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AN EXTENSION OF THE FUGLEDE-PUTNAM THEOREM TO (p,k)-QUASIHYPONORMAL OPERATORS

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Abstract. The equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal operators is known as the familiar Fuglede-Putnam theorem. In this paper, the hypothesis on A and B can be relaxed by using a Hilbert-Schmidt operator X: Let A be a (p,k)-quasihyponormal operator and B^* be an invertible (p,k)-quasihyponormal operator such that $AX = XB$ for a Hilbert Schmidt operators X, then $A^*X = XB^*$. As a consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

1 Introduction Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. For any operator A in $B(H)$ set, as usual, $\mid A \mid = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = \mid A \mid^2 - \mid A^* \mid^2$ (the self commutator of A), and consider the following standard definitions: A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-quasihyponormal if $A^*((A^*A)^p - (AA^*)^p)A \geq 0$ $(0 < p \leq 1)$, (p,k)-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0$ $(0 < p \leq 1, k \in N)$. If $p = 1$, $k = 1$ and $p = k = 1$, then A is k-quasihyponormal, p-quasihyponormal and quasihyponormal respectively. A is said to be normaloid if $\|A\| = r(A)$ (the spectral radius of A). Let (N), (HN), $(Q(p))$, $(Q(p,k))$ and (NL) denote the classes constituting of normal, hyponormal, p-quasihyponormal, (p,k)-quasihyponormal, and normaloid operators. These classes are related by proper inclusion:

$$(N) \subset (HN) \subset (Q(p)) \subset (Q(p,k)) \subset (NL).$$

A (p,k)-quasihyponormal operator is an extension of hyponormal, p-hyponormal, p-quasihyponormal and k- quasihyponorma. For an example of an operator in each these classes that does not belong to the smaller classes (see [12, 13, 14, 24]). Here we present some examples.

1) If T is hyponormal and invertible and $0 < p < 1$ and T^p exists, then T is p-hyponormal and need not be hyponormal.

and

2) If M is the closure of the range of T^k, then T is (p,k)-quasihyponormal if and only if $T|_M$ is p-hyponormal. Thus, if T has dense range, then T is (p,k)-quasihyponormal if and only if T is p-hyponormal.

and

3) It follows easily from (2) above that: A unilateral weight shift with weight sequence $\{a_n\}_{n=0}^{\infty}$ is (p,k)-quasihyponormal if and only if the sequence $\{a_n\}_{n=k}^{\infty}$ is non-decreasing, (so the first k terms can be arbitrary). However, a bilateral weighted shift that is (p,k)-quasihyponormal must actually be hyponormal (by item (2) above).

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A 1-hyponormal operator is called hyponormal operator, which has been studied by many authors and it is known that hyponormal operators have many intersting properties similar to those of normal operators (see[27]). A. Aluthge, B.C.Gupta, A.C. Arora and P.Arora introduced p-hyponormal, p-quasihyponormal and k-quasihyponormal operators, respectively (see[1, 2, 8]), and now it is known that these operators have many interesting properties (see[7, 13, 21, 23]). It is obvious that p-hyponormal operators are q-hyponormal for $0 < q \leq p$ by Lowner-Heinz's inequality (See [10, 15]). But $(p,1)$-quasihyponormal operators are not always $(q,1)$-quasihyponormal operators for for $0 < q \leq p$ (see[24]). Also, it is obvious that (p,k)-quasihyponormal operators are $(p,k+1)$-quasihyponormal.

The familiar Fuglede-Putnam Theorem is as follows (see [4], [9] and [11]):

Theorem 1.1 *If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

S.K.Berberian [3] relaxes the hypothesis on A and B in Theorem 1.1 at the cost of requiring X to be Hilbert-Schmidt class. H.K.Cha [5] showed that the hyponormality in the result of Berberian [3] can be replaced by the quasihyponormality of A and B^* under some additional conditions. Recently M.Y.Lee [12] proved that if A is p-quasihyponormal operator and B^* is an invertible p-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and $\||A|^{1-p}\|$.
$\||B^{-1}|^{1-p}\| \leq 1$, then $A^*X = XB^*$. In this paper we will show that this result remains true without the condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. We also prove that the above result remains true for (p,k)-quasihyponormal without the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$ showing that we don't need this additional condition as in ([13], Theorem 4). Let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq ... \geq 0$ denote the singular values of T i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if $\|T\|_p = [\sum_{i=1}^{\infty} s_j(T)^p]^{\frac{1}{p}} = [tr|T|^p]^{\frac{1}{p}} < \infty$, $1 \leq p < \infty$, where tr denotes the trace functional. Hence $C_1(H)$ is the trace class, $C_2(H)$ is the Hilbert -Schmidt class, and C_∞ is the class of compact operators with $\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$ denoting the usual operator norm. For the general theory of the Schatten $p-$ classes the reader is referred to [17], [18]. Let $\delta_{A,B}$ be the generalized derivation defined on $B(H)$ by $\delta_{A,B}(X) = AX - XB$. It is clear that $\delta_{A,B}(C_p) \subseteq C_p$. However it can also happen that $\delta_{A,B}(X) \in C_p$ for some $X \in B(H) \setminus C_p$, hence $ran(\delta_{A,B}\mid_{C_p}) \subseteq ran\delta_{A,B} \cap C_p$ and then we also have $\overline{ran(\delta_{A,B}\mid_{C_p})}^{C_p} \subseteq \overline{ran\delta_{A,B} \cap C_p}^{C_p}$, where $\overline{(.)}^{C_p}$ denotes the closure of the C_p norm . A.Turnsek [19] asked the following question: When the reverse inclusion is possible ? In this paper we consider the question when

$$\overline{ran(\delta_{A,B}\mid_{C_2})}^{C_2} = \overline{ran\delta_{A,B} \cap_{C_2}}^{C_2}. \tag{1.1}$$

Or equivalently, if $\delta_{A,B}(X) \in C_2$, then $\delta_{A,B}(X) = Lim_n \delta_{A,B}(X_n)$, and $X_n \in C_2$. We prove that (1.1) holds in the case when A is (p,k)-quasihyponormal and B^* is invertible (p,k)-quasihyponormal

2 Main results

Lemma 2.1 *[25, Lemma 3] Let A be a (p,k)-quasihyponormal operator on Hilbert space H. if $\lambda \in C$, $x \in H$ and $Ax = \lambda x$, then $A^*x = \overline{\lambda}x$.*

Theorem 2.1 *Let A and B operators in $B(H)$. If A and B^* are p-quasihypormal operators, then the operator $\mathcal{K} : C_2(H) \to C_2(H)$ defined by $\mathcal{K}X = AXB$ is p-quasihyponormal.*

Proof. It is known [3] that $\mathcal{K}^*X = A^*XB^*$. Note that
by the uniqueness of the square root of a positive operators we have

$$(\mathcal{K}^*\mathcal{K})^{\frac{1}{2}}X = |\mathcal{K}|X = |A|X|B^*|,\ (\mathcal{K}\mathcal{K}^*)^{\frac{1}{2}}X = |\mathcal{K}^*|X = |A^*|X|B|.$$

Thus for $X \in C_2(H)$

$$\mathcal{K}^*\left(|\mathcal{K}|^{2p} - |\mathcal{K}^*|^{2p}\right)\mathcal{K}X = \mathcal{K}\left(|\mathcal{K}|^{2p} - |\mathcal{K}^*|^{2p}\right)AXB =$$

$$A^*|A|^{2p}AXB|B^*|^{2p}B^* - A^*|A^*|^{2p}AXB|B|^{2p}B^* =$$

$$A^*\left(|A|^{2p} - |A^*|^{2p}\right)AXB|B^*|^{2p}B^* + A^*|A|^{2p}AXB\left(|B^*|^{2p} - |B|^{2p}\right)B^*.$$

Now since A and B^* are p-quasi hyponormal,

$$\mathcal{K}^*\left(|\mathcal{K}|^{2p} - |\mathcal{K}^*|^{2p}\right)\mathcal{K} \geq 0$$

and so, $\mathcal{K}$ is p-quasihyponormal. by this we complete the proof. □

Now we are ready to extend Fuglede-Putnam theorem to p-quasihyponormal operators.

Theorem 2.2 *Let A be p-quasihyponormal operator and B^* be an invertible p-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.*

Proof. Let $\mathcal{K}$ be defined on $C_2(H)$ by $\mathcal{K}Y = AYB^{-1}$ for all $Y \in C_2(H)$. Since B^* is p-quasihyponormal, $(B^*)^{-1}$ is p-quasihyponormal (see [12]). Then it follows from Lemma 2.1 that $\mathcal{K}$ is p-quasihyponormal, furthermore, $\mathcal{K}X = AXB^{-1} = X$ and so, X ia an eigenvector of $\mathcal{K}$. Now by applying Lemma 2.1 we get $\mathcal{K}^*X = A^*X(B^{-1})^* = X$, that is, $A^*X = XB^*$ and the proof is achieved. □

Remark 2.1 It is shown in [12] that if A is p-quasihyponormal operator and B^* is an invertible p-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$, then $A^*X = XB^*$. We proved in Theorem 2.2 that we don't need the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. Also M. Young Lee showed in [13] if A is (p,k)-quasihyponormal operator and B^* is an invertible (p,k)-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$, then $A^*X = XB^*$. Here also we don't need the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. Indeed, by a slight modification in the proof of Theorem 2.1 we show that this theorem remains true if we consider a (p,k)-quasihyponormal operator instead of a p-quasihyponormal operator. Since an invertible (p,k)-quasihyponormal is (p,k)-quasihyponormal (see [12]), Theorem 2.2 remains true with (p,k)-quasihyponormal operators without the additional condition $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. Thus we have proved the following theorem:

Theorem 2.3 *Let A be (p,k)-quasihyponormal operator and B^* be an invertible (p,k)-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.*

As a consequences of Theorem 2.3, we obtain

Corollary 2.1 *[3] Assume that A, B and X are operators in an Hilbert space H such that $AX = XB$. Assume also that X is an operator of Hilbert-Schmidt class. Then $A^*X = XB^*$ under either of the following hypothesis*

(1) A and B^* are hyponormal;

(2) B is invertible and $\|A\|.\|B^{-1}\| \leq 1$.

Corollary 2.2 *[5] Let A be quasihyponormal operator and B^* be an invertible quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.*

Corollary 2.3 *[12] Let A be p-quasihyponormal operator and B^* be an invertible p-quasihyponormal ($0 < p \leq 1$) operator such that $AX = XB$ for $X \in C_2(H)$ and $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. Then $A^*X = XB^*$.*

Corollary 2.4 *[13] Let A be a (p,k)-quasihyponormal operator and B^* be an invertible (p,k)-quasihyponormal operator such that $AX = XB$ for $X \in C_2(H)$ and $\||A|^{1-p}\|.\||B^{-1}|^{1-p}\| \leq 1$. Then $A^*X = XB^*$.*

Corollary 2.5 *Let A be a p-hyponormal operator and B^* be an invertible p-hyponormal operator such that $AX = XB$ for $X \in C_2(H)$. Then $A^*X = XB^*$.*

Theorem 2.4 *Let A, B be operators in $B(H)$ and $S \in C_2$. Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 \tag{2.1}$$

and

$$\left\|\delta^*_{A,B}(X) + S\right\|_2^2 = \left\|\delta^*_{A,B}(X)\right\|_2^2 + \|S\|_2^2 \tag{2.2}$$

if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, for all $X \in C_2(H)$.

Proof. It is well known that the Hilbert-Schmidt class $C_2(H)$ is a Hilbert space under the inner product

$$\langle Y, Z\rangle = tr(Z^*Y) = tr(YZ^*).$$

Note that

$$\begin{aligned}\|\delta_{A,B}(X) + S\|^2 &= \|\delta_{A,B}(X)\|^2 + \|S\|^2 + 2Re\,\langle \delta_{A,B}(X), S\rangle \\ &= \|\delta_{A,B}(X)\|^2 + \|S\|^2 + 2Re\,\left\langle X, \delta^*_{A,B}(S)\right\rangle\end{aligned}$$

and

$$\|\delta^*_{A,B}(X) + S\|^2 = \|\delta^*_{A,B}(X)\|^2 + \|S\|^2 + 2Re\,\langle X, \delta_{A,B}(S)\rangle\,.$$

Hence by the equality $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$ we obtain (2.1) and (2.2). □

Corollary 2.6 *Let A, B be operators in $B(H)$ and $S \in C_2$. Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$

and

$$\left\|\delta^*_{A,B}(X) + S\right\|_2^2 = \left\|\delta^*_{A,B}(X)\right\|_2^2 + \|S\|_2^2$$

if and only if either of the following hypotheses hold:

(1) A and B^* hyponormal operators
(2) $A, B \in B(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in \mathcal{H}$
(3) A is p-hyponormal and B^* is invertible p-hyponormal.
(4) A is k-quasihyponormal and B^* is invertible k-quasihyponormal.
(5) A is p-quasihyponormal and B^* is invertible p-quasihyponormal.

Now we will answer the question when

$$\overline{ran(\delta_{A,B}\mid_{C_2})}^{C_2} = \overline{ran\delta_{A,B}\cap_{C_2}}^{C_2}.$$

Theorem 2.5 *Let A and B be operators in $B(H)$ such that $Ker\delta_{A,B} \subseteq Ker\delta_{A^*,B^*}$. Then*

$$\overline{ran(\delta_{A,B} \mid_{C_2(H)})}^{C_2(H)} = \overline{ran\delta_{A,B} \cap_{C_2(H)}}^{C_2(H)}.$$

Proof. Since

$$ran^{\perp}(\delta_{A,B} \mid_{C_2(H)}) = Ker(\delta_{A,B} \mid_{C_2(H)}).$$

Thus if $S \in ran^{\perp}(\delta_{A,B} \mid_{C_2(H)})$, then $S \in Ker(\delta_{A,B} \mid_{C_2(H)})$. Now Theorem 2.4 would imply that $S \in (ran\delta_{A,B} \cap C_2(H))^{\perp}$, that is,

$$ran^{\perp}(\delta_{A,B} \mid_{C_2(H)}) \subseteq (ran\delta_{A,B} \cap C_2(H))^{\perp}.$$

Since trivially

$$(ran\delta_{A,B} \cap C_2(H))^{\perp} \subseteq ran^{\perp}(\delta_{A,B} \mid_{C_2(H)}).$$

Thus

$$ran^{\perp}(\delta_{A,B} \mid_{C_2(H)}) = (ran\delta_{A,B} \cap C_2(H))^{\perp}.$$

Consequently

$$\overline{ran(\delta_{A,B} \mid_{C_2(H)})}^{C_2(H)} = \overline{ran\delta_{A,B} \cap C_2(H)}^{C_2(H)}.$$

□

Corollary 2.7 *Let A and B be operators in $B(H)$. Then*

$$\overline{ran(\delta_{A,B} \mid_{C_2(H)})}^{C_2(H)} = \overline{ran\delta_{A,B} \cap_{C_2(H)}}^{C_2(H)}.$$

under any of the hypotheses (1)-(5) in Corollary 2.6.

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