C-SEMIGROUPS AND INTEGRAL OPERATORS

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Abstract. Let $T(t), 0 \leq t < \infty$, be a one parameter C-semigroup of bounded linear operators on a Banach space X with generator A. The object of this paper is to discuss when can a C-semigroup be an ideal of integral operators.

Introduction. Let X^* be the dual of the Banach space X, $L(X)$ be the space of all bounded linear operators from X into X and for $T \in L(X)$, $\|T\|$ denote the operator norm of T.

An operator $T \in L(X)$ is called **integral operator** if T admits a factorization:

$$\begin{array}{ccccc} & X & \xrightarrow{T} & X & \xrightarrow{i} & X^{**} \\ P & \downarrow & & & & \uparrow & Q \\ & L^\infty(\Omega,\mu) & & \xrightarrow{j} & & L^1(\Omega,\mu), \end{array}$$

where μ is a finite regular Borel measure on some compact Hausdorff space Ω, $j : L^\infty(\Omega,\mu) \to L^1(\Omega,\mu)$ is the canonical inclusion of $L^\infty(\Omega,\mu)$ into $L^1(\Omega,\mu)$, $Q \in L(L^1(\Omega,\mu), X^{**})$, $P \in L(X, L^\infty(\Omega,\mu))$ and i is the canonical embedding of X into X^{**}. Let $I(X)$ denote the set of all integral operators in $L(X)$. For $T \in I(X)$, set $\|T\|_{int} = |\mu|(\Omega)$. It is known that $I(X)$ is an ideal of operators in $L(X)$ and $\|.\|_{int}$ is an ideal norm on $I(X)$.

A one parameter family $T(t)$, $t \in [0,\infty)$, of bounded linear operators from X into X is called a one parameter C-semigroup of operators on X if : (i) $T(0) = C$ and (ii) $CT(s+t) = T(s)T(t)$ for all s,t in $[0,\infty)$, where C is an injective bounded linear operator on X. A C-semigroup, $T(t)$, is called strongly continuous if $\lim_{t\to 0^+} T(t)x = Cx$ for every $x \in X$. A C-semigroup for which there exist constants $M > 0$ and $\omega \in R$ (the set of real numbers) such that $\|T(t)\| \leq Me^{\omega t}$ is called an exponentially bounded C-semigroup. The linear operator A defined by:

$$\begin{aligned} D(A) &= \{x \in X : C^{-1} \lim_{t\to 0^+} \frac{T(t)x - Cx}{t} exists\} \\ Ax &= C^{-1} \lim_{t\to 0^+} \frac{T(t)x - Cx}{t} \end{aligned}$$

for $x \in D(A)$ is called the **generator** of the C-semigroup $T(t)$ and $D(A)$ is the domain of A. The resolvent set of A is denoted by $\rho(A)$ and for $\lambda \in \rho(A)$, the operator $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent operator of A. It is known,[3] , for an exponentially bounded C-semigroup, that $D(A)$ is dense in Range(C), A is a closed operator and the resolvent operator $R(\lambda, A)$ is a bounded operator for all $\lambda \in \rho(A)$. We refer to [2] and [3] for excellent monographs on C-semigroups.

The object of this paper is to discuss when can a C-semigroup be an ideal of integral operators $I(X)$?

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Pazy,[10] , studied the problem for c_0-semigroups and the case of the ideal of compact operators, $K(X) \subseteq L(X)$ for any Banach space X, Khalil and Deeb,[7], studied the problem for the ideal of Schatten Classes $C_p(H) \subseteq L(H)$, where H is a Hilbert space and the case for C-semigroup and the ideal of (p,q)-summing operators was studied by Al-Sharif [1] .

Throughout this paper, the dual of a Banach X is denoted by X^*, and $B_1(X)$ is the open unit ball of X. For $x^* \in X^*$ and $x \in X$ the value of x^* at x is denoted by $< x^*, x >$. The set of real numbers will be denoted by R, and the set of natural numbers by N.

I. C-Semigroups and Integral Operator.

Let X be a Banach space and $T(t)$ be a strongly continuous exponentially bounded C-semigroup on X with generator A. Assume that Range(C) is dense in X. Thus by Theorem 2.4 in [3] , $D(A)$ is dense in X.

Lemma 1.1. Let $T_n \in I(X)$. for which $\sup_n \|T_n\|_{int} \leq \xi$ for some $\xi > 0$. If $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$, then $T \in I(X)$ and $\|T\|_{int} \leq \xi$.

Proof. Let (T_n) be a sequence in $I(X)$ such that $\sup_n \|T_n\|_{int} \leq \xi$ for some $\xi > 0$ and $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$. By Proposition 17.5.2 in [6] we have :

$$|tr(T_n S)| \leq \|T_n\|_{int} \|S\|$$

for every finite rank operator S in $L(X)$. If $S = \sum_{i=1}^{m} x_i^* \otimes x_i$, then

$$|tr(T_n S)| = \left| \sum_{i=1}^{m} \langle T_n x_i \, , \, x_i^* \rangle \right| \leq \|T_n\|_{int} \|S\| \leq \xi \|S\| .$$

Since $\lim_{n\to\infty} T_n x = Tx$ for all $x \in X$ we get :

$$|tr(TS)| = \left| \sum_{i=1}^{m} \langle Tx_i \, , x_i^* \rangle \right| \leq \xi \|S\| .$$

Another application of Proposition 17.5.2 in [6] implies T is an integral operator and $\|T\|_{int} \leq \xi$. ■

Lemma 1.2. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup of bounded linear operators on X with generator A. If $T(t) \in I(X)$ for all $t > 0$ and $\|T(t)\|_{int} < \xi$ in $(0, \epsilon)$ for some $\epsilon > 0$, then for $0 < a < \epsilon$, the operator $G_a : X \to X$, $G_a x = C \int_0^a e^{-\lambda s} T(s) x \, ds$, for any $x \in X$, belongs to $I(X)$ and $\|G_a\|_{int} < \|C\| \xi \frac{1-e^{-\lambda a}}{\lambda}$.

Proof. For all $n \in N$, since $T(t) \in I(X)$ for all $t > 0$ and $\|T(t)\|_{int} < \xi$ in $(0, \epsilon)$ for some $\epsilon > 0$, the operators G_{an} defined by $G_{an} x = \sum_{k=1}^{n} \frac{aCe^{-\lambda t_k} T(t_k) x}{n}$ is in $I(X)$ and

$$\begin{aligned} \|G_{an}\|_{int} &= \left\| \sum_{k=1}^{n} \frac{ae^{-\lambda t_k} CT(t_k)}{n} \right\|_{int} \\ &\leq \sum_{k=1}^{n} \frac{ae^{-\lambda t_k} \|C\| \|T(t_k)\|_{int}}{n} \\ &< \xi \|C\| \sum_{k=1}^{n} \frac{ae^{-\lambda t_k}}{n} < \xi \|C\| \frac{1 - e^{-\lambda a}}{\lambda}, \end{aligned}$$

where, $\frac{(k-1)a}{n} < t_k < \frac{ka}{n}$.

Since $T(s)$ is strongly continuous, the operators G_{na} converge strongly to the operator G_a. By Lemma 1.1, the operator $G_a \in I(X)$ and $\|G_a\|_{int} < \xi \|C\| \frac{1-e^{-\lambda a}}{\lambda}$. ■

Lemma 1.3 Let $T(t)$ be a strongly continuous C-semigroup of bounded linear operators on X. If $T(t_0) \in I(X)$ for some $t_0 > 0$, then $CT(t) \in I(X)$ for all $t > t_0$.

Proof. Suppose $T(t_0) \in I(X)$ for some $t_0 > 0$. Then,

$$CT(t) = CT(t - t_0 + t_0) = T(t - t_0)T(t_0).$$

Thus $CT(t) \in I(X)$ for all $t > t_0$. ■

Now we prove one of the main results of this paper.

Theorem 1.4. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup of bounded linear operators on X with generator A. If $T(t) \in I(X)$ and $\|T(t)\|_{int} \leq \xi$ in $[0, \epsilon)$ for some $\epsilon > 0$, then $C^2R(\lambda, A) \in I(X)$ for all $\lambda \in \rho(A)$ and for $\lambda > \omega > 0$, $\left\|C^2R(\lambda, A)\right\|_{int} \leq \frac{\beta}{\lambda-\omega}$ for some $\beta > 0$.

Proof. Let $x \in X$ and $\lambda \in \rho(A)$, $\lambda \in R$, $\lambda > \omega > 0$. Then by Theorem 3.3, [3] we have :

$$CR(\lambda, A)x = R(\lambda, A)Cx = \int_0^\infty e^{-\lambda s}T(s)x\,ds.$$

For $t \in (0, \epsilon)$ and $\lambda > \omega > 0$, define:

$$\begin{aligned} R_t(\lambda, A)x &= C\int_t^\infty e^{-\lambda s}T(s)x\,ds \\ &= \int_t^\infty e^{-\lambda s}CT(s-t+t)x\,ds \\ &= \int_t^\infty e^{-\lambda s}T(t)T(s-t)x\,ds \\ &= T(t)\int_t^\infty e^{-\lambda s}T(s-t)x\,ds. \end{aligned}$$

Since $I(X)$ is an ideal in $L(X)$, $T(t) \in I(X)$ for $t \in (0, \epsilon)$ and the operator P defined by $P(x) = \int_t^\infty e^{-\lambda s}T(s-t)\,xds$ is a bounded linear operator in $L(X)$ for $\lambda > \omega$, the operators $R_t(\lambda, A) \in I(X)$ for $\lambda \in R$, $\lambda > \omega > 0$ and $t \in (0, \epsilon)$. Further more, since $T(t)$ is an exponentially bounded C-semigroup, for $\lambda > \omega$ and $t \in (0, \epsilon)$, we get:

$$\|R_t(\lambda, A)\|_{int} = \|T(t)P\|_{int} \leq \|T(t)\|_{int}\|P\| \leq \xi\, e^{-\omega t}\int_t^\infty e^{-\lambda s}Me^{\omega s}\,ds = \frac{M\xi}{\lambda-\omega}e^{-\lambda t}.$$

Since $T(t) \in I(X)$, by Lemma 1.2 the operator $R_t(\lambda, A) - C^2R(\lambda, A)$,

$$\left(R_t(\lambda, A) - C^2R(\lambda, A)\right)x = C\int_0^t e^{-\lambda s}T(s)x\,ds$$

is in $I(X)$ and $\left\|R_t(\lambda, A) - C^2 R(\lambda, A)\right\|_{int} \leq \xi \|C\| \frac{1-e^{-\lambda t}}{\lambda}$ for $t \in (0, \epsilon)$.

Since $R_t(\lambda, A) \in I(X)$ for $t \in (0, \epsilon)$, $\lim_{t \to 0^+} \xi \|C\| \frac{1-e^{-\lambda t}}{\lambda} = 0$ and $I(X)$ is a Banach space it follows that $C^2\ R(\lambda, A) \in I(X)$ for $\lambda > \omega > 0$ and

$$\begin{aligned}
\left\|C^2 R(\lambda, A)\right\|_{int} &= \left\|\lim_{t \to 0^+} R_t(\lambda, A)\right\|_{int} \\
&= \lim_{t \to 0^+} \|R_t(\lambda, A)\|_{int} \\
&\leq \lim_{t \to 0^+} \frac{M\xi}{\lambda - \omega} e^{-\lambda t} = \frac{\beta}{\lambda - \omega}.
\end{aligned}$$

The resolvent identity,

$$C^2 R(\mu, A) = C^2 R(\lambda, A) + (\lambda - \mu) C^2 R(\lambda, A) R(\mu, A)$$

implies that $C^2 R(\mu, A) \in I(X)$ for all $\mu \in \rho(A)$. ■

Theorem 1.5. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup of bounded linear operators on X with generator A. If $R(\lambda, A) \in I(X)$ for all $\lambda \in \rho(A)$ and $\|R(\lambda, A)\|_{int} \leq \frac{\beta}{\lambda - \omega}$ for $\lambda > \omega$, then $T(t) \in I(X)$ and $\|T(t)\|_{int} \leq \xi$ in $(0, \varepsilon)$ for some $\varepsilon > 0$. Further $CT(t) \in I(X)$ for all $t > 0$.

Proof. Let $\lambda \in \rho(A)$, $\lambda > \omega > 0$ and $x \in D(A)$. then

$$\begin{aligned}
\|\lambda R(\lambda, A) T(t)x - T(t)x\| &= \|AR(\lambda, A)T(t)x\| \\
&= \|R(\lambda, A) AT(t)x\| \\
&\leq \|R(\lambda, A)\| \, \|AT(t)x\|. \\
&\leq \|R(\lambda, A)\|_{int} \|AT(t)x\| \\
&\leq \frac{\beta}{\lambda - \omega} \|AT(t)x\|.
\end{aligned}$$

Taking the limit as $\lambda \to \infty$, we get $\lim_{\lambda \to \infty} \lambda R(\lambda, A) T(t)x = T(t)x$. But $D(A)$ is dense in X. Therefore $\lim_{\lambda \to \infty} \lambda R(\lambda, A) T(t)x = T(t)x$ for all $x \in X$.

Since $R(\lambda, A) \in I(X)$ and $T(t) \in L(X)$ for all $t > 0$, it follows that $\lambda R(\lambda, A) T(t) \in I(X)$ and so,

$$\|\lambda R(\lambda, A) T(t)\|_{int} \leq \|\lambda R(\lambda, A)\|_{int} \|T(t)\| \leq \frac{\beta \lambda}{\lambda - \omega} \|T(t)\|.$$

Consequently, since $T(t)$ is exponentially bounded there exist $\gamma > 0$ and $\varepsilon > 0$, such that $\|\lambda R(\lambda, A) T(t)\|_{int} \leq \gamma \frac{\beta\lambda}{\lambda - \omega}$ for all $t \in (0, \varepsilon)$, and $\lambda > \omega$. Lemma 1.1 implies that $T(t) \in I(X)$ for all $t \in (0, \varepsilon)$ and Lemma 1.3 then implies $CT(t) \in I(X)$ for all $t > 0$. Further, since $\{\frac{\lambda}{\lambda - \omega} : \lambda > \omega \geq 0\}$ is a bounded set, it follows that $\|T(t)\|_{int} \leq \gamma \sup_{\lambda} \frac{\beta\lambda}{\lambda - \omega}$ for $t \in (0, \varepsilon)$. ■

Theorem 1.6. Let $T(t)$ be an exponentially bounded strongly continuous C-semigroup on X, with generator A. If $T(t) \in I(X)$ for all $t > 0$, and $T \in L^1((0, t_0), I(X))$ for some $t_0 > 0$, then $C^2 R(\lambda, A) \in I(X)$ for all $\lambda \in \rho(A)$, and for $\lambda > \omega > 0$, $\left\|C^2 R(\lambda, A)\right\|_{int}$ is bounded.

Proof. Since $T \in L^1((0, t_0), I(X))$, then for $0 < t < t_0$

$$\int_0^{t_0} \|T(s)\|_{int} \, ds = \int_0^t \|T(s)\|_{int} \, ds + \int_t^{t_0} \|T(s)\|_{int} \, ds < \infty.$$

Thus, $T \in L^1((0,t), I(X))$ for all $t < t_0$ and so $\lim\limits_{t\to 0}\int\limits_0^t \|T(s)\|_{int}\, ds = 0$.

For $t > 0$ and $\lambda \in \rho(A)$, $\lambda > \omega > 0$, define :

$$\begin{aligned} R_t(\lambda, A)x &= C\int\limits_t^\infty e^{-\lambda s}T(s)x\, ds \\ &= T(t)\int\limits_t^\infty e^{-\lambda s}T(s-t)x\, ds. \end{aligned}$$

Since $T(t) \in I(X)$ and the operator P, $P(x) = \int\limits_0^\infty e^{-\lambda s}T(s-t)\, xds$ is a bounded operator in $L(X)$ for $\lambda > \omega > 0$, then $R_t(\lambda, A) \in I(X)$, and

$$\begin{aligned} \left\|R_t(\lambda, A) - C^2R(\lambda, A)\right\|_{int} &= \left\|C\int\limits_0^t e^{-\lambda s}T(s)\, ds\right\|_{int} \\ &\le \|C\|\int\limits_0^t \|T(s)\|_{int}\, ds, \end{aligned}$$

noting that $\sup\limits_{s\in(0,t)} e^{-\lambda s} \le 1$. Consequently;

$$\lim_{t\to 0}\left\|R_t(\lambda, A) - C^2R(\lambda, A)\right\|_{int} = 0.$$

Since $I(X)$ is a Banach space, it follows that $C^2R(\lambda, A) \in I(X)$ for $\lambda \in R$, $\lambda > \omega > 0$, and

$$\begin{aligned} \left\|C^2R(\lambda, A)\right\|_{int} &= \left\|C\int\limits_0^\infty e^{-\lambda s}T(s)\, ds\right\|_{int} \\ &= \left\|\sum_{n=0}^\infty C\int\limits_{nt_0}^{(n+1)t_0} e^{-\lambda s}T(s)\, ds\right\|_{int} \\ &= \left\|\sum_{n=0}^\infty T(nt_0)\int\limits_{nt_0}^{(n+1)t_0} e^{-\lambda s}T(s-nt_0)\, ds\right\|_{int} \\ &= \left\|\sum_{n=0}^\infty T(nt_0)\int\limits_0^{t_0} e^{-n\lambda t_0}e^{-\lambda s}T(s)\, ds\right\|_{int} \\ &\le \sum_{n=0}^\infty \|T(nt_0)\|\, e^{-n\lambda t_0}\int\limits_0^{t_0} e^{-\lambda s}\, \|T(s)\|_{int}\, ds \\ &= M\int\limits_0^{t_0} e^{-\lambda s}\, \|T(s)\|_{int}\, ds\ \sum_{n=0}^\infty e^{n\omega t_0}e^{-n\lambda t_0} \\ &= M\int\limits_0^{t_0} e^{-\lambda s}\, \|T(s)\|_{int}\, ds\left(\sum_{n=0}^\infty e^{(\omega-\lambda)t_0 n}\right) \\ &\le \frac{M}{1-e^{(\omega-\lambda)t_0}}\int\limits_0^{t_0} e^{-\lambda s}\, \|T(s)\|_{int}\, ds. \end{aligned}$$

noting that for large λ, $e^{(\omega-\lambda)t_0} < 1$ and $\int\limits_0^{t_0} e^{-\lambda s}\, \|T(s)\|_{int}\, ds \le \int\limits_0^{t_0} \|T(s)\|_{int}\, ds$ for all $n \ge 1$ and all $\lambda > 0$. Since $T(s) \in L^1((0,t_0), I(X))$, it follows that $\left\|C^2R(\lambda, A)\right\|_{int}$ is bounded for $\lambda > \omega > 0$.

Now, let μ be any element in $\rho(A)$. Then from the resolvent identity we have :

$$C^2R(\mu, A) = C^2R(\lambda, A) + (\lambda - \mu)C^2R(\lambda, A)R(\mu, A)$$

for any $\lambda \in \rho(A)$. Thus, if $\lambda \in \rho(A)$ and $\lambda > \omega$, we get $C^2R(\mu, A) \in I(X)$. Hence $C^2R(\mu, A) \in I(X)$ for all $\mu \in \rho(A)$. ■

II. Differentiable C-Semigroup and $I(X)$.

Let X be a Banach space and $T(t)$ be a differentiable strongly continuous exponentially bounded C-semigroup on X with generator A. Assume that Range(C) is dense in X. Thus by Theorem 2.4 in [3], $D(A)$ is dense in X.

Theorem 2.1 Let $T(t)$ be a differentiable strongly continuous exponentially bounded C-semigroup on X with generator A. If there exists $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A) \in I(X)$, then $T(t) \in I(X)$ for all $t > 0$.

Proof. Let $\lambda_0 \in \rho(A)$ and $\lambda_0 = 0$. Define $B(t)x = \int_0^t T(s)x\, ds$. Then $B \in L(X)$ and

$$AB(t)x = A\int_0^t T(s)x\,ds = T(t)x - Cx = (T(t) - C)x$$

for all $x \in X$, (Lemma 2.7,[3]). Hence

$$-AB(t)x = (0 - A)B(t)x = (C - T(t))x.$$

So $B(t)x = R(0, A)(C - T(t))x$ for all $x \in X$. Thus $B(t) = R(0, A)(C - T(t))$. But $R(0, A) \in I(X)$. So $B(t) \in I(X)$ for all $t > 0$.

Now : since $T(t)$ is strongly continuous, then for $x \in D(A)$, $B'(t)x$ exists and

$$\begin{aligned} B'(t)x &= \lim_{h\to 0}\frac{B(t+h)x - B(t)x}{h} \\ &= \lim_{n\to\infty} n\left(B(t+\frac{1}{n})x - B(t)x\right) \\ &= \lim_{n\to\infty} n\left(R(0,A)(C - T(t+\frac{1}{n}))x - R(0,A)(C - T(t))x\right) \\ &= \lim_{n\to\infty} nR(0,A)\left(T(t+\frac{1}{n})x - T(t)x\right). \end{aligned}$$

Define $D_n(t)x = nR(0,A)\left(T(t)x - T(t+\frac{1}{n})x\right)$. Since $R(0, A) \in I(X)$, it follows that $D_n(t) \in I(X)$ for all $t > 0$ and all $n \in N$. But

$$B'(t)x = \frac{d}{dt}\int_0^t T(s)x\,ds = T(t)x.$$

Consequently; since $T(t)$ is differentiable, then $\lim_{n\to\infty} n\left(T(t) - T(t+\frac{1}{n})\right) = -T'(t)$ and

$$T(t)x = \lim_{n\to\infty} D_n(t)x = -R(0, A)T'(t)x.$$

But $D(A)$ is dense in X. Thus $T(t) \in I(X)$. Further :

$$\|T(s)\|_{int} \le \left\|-R(0,A)T'(t)\right\|_{int} \le \|R(0,A)\|_{int}\left\|T'(t)\right\| < \infty.$$

For $\lambda_0 \neq 0$, define $S(t) = e^{\lambda_0 t}T(t)$. Then if G is the generator of $T(t)$, then $G - \lambda_0$ is the generator of $e^{-\lambda_0 t}T(t)$. So if $\lambda_0 \in \rho(G - \lambda_0)$, then $0 \in \rho(G)$. ■

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