ON IMPLICATIVE IDEALS OF PSEUDO *MV*-ALGEBRAS

Andrzej Walendziak

Received April 26, 2005; revised July 1, 2005

Abstract. In the theory of MV-algebras, implicative ideals are studied by Hoo and Sessa. In this paper we investigate implicative ideals of pseudo MV-algebras. (Pseudo MV-algebras are a noncommutative generalization of MV-algebras.) Some characterizations of such ideals are given. We prove that every ideal extension of an implicative ideal is implicative. It is also shown that if an ideal is prime and implicative, then it is maximal.

1. Preliminaries

The notion of pseudo MV-algebra has been introduced independently by Georgescu and Iorgulescu [1] and by Rachůnek [7] (in [7], the term "generalized MV-algebra" has been applied).

Let $\mathcal{A} = (A; \oplus, -, \sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. For $x, y \in A$ we put $x \cdot y = (y^- \oplus x^-)^\sim$. We consider that the operation " $\cdot$ " has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra $\mathcal{A}$ is called a *pseudo MV-algebra* if the following conditions are satisfied for each $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$;
(A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$;
(A8) $(x^-)^\sim = x$.

Throughout this paper, $\mathcal{A}$ shall denote a pseudo MV-algebra. For any $x \in A$ we put

$$x^0 = 1 \text{ and } x^{n+1} = x^n \cdot x = x \cdot x^n (n = 0, 1, 2, ...).$$

We will denote by $\mathrm{B}(A)$ the set of all idempotents of $\mathcal{A}$, that is,

$$B(A) = \{x \in A : x \oplus x = x\}.$$

By Proposition 4.2 of [1], $\mathrm{B}(A) = \{x \in A : x \wedge x^- = 0\}$.

Proposition 1.1. (Georgescu and Iorgulescu [1].) *The following properties hold for any* $x, y, z \in A$:

2000 *Mathematics Subject Classification.* 06D35.
Key words and phrases. Pseudo MV-algebra, ideal, implicative ideal.

(i) $(x^{\sim})^{-} = x$;
(ii) $x \cdot 1 = 1 \cdot x = x$;
(iii) $x \oplus x^{\sim} = 1$; $x^{-} \oplus x = 1$;
(iv) $x \cdot x^{-} = 0$; $x^{\sim} \cdot x = 0$;
(v) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

As always in algebra, associativity of $\cdot$ (see Proposition 1.1(v)) makes it possible to write $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ without using parenthess.

We define

$$x \leq y \Leftrightarrow x^{-} \oplus y = 1.$$

In [1] it is proved that $(A; \leq)$ is a lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$x \vee y = x \oplus x^{\sim} \cdot y = x \cdot y^{-} \oplus y, \tag{1}$$

$$x \wedge y = x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y. \tag{2}$$

Proposition 1.2. (Georgescu and Iorgulescu [1].) *For every pseudo MV-algebra $\mathcal{A}$ the algebra $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice.*

Proposition 1.3. (Georgescu and Iorgulescu [1].) *Let $x, y, z \in A$. Then the following statements hold:*

(i) $x \cdot y \leq x \wedge y \leq x \vee y \leq x \oplus y$;
(ii) $(x \vee y)^{-} = x^{-} \wedge y^{-}$;
(iii) $x \leq y \Rightarrow z \cdot x \leq z \cdot y$, $x \cdot z \leq y \cdot z$;
(iv) $z \oplus (x \wedge y) = (z \oplus x) \wedge (z \oplus y)$;
(v) $z \cdot (x \oplus y) \leq z \cdot x \oplus y$.

2. Ideals. Definition and properties

Definition 2.1. An *ideal* of $\mathcal{A}$ is a subset I of A satisfying the following conditions:
(I1) $0 \in I$,
(I2) If $x, y \in I$, then $x \oplus y \in I$,
(I3) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Proposition 2.2. *Let I be a nonvoid subset of A. I is an ideal of $\mathcal{A}$ if and only if I satisfies conditions* (I2) *and*

(I3') If $x \in I$, $y \in A$, then $x \wedge y \in I$.

Proof. If I is an ideal of $\mathcal{A}$, then it is clear that I satisfies (I3'). Let I satisfy (I2) and (I3'). Let $x \in I$, $y \in A$ and $y \leq x$. Then $0 = x \wedge 0 \in I$ and $y = x \wedge y \in I$.
Thus I is an ideal of $\mathcal{A}$. □

We denote by $\mathrm{Id}(\mathcal{A})$ the set of ideals of $\mathcal{A}$.

Proposition 2.3. Let $I \in \mathrm{Id}(\mathcal{A})$.

(i) If $x, y \in I$, then $x \cdot y$, $x \wedge y$, $x \vee y \in I$.
(ii) I is an ideal of the lattice $\mathcal{L}(\mathcal{A})$.
(iii) For $x, y \in A$, $x, y \in I$ if and only if $x \vee y \in I$.

Proof. Let $x, y \in I$. We have (i), because $x \cdot y \le x \wedge y \le x \vee y \le x \oplus y$ and $x \oplus y \in I$. (ii) and (iii) are obvious. □

Theorem 2.4. *Let I be a subset of A containing* 0. *Then the following statements are equivalent:*

(i) $I \in \mathrm{Id}(\mathcal{A})$;
(ii) $\forall x \in I\ \forall y \in A\ (x^{\sim} \cdot y \in I \Rightarrow y \in I)$;
(iii) $\forall x \in I\ \forall y \in A\ (y \cdot x^{-} \in I \Rightarrow y \in I)$.

Proof. (i) $\Rightarrow$ (ii). Let $x \in I$ and $x^{\sim} \cdot y \in I$. Applying (1) we have $y \le x \vee y = x \oplus x^{\sim} \cdot y \in I$, and hence $y \in I$.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds. We show first that the implication

$$(x \in I \text{ and } y \le x) \Rightarrow y \in I \tag{3}$$

is satisfied. If $y \le x$, then by Propositions 1.3(iii) and 1.1(iv) we get $x^{\sim} \cdot y \le x^{\sim} \cdot x = 0$, and hence $x^{\sim} \cdot y = 0 \in I$. From (ii) it follows that $y \in I$. Thus (3) holds. Let now $x \in I$ and $y \cdot x^{-} \in I$. We set $a = y \cdot x^{-}$. Applying Propositions 1.3(v) and 1.1(iv) we obtain

$$a^{\sim} \cdot (a \oplus x) \le a^{\sim} \cdot a \oplus x = 0 \oplus x = x.$$

Therefore, $a^{\sim} \cdot (a \oplus x) \in I$, by (3). We conclude from (ii) that $a \oplus x \in I$, i.e., $y \cdot x^{-} \oplus x \in I$. But $y \cdot x^{-} \oplus x = x \oplus x^{\sim} \cdot y \ge x^{\sim} \cdot y$, and hence $x^{\sim} \cdot y \in I$. (ii) now shows that $y \in I$.

(iii) $\Rightarrow$ (i). Let $y \le x \in I$. Then, by Propositions 1.3(iii) and 1.1(iv), we get $y \cdot x^{-} \le x \cdot x^{-} = 0$. This gives $y \cdot x^{-} = 0 \in I$. From (iii) we see that $y \in I$. Thus (I3) holds. Observe that (I2) is also satisfied. Let $x, y \in I$. Then $x \wedge y^{-} = (x \oplus (y^{-})^{\sim}) \cdot y^{-} = (x \oplus y) \cdot y^{-}$, by (2) and (A8). Consequently, $(x \oplus y) \cdot y^{-} \le x \in I$. Applying (I3) we deduce that $(x \oplus y) \cdot y^{-} \in I$, and hence $x \oplus y \in I$. Therefore, (I2) holds, and thus I is an ideal of $\mathcal{A}$. □

Definition 2.5. Let I be a proper ideal of $\mathcal{A}$ (i.e., $I \ne A$).
(i) I is called *prime* if, for all $I_1, I_2 \in \mathrm{Id}(\mathcal{A})$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.
(ii) I is *maximal* iff whenever J is an ideal such that $I \subseteq J \subseteq A$, then either $I = J$ or $J = A$.

Proposition 2.6. (Georgescu and Iorgulescu [1].) *For $I \in \mathrm{Id}(\mathcal{A})$, the following are equivalent:*

(i) *I is prime;*
(ii) *If $x \wedge y \in I$, then $x \in I$ or $y \in I$.*

Lemma 2.7. *If $I \in \mathrm{Id}(\mathcal{A})$ is maximal, then I is prime.*

Proof. Let I be a maximal ideal of $\mathcal{A}$ and let $I = I_1 \cap I_2$ for $I_1, I_2 \in \mathrm{Id}(\mathcal{A})$. Then $I \subseteq I_1$ and $I \subseteq I_2$. Since I is proper, $I_1 \ne A$ or $I_2 \ne A$. Let, for example $I_1 \ne A$. As I is maximal we have $I = I_1$. Thus I is prime. □

Definition 2.8. An ideal I of a pseudo MV-algebra $\mathcal{A}$ is called *normal* if it satisfies the condition:

(N) For all $x, y \in A$, $x \cdot y^- \in I \Longleftrightarrow y^\sim \cdot x \in I$.

To any normal ideal I of $\mathcal{A}$ we can associate a binary relation on A defined by:

$$x \sim_I y \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I. \tag{4}$$

By Theorem 3.8 of [1], $\sim_I$ is a congruence of $\mathcal{A}$. We denote by x/I the congruence class of an element $x \in A$. Let $A/I = \{x/I : x \in A\}$ and we define on the set A/I the operations:

$$x/I \oplus y/I = (x \oplus y)/I, (x/I)^- = (x^-)/I, (x/I)^\sim = (x^\sim)/I.$$

We remark that the algebra $\mathcal{A}/I = (A/I; \oplus, -, \sim, 0/I, 1/I)$ becomes a pseudo MV-algebra, called the *quotient algebra of* $\mathcal{A}$ *by the normal ideal* I. Observe that for all $x, y \in A$,

$$x/I \cdot y/I = (x \cdot y)/I. \tag{5}$$

Indeed, $x/I \cdot y/I = (y^-/I \oplus x^-/I)^\sim = ((y^- \oplus x^-)/I)^\sim = (y^- \oplus x^-)^\sim/I = (x \cdot y)/I$. Similarly,

$$x/I \wedge y/I = (x \wedge y)/I \tag{6}$$

and

$$x/I \vee y/I = (x \vee y)/I. \tag{7}$$

For $x, y \in A$ we have

$$\begin{array}{rcll} x \sim_I y & \Leftrightarrow & x \cdot y^- \vee y \cdot x^- \in I & \text{(by (4))} \\ & \Leftrightarrow & x \cdot y^-,\ y \cdot x^- \in I & \text{(use Proposition 2.3(iii))} \\ & \Leftrightarrow & y^\sim \cdot x,\ x^\sim \cdot y \in I & \text{(by (N))} \\ & \Leftrightarrow & x^\sim \cdot y \vee y^\sim \cdot x \in I. & \end{array}$$

Now it is easily seen that:

$$\begin{array}{lrcl} (8) & x/I & = & y/I \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I \Longleftrightarrow x^\sim \cdot y \vee y^\sim \cdot x \in I; \\ (9) & x/I & = & 0/I \Longleftrightarrow x \in I; \\ (10) & x/I & = & 1/I \Longleftrightarrow x^- \in I \Longleftrightarrow x^\sim \in I. \end{array}$$

3. Implicative ideals

In the theory of MV-algebras, implicative ideals are defined and studied in [2]-[6].

Definition 3.1. An ideal I of a pseudo MV-algebra $\mathcal{A}$ is called *implicative* if it satisfies the following condition ($x, y, z \in A$):

(Im) $(x \cdot y \cdot z \in I$ and $z^\sim \cdot y \in I) \Rightarrow x \cdot y \in I$.

Proposition 3.2. *Let I be a subset of $\mathcal{A}$ containing* 0. *If the condition* (Im) *holds for all $x, y, z \in A$, then I is an ideal.*

Proof. Let $y \cdot x^- \in I$ and $x \in I$. Then, by (A8) and Proposition 1.1(ii), we conclude that $y \cdot 1 \cdot x^- \in I$ and $(x^-)^\sim \cdot 1 = x \cdot 1 = x \in I$. From (Im) it follows that $y = y \cdot 1 \in I$. Therefore I is an ideal by Theorem 2.4. $\square$

Proposition 3.3. *The implication* (Im) *is equivalent to*

(Im′) *For all $x, y, z \in A$, if $x \cdot y \cdot z^- \in I$ and $z \cdot y \in I$, then $x \cdot y \in I$.*

Proof. (Im) $\Rightarrow$ (Im′). Let $x \cdot y \cdot z^- \in I$ and $z \cdot y \in I$. We put $t = z^-$. Then $x \cdot y \cdot t \in I$ and by (A8), $t^\sim \cdot y \in I$. Applying (Im) we see that $x \cdot y \in I$.

(Im′) $\Rightarrow$ (Im). If $x \cdot y \cdot z$, $z^\sim \cdot y \in I$, then $x \cdot y \cdot (z^\sim)^- \in I$ and $z^\sim \cdot y \in I$. From (Im′) we deduce that $x \cdot y \in I$. Therefore (Im) holds. □

Theorem 3.4. *Let $I \in \mathrm{Id}(\mathcal{A})$. Then the following statements are equivalent:*

(i) *I is implicative;*
(ii) *If $x, y \in A$ and $x \cdot y \cdot y \in I$, then $x \cdot y \in I$;*
(iii) *If $x^n \in I$ for some $n \geq 1$, then $x \in I$;*
(iv) $\{x \in A : x^2 = 0\} \subseteq I$;
(v) *$x \wedge x^- \in I$ for all $x \in A$;*
(vi) *$x \wedge x^\sim \in I$ for all $x \in A$.*

Proof. (i) $\Rightarrow$ (ii). Let $x \cdot y \cdot y \in I$. By Proposition 1.1(iv), $y^\sim \cdot y = 0 \in I$. From (Im) we conclude that $x \cdot y \in I$, and hence (ii) holds.

(ii) $\Rightarrow$ (iii). (iii) is obviously true in case $n = 1$. If $x^2 \in I$, then $1 \cdot x \cdot x \in I$. By (ii), $x = 1 \cdot x \in I$. We may now assume that $n \geq 2$, and suppose that (iii) holds for all positive intergers $\leq n$. Let $x^{n+1} \in I$. We have $x^{n+1} = (x^{n-1} \cdot x) \cdot x$. From (ii) it follows that $x^n = x^{n-1} \cdot x \in I$, and hence, by induction hypothesis, $x \in I$.

(iii) $\Rightarrow$ (iv). If $x^2 = 0$, then $x^2 \in I$, and we conclude from (iii) that $x \in I$.

(iv) $\Rightarrow$ (v). By Propositions 1.3(iii) and 1.1(iv), $(x \wedge x^-)^2 = (x \wedge x^-) \cdot (x \wedge x^-) \leq x \cdot x^- = 0$. Consequently, $(x \wedge x^-)^2 = 0$. Hence $x \wedge x^- \in I$.

(v) $\Rightarrow$ (vi). $x \wedge x^\sim = x^\sim \wedge x = x^\sim \wedge (x^\sim)^-$, by Proposition 1.1(i). Since $x^\sim \wedge (x^\sim)^- \in I$, we have (vi).

(vi) $\Rightarrow$ (i). Let $x \cdot y \cdot z \in I$ and $z^\sim \cdot y \in I$. By Proposition 1.3 and (1) we have

$$x \cdot y \cdot y \leq x \cdot y \cdot (z \vee y) = x \cdot y \cdot (z \oplus z^\sim \cdot y) \leq x \cdot y \cdot z \oplus z^\sim \cdot y.$$

Since I is an ideal we deduce that $x \cdot y \cdot y \in I$. From (vi) it follows that $y \wedge y^\sim \in I$. Then $a = x \cdot y \cdot y \oplus (y \wedge y^\sim) \in I$. Applying Proposition 1.3(iv) we get $a = (x \cdot y \cdot y \oplus y) \wedge (x \cdot y \cdot y \oplus y^\sim) \geq y \wedge (x \cdot y \cdot y \oplus y^\sim)$. But $x \cdot y \cdot y \oplus y^\sim = x \cdot y \cdot (y^\sim)^- \oplus y^\sim = x \cdot y \vee y^\sim$, by Proposition 1.1(i) and (1). Therefore,

$$x \cdot y \leq y \wedge (x \cdot y \cdot y \oplus y^\sim) \leq a \in I.$$

Hence $x \cdot y \in I$, and we obtain (Im). Thus I is implicative. □

Let $\mathrm{Inf}(A) = \{x \in A : x^2 = 0\}$ and $\mathrm{N}(A) = \{x \in A : x^n = 0$ for some interger $n \geq 1\}$. Elements of $\mathrm{N}(A)$ are called the *nilpotent* elements of $\mathcal{A}$. It is obvious that $\mathrm{Inf}(A) \subseteq \mathrm{N}(A)$. It was shown in Corollary 3.2 of [3] that if $\mathcal{A}$ is a MV-algebra, then $\mathrm{N}(A)$ is contained in every implicative ideal of $\mathcal{A}$. Theorem 3.4 implies

Corollary 3.5. *Let I be an ideal of a pseudo MV-algebra $\mathcal{A}$. Then the following three conditions are equivalent:*

(i) *I is implicative;*
(ii) $\mathrm{N}(A) \subseteq I$;
(iii) $\mathrm{Inf}(A) \subseteq I$.

Hence we obtain

Corollary 3.6. *If* $\mathrm{Inf}(A)$ *is an ideal, then* $\mathrm{Inf}(A)$ *is implicative.*

Corollary 3.7. (Extension property for implicative ideals.) *Let I be implicative and suppose that J is an ideal of $\mathcal{A}$ such that $I \subseteq J$. Then J is implicative.*

Proof. Follows from Corollary 3.5. □

Remark 3.8. The preceding result generalizes Corollary 2.14 of [4], since MV-algebras are pseudo MV-algebras.

Theorem 3.9. *Let I be a proper ideal of $\mathcal{A}$. Then the following statements are equivalent:*

(i) *I is maximal and implicative;*
(ii) *I is prime and implicative;*
(iii) $\forall x \in A$ ($x \in I$ *or* $x^- \in I$);
(iv) $\forall x \in A$ ($x \in I$ *or* $x^\sim \in I$);
(v) $\forall x \in A$ ($x \in I$ *or* $x^- \in I$ *or* $x^\sim \in I$).

Proof. (i) $\Rightarrow$ (ii). Follows from Lemma 2.7.

(ii) $\Rightarrow$ (iii). Let $x \in A$. Since I is implicative, Theorem 3.4 implies $x \wedge x^- \in I$. By Proposition 2.6, $x \in I$ or $x^- \in I$.

(iii) $\Rightarrow$ (iv). Let $x \in A$ and suppose that $x \notin I$. Then $(x^\sim)^- = x \notin I$. From (iii) it follows that $x^\sim \in I$.

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (i). Let I satisfy (v) and let $x \in A$. Suppose that $x \wedge x^- \notin I$. Therefore $x \notin I$ and $x^- \notin I$. From (v) we conclude that $x^\sim \in I$. We have $x \wedge x^\sim \in I$, because $x \wedge x^\sim \leq x^\sim$. By Theorem 3.4, I is implicative. Observe that I is also maximal. Let J be an ideal of $\mathcal{A}$ such that $I \subset J$. For every $z \in J - I$, we have by (v) that $z^- \in I$ or $z^\sim \in I$. Consequently, $z^- \in J$ or $z^\sim \in J$. Then $z^- \oplus z \in J$ or $z \oplus z^\sim \in J$. But $z^- \oplus z = z \oplus z^\sim = 1$ (see Proposition 1.1(iii)). Hence $1 \in J$ and thus $J = A$. □

Remark 3.10. Theorem 3.9 implies Theorem 3.7 of [2] and Proposition 3.7 of [3] (see also [5], p. 463).

As an immediate consequence of Theorem 3.9 we obtain

Corollary 3.11. *Let I be a proper implicative ideal. Then I is prime if and only if I is maximal.*

Lemma 3.12. *Suppose that an ideal I of $\mathcal{A}$ is normal and implicative. Then $x^-/I = x^\sim/I$ for all $x \in A$.*

Proof. Let $x \in A$. Since I is implicative and $x^- \cdot x \leq x \wedge x^-$, we conclude that $x^- \cdot x \in I$. Similarly, $(x^-)^- \cdot x^- \leq x^- \wedge (x^-)^- \in I$, and hence $(x^-)^- \cdot x^- \in I$. Then we have $x^\sim \cdot (x^-)^- \in I$, because I is normal. Therefore,

$$x^- \cdot (x^\sim)^- \vee x^\sim \cdot (x^-)^- = x^- \cdot x \vee x^\sim \cdot (x^-)^- \in I.$$

From (8) we obtain $x^-/I = x^\sim/I$. □

Theorem 3.13. *Let I be a normal ideal of a pseudo MV-algebra $\mathcal{A}$. Then the following statements are equivalent:*

(i) *$\mathcal{L}(\mathcal{A}/I)$ is a Boolean lattice in which $x^-/I = x^\sim/I$ is the complement of x/I;*

(ii) $\mathrm{B}(A/I) = A/I$;
(iii) *I is implicative.*

Proof. (i) $\Rightarrow$ (ii).We choose to note by x^-/I the complement of x/I in $\mathcal{L}(\mathcal{A}/I)$. We have $x/I \wedge x^-/I = 0/I$, and hence $x/I \in \mathrm{B}(A/I)$ for all $x \in A$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii). Let $x \in A$. Since $x/I \in \mathrm{B}(A/I)$, we conclude that $x/I \wedge x^-/I = 0/I$. From (6) we obtain $(x \wedge x^-)/I = 0/I$. By (9), $x \wedge x^- \in I$. Theorem 3.4 shows that I is implicative.

(iii) $\Rightarrow$ (i). Proposition 1.2 implies that $\mathcal{L}(\mathcal{A}/I)$ is a bounded distributive lattice. By Lemma 3.12, $x^-/I = x^\sim/I$ for all $x \in A$. Since I is implicative, we see that $x \wedge x^- \in I$. Applying (6) we get $(x \wedge x^-)/I = 0/I$, and therefore $x/I \wedge x^-/I = 0/I$. Observe that

$$(x \vee x^\sim)^- \in I. \tag{11}$$

Indeed, applying Propositions 1.3(ii) and 1.1(i) we obtain

$$(x \vee x^\sim)^- = x^- \wedge (x^\sim)^- = x^- \wedge x \in I.$$

Applying (7) and combining (10) with (11) we get

$$x/I \vee x^-/I = x/I \vee x^\sim/I = (x \vee x^\sim)/I = 1/I.$$

Consequently, $x^-/I = x^\sim/I$ is the complement of x/I in $\mathcal{L}(\mathcal{A}/I)$ and $\mathcal{L}(\mathcal{A}/I)$ is a Boolean lattice. $\square$

Acknowledgements

The author thanks the referee for his remarks which were incorporated into this revised version.

References

[1] G. Georgescu and A. Iorgulescu, *Pseudo MV-algebras*, Multi. Val. Logic **6** (2001), 95-135.

[2] C. S. Hoo, *MV-algebras, ideals, and semisimplicity*, Math. Japonica **34** (1989), 563-584.

[3] C. S. Hoo, *Fuzzy ideals of BCI and MV-algebras*, Fuzzy sets and Systems **62** (1994), 111-114.

[4] C. S. Hoo, *Fuzzy implicative and Boolean ideals of MV-algebras*, Fuzzy sets and Systems **66** (1994), 315-327.

[5] C. S. Hoo, *Molecules and linearly ordered ideals of MV-algebras*, Publications Matemàtiques, **41** (1997), 455-465.

[6] C. S. Hoo and S. Sessa, *Implicative and Boolean ideals of MV-algebras*, Math. Japonica **39** (1994), 215-219.

[7] J. Rachůnek, *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255-273.

Andrzej Walendziak
Institute of Mathematics and Physics
University of Podlasie
08-110 Siedlce, POLAND
walent@interia.pl