a_1}\Big)-N\Big(r,\frac{1}{f'}\Big)-N\Big(r,\frac{1}{g'}\Big)+S(r)\\
&\le 4\Big\{\overline{N}(r,f)+\overline{N}(r,g)\Big\}+5\Big\{N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big\}+\Big\{N\Big(r,\frac{1}{f'}\Big)+N\Big(r,\frac{1}{g'}\Big)\Big\}\\
&\quad +N\Big(r,\frac{1}{f^m+a_1}\Big)+N\Big(r,\frac{1}{g^m+a_1}\Big)+2\Big\{\overline{N}_{(k+1}\Big(r,\frac{1}{F-1}\Big)+\overline{N}_{(k+1}\Big(r,\frac{1}{G-1}\Big)\Big\}\\
&\quad +N\Big(r,\frac{1}{f^m+a}\Big)+N\Big(r,\frac{1}{g^m+a}\Big)+S(r)\\
&\le 4\Big\{\overline{N}(r,f)+\overline{N}(r,g)\Big\}+5\Big\{N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big\}\\
&\quad +\Big\{T(r,f')+T(r,g')\Big\}+N\Big(r,\frac{1}{f^m+a_1}\Big)+N\Big(r,\frac{1}{g^m+a_1}\Big)\\
&\quad +\frac{2}{k+1}\Big\{N\Big(r,\frac{1}{F-1}\Big)+N\Big(r,\frac{1}{G-1}\Big)\Big\}+N\Big(r,\frac{1}{f^m+a}\Big)+N\Big(r,\frac{1}{g^m+a}\Big)+S(r)\\
&\le 4\Big\{\overline{N}(r,f)+\overline{N}(r,g)\Big\}+5\Big\{N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{g}\Big)\Big\}\\
&\quad +2\Big\{T(r,f)+T(r,g)\Big\}+\frac{2}{k+1}\Big\{T(r,F)+T(r,G)\Big\}\\
&\quad +2m\Big\{T(r,f)+T(r,g)\Big\}+S(r)\\
&\le \Big\{11+2m+\frac{2(m+n+2)}{k+1}\Big\}\Big\{T(r,f)+T(r,g)\Big\}+S(r).
\end{aligned}$$

Hence we have,

$$(n+m+1)\Big(T(r,f)+T(r,g)\Big)\le\Big\{11+2m+\frac{2(m+n+2)}{k+1}\Big\}\Big(T(r,f)+T(r,g)\Big)+S(r).$$

Thus we have $n+m+1\le 11+2m+\{2(m+n+2)/(k+1)\}$, which contradicts $(k-1)n>14+3m+(10+m)k$. Therefore we have $H\equiv 0$, that is,

$$\frac{F''}{F'}-\frac{2F'}{F-1}\equiv\frac{G''}{G'}-\frac{2G'}{G-1}.$$

Hence we see

$$\frac{1}{G-1} \equiv \frac{A}{F-1} + B,$$

where $A \neq 0$ and B are constants. Thus $E(F-1) = E(G-1)$, and

$$T(r,F) = T(r,G) + S(r). \tag{10}$$

Since

$$\begin{aligned}\overline{N}(r,F) &+ \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}(r,G) + \overline{N}\Big(r,\frac{1}{G}\Big)\\ &\leq \overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f'}\Big) + \overline{N}\Big(r,\frac{1}{f^m+a_1}\Big)\\ &\quad +\overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{g'}\Big) + \overline{N}\Big(r,\frac{1}{g^m+a_1}\Big) + S(r)\\ &\leq (m+4)\{T(r,f)+T(r,g)\} + S(r)\\ &\leq \frac{2(m+4)}{n+m-2}T(r) + S(r),\end{aligned}$$

we have

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{\overline{N}\Big(r,1/F\Big) + \overline{N}\Big(r,1/G\Big) + \overline{N}(r,F) + \overline{N}(r,G)}{T(r)} < 1,$$

by Lemma 4 we get $FG \equiv 1$ or $F \equiv G$.

We next discuss the following two cases.

Case 1. Suppose that $FG \equiv 1$, that is,

$$f^n(f^m+a_1)f'g^n(g^m+a_1)g' \equiv \alpha^2(z). \tag{11}$$

(a) Let z_0 be a zero of f of order p such that $\alpha(z_0) \not\equiv 0, \infty$. From (11) we know that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. From (11) we obtain that

(i) If $p = 1$, then $n = nq + mq + q + 1$. This is a contradiction.

(ii) If $p > 1$, then $np + p - 1 = nq + mq + q + 1$. This implies $(n+1)(p-q) = mq + 2 > 0$. Hence $p \geq q + 1$. Thus we have $np + p - 1 < (n+m+1)(p-1) + 1$. Therefore we see $p \geq (n+m-1)/m$.

(b) Let z_1 be a zero of $f^m + a_1$ of order p_1 such that $\alpha(z_1) \not\equiv 0, \infty$. From (11) we know that z_1 is a pole of g. From (11) we obtain that

(i) If $p_1 = 1$, then $1 = nq_1 + mq_1 + q_1 + 1$. This is a contradiction.

(ii) If $p_1 > 1$, then $p_1 + p_1 - 1 = nq_1 + mq_1 + q_1 + 1$. Thus $p_1 \geq (n+m+3)/2$.

(c) Let z_2 be a zero of f' of order p_2 such that $\alpha(z_2) \not\equiv 0, \infty$ that is not a zero of $f(f^m + a_1)$. From (11) we know that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2. From (11) we obtain that $p_2 = nq_2 + mq_2 + q_2 + 1$. Thus $p_2 \geq n + m + 2$.

Moreover, in the same method as above, we have the similar results for the zeros of $g(g^m + a_1)g'$. On the other hand, we suppose that z_3 is a pole of f such that $\alpha(z_3) \not\equiv 0, \infty$.

From (11) we obtain that z_3 is a zero of $g(g^m+a_1)g'$.Thus we have

$$\begin{aligned}\overline{N}(r,f) &\le \overline{N}\Big(r,\frac{1}{g}\Big)+\overline{N}\Big(r,\frac{1}{g^m+a_1}\Big)+\overline{N}_\star\Big(r,\frac{1}{g'}\Big)\\ &\le \frac{m}{n+m-1}N\Big(r,\frac{1}{g}\Big)+\frac{2}{n+m+3}N\Big(r,\frac{1}{g^m+a_1}\Big)+\frac{1}{n+m+2}N\Big(r,\frac{1}{g'}\Big)\\ &\le \Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}+\frac{2}{n+m+2}\Big)T(r,g)+S(r,g),\end{aligned}$$

where $n_\star(r,g)$ is defined the number of zeros of g' that is not zero of $g(g^m+a_1)$ in $\mid z\mid\le r$, a zero point with multiplicity m is counted m times in the set. $N_\star(r,1/g)$ is defined in the terms of $n_\star(r,1/g)$ in the usual manner.

Hence

$$\begin{aligned}mT(r,f) &< \overline{N}(r,f)+\sum_{j=1}^{m}\overline{N}\Big(r,\frac{1}{f-c_j}\Big)+\overline{N}\Big(r,\frac{1}{f}\Big)+S(r)\\ &\le \Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}+\frac{2}{n+m+2}\Big)T(r,g)\\ &\quad+\frac{m}{n+m-1}N\Big(r,\frac{1}{f}\Big)+\sum_{j=1}^{m}\frac{2}{n+m+3}N\Big(r,\frac{1}{f-c_j}\Big)+S(r)\\ &= \Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}+\frac{2}{n+m+2}\Big)T(r,g)\\ &\quad+\Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}\Big)T(r,f)+S(r),\end{aligned}$$

where $f^m-a_1=(f-c_1)(f-c_2)\cdots(f-c_m)$. Similarly we have

$$\begin{aligned}mT(r,g) &< \Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}+\frac{2}{n+m+2}\Big)T(r,f)\\ &\quad+\Big(\frac{m}{n+m-1}+\frac{2m}{n+m+3}\Big)T(r,g)+S(r).\end{aligned}$$

Thus we have

$$m\Big(T(r,f)+T(r,g)\Big)\le\Big(\frac{2m}{n+m-1}+\frac{4m}{n+m+3}+\frac{2}{n+m+2}\Big)\Big(T(r,f)+T(r,g)\Big)+S(r).$$

Hence we have

$$m<\frac{2m}{n+m-1}+\frac{4m}{n+m+3}+\frac{2}{n+m+2},$$

which contradicts with $n>m+10$.

Case 2. Suppose that $F\equiv G$, then

$$F_1\equiv G_1+C, \tag{12}$$

where C is a constant and

$$F_1=\frac{1}{n+m+1}f^{n+m+1}+\frac{a_1}{n+1}f^{n+1},\quad G_1=\frac{1}{n+m+1}g^{n+m+1}+\frac{a_1}{n+1}g^{n+1}.$$

By Lemma 1 we have

$$T(r,F_1)=(n+m+1)T(r,f)+S(r),\qquad T(r,G_1)=(n+m+1)T(r,g)+S(r).$$

It follows that

$$T(r,f) = T(r,g) + S(r). \tag{13}$$

Suppose that $C \neq 0$. By (13) we have

$$\begin{aligned}
(n+m+1)T(r,g) &= T(r,G_1) \\
&< \overline{N}\Big(r,\frac{1}{G_1}\Big) + \overline{N}\Big(r,\frac{1}{G_1+C}\Big) + \overline{N}(r,G_1) + S(r) \\
&\leq \overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{g^m+a}\Big) + \overline{N}\Big(r,\frac{1}{f^m+a}\Big) \\
&\quad + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{g}\Big) + S(r) \\
&\leq (2m+3)T(r,g) + S(r).
\end{aligned}$$

Thus $n+m+1 \leq 2m+3$, which contradicts with $n > m+10$. Therefore $F_1 \equiv G_1$, that is,

$$f^{n+1}\Big(f^m + a\Big) \equiv g^{n+1}\Big(g^m + a\Big). \tag{14}$$

Thus f and g share ∞ CM. Let $h = f/g$. If $h \not\equiv 1$, we have

$$g^m \equiv \frac{-a(h^{n+1}-1)}{h^{n+m+1}-1}.$$

If $m \geq 2$, we have

$$\begin{aligned}
(n-1)T(r,h) &\leq \sum_{j=1}^{n+1} \overline{N}\Big(r,\frac{1}{h-d_j}\Big) + S(r,h) \\
&\leq \frac{n+1}{m} T(r,h) + S(r,h),
\end{aligned}$$

where $h^{n+m+1} - 1 = (h-1)(h-d_1)\cdots(h-d_{n+m})$. In fact, since each zero point of $h-d_i$ has multiplicity at least m, $\overline{N}(r,1/(h-d_i)) \leq (1/m)N(r,1/(h-d_i)) \leq (1/m)T(r,h)$. Thus $(n-1) \leq (n+1)/m$, which contradicts with $n > m+10$. Therefore $h \equiv 1$. Then $f \equiv g$.

If $m = 1$, by (14), f and g satisfy the algebraic relation $R(f,g) \equiv 0$, where $R(\varpi_1,\varpi_2) = (n+1)(\varpi_1^{n+2} - \varpi_2^{n+2}) - (n+2)(\varpi_1^{n+1} - \varpi_2^{n+1})$. This completes the proof of Theorem 2.

(II) Proof of Theorem 1 and Theorem 3

By making use of Lemma 3 and a similar method to the proof of Theorem 2, we easily obtain the proof of Theorem 1 and Theorem 3.

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