# AVERAGING FUNCTIONS AND INEQUALITIES DUE TO HÖLDER-ROGERS AND MINKOWSKI

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ABSTRACT. We introduce two notations of functions, which we call "subaveraging function" and "superaveraging function". Both the well-known Hölder-Rogers inequality and the Minkowski inequality can be seen simultaneously through these functions. Our study was inspired by the work of L. Maligranda [2].

## 1. SUBAVERAGING FUNCTION AND SUPERAVERAGING FUNCTION

Let $X$ be a set and $S$ a real linear space consisting of real functions on $X$. Let $D$ be a domain in $\mathbf{R}^n$ and $S_0$ a subset of $S$ such that

(0) $(f_1(x), \cdots, f_n(x)) \in D$ for all $f_1, \cdots, f_n \in S_0$ and $x \in X$.

We consider two functions $m, M\colon D \to \mathbf{R}$ such that

(1) $m \circ (f_1, \cdots, f_n)$, $M \circ (f_1, \cdots, f_n) \in S$ for all $f_1, \cdots, f_n \in S_0$.

(2) $m(Lf_1, \cdots, Lf_n) \le L(m \circ (f_1, \cdots, f_n))$, $M(Lf_1, \cdots, Lf_n) \ge L(M \circ (f_1, \cdots, f_n))$ for all $f_1, \cdots, f_n \in S_0$ and all positive linear functionals $L$ from $S$ into $\mathbf{R}$ such that $(Lf_1, \cdots, Lf_n) \in D$ for all $f_1, \cdots, f_n \in S_0$.

Here we say that $L$ is positive if $Lf \ge 0$ for all positive functions $f \in S$.

**Definition 1.** We say that the above functions $m$ and $M$ are subaveraging on $D$ and superaveraging on $D$ with respect to the couple $(S, S_0)$, respectively.

**Remark 1.** Let $\alpha_1, \cdots, \alpha_n \in \mathbf{R}$. Then the following function on $D$ is subaveraging and superaveraging :

$$f(a_1, \cdots, a_n) = \alpha_1 a_1 + \cdots + \alpha_n a_n \quad ((a_1, \cdots, a_n) \in D)$$

This is a trivial case but it gives us an important suggestion for a construction of subaveraging functions and superaveraging functions. We next give non-trivial examples of such functions, which give us useful suggestions.

Let $D = \mathbf{R}^+ \times \mathbf{R}^+$, $S = \mathbf{R}^2$, $S_0 = \mathbf{R}^+ \times \mathbf{R}^+$, where $\mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$. Of course, we regard $S$ as a real linear space consisting of all real functions on the set $\{1, 2\}$. Note that the couple $(D, S_0)$ satisfies the condition (0). In this case, we have

(i) Both $M_1(a, b) = \left(\sqrt{a} + \sqrt{b}\right)^2$ and $M_2(a, b) = \sqrt{ab}$ are superaveraging functions on $D$ with respect to $(S, S_0)$.

(ii) Both $m_1(a, b) = \sqrt{a^2 + b^2}$ and $m_2(a, b) = a^2/b$ are subaveraging functions on $D$ with respect to $(S, S_0)$.

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In fact, let $L$ be a positive linear functional on $S$ such that $(Lf_1, Lf_2) \in D$ for all $f_1, f_2 \in S_0$. Then we can write $L(x,y) = \alpha x + \beta y$ for all $(x,y) \in S$ and some $\alpha, \beta \geq 0$ with $\alpha^2 + \beta^2 \neq 0$. Let $a, b, c, d > 0$, and set $f_1 = (a,b)$ and $f_2 = (c,d)$, hence $f_1, f_2 \in S_0$.

(i) Note that

$$M_1(Lf_1, Lf_2) = M_1(\alpha a + \beta b, \alpha c + \beta d) = \left(\sqrt{\alpha a + \beta b} + \sqrt{\alpha c + \beta d}\right)^2$$

and

$$\begin{aligned} L(M_1 \circ (f_1, f_2)) &= L(M_1(a,c), M_1(b,d)) = \alpha M_1(a,c) + \beta M_1(b,d) \\ &= \alpha\left(\sqrt{a} + \sqrt{c}\right)^2 + \beta\left(\sqrt{b} + \sqrt{d}\right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \left(\sqrt{\alpha a + \beta b} + \sqrt{\alpha c + \beta d}\right)^2 &\geq \alpha\left(\sqrt{a} + \sqrt{c}\right)^2 + \beta\left(\sqrt{b} + \sqrt{d}\right)^2 \quad (\forall \alpha, \beta \geq 0) \\ &\iff ad + bc \geq 2\sqrt{abcd} \end{aligned}$$

and hence $M_1(Lf_1, Lf_2) \geq L(M_1 \circ (f_1, f_2))$ holds, that is, $M_1$ is superaveraging on $D$.

Note that

$$M_2(Lf_1, Lf_2) = M_2(\alpha a + \beta b, \alpha c + \beta d) = \sqrt{(\alpha a + \beta b)(\alpha c + \beta d)}$$

and

$$\begin{aligned} L(M_2 \circ (f_1, f_2)) &= L(M_2(a,c), M_2(b,d)) \\ &= \alpha M_2(a,c) + \beta M_2(b,d) = \alpha\sqrt{ac} + \beta\sqrt{bd}. \end{aligned}$$

Moreover,

$$\sqrt{(\alpha a + \beta b)(\alpha c + \beta d)} \geq \alpha\sqrt{ac} + \beta\sqrt{bd} \quad (\forall \alpha, \beta \geq 0) \iff ad + bc \geq 2\sqrt{abcd}$$

and hence $M_2(Lf_1, Lf_2) \geq L(M_2 \circ (f_1, f_2))$ holds, that is, $M_2$ is superaveraging on $D$.

(ii) Note that

$$m_1(Lf_1, Lf_2) = m_1(\alpha a + \beta b, \alpha c + \beta d) = \sqrt{(\alpha a + \beta b)^2 + (\alpha c + \beta d)^2}$$

and

$$\begin{aligned} L(m_1 \circ (f_1, f_2)) &= L(m_1(a,c), m_1(b,d)) = \alpha m_1(a,c) + \beta m_1(b,d) \\ &= \alpha\sqrt{a^2 + c^2} + \beta\sqrt{b^2 + d^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \sqrt{(\alpha a + \beta b)^2 + (\alpha c + \beta d)^2} &\leq \alpha\sqrt{a^2 + c^2} + \beta\sqrt{b^2 + d^2} \quad (\forall \alpha, \beta \geq 0) \\ &\iff 2abcd \leq (ad)^2 + (bc)^2 \end{aligned}$$

and hence $m_1(Lf_1, Lf_2) \leq L(m_1 \circ (f_1, f_2))$ holds, that is, $m_1$ is subaveraging on $D$.

Note that

$$m_2(Lf_1, Lf_2) = m_2(\alpha a + \beta b, \alpha c + \beta d) = \frac{(\alpha a + \beta b)^2}{\alpha c + \beta d}$$

and

$$\begin{aligned} L(m_2 \circ (f_1, f_2)) &= L(m_2(a,c), m_2(b,d)) = \alpha m_2(a,c) + \beta m_2(b,d) \\ &= \alpha\frac{a^2}{c} + \beta\frac{b^2}{d}. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{(\alpha a+\beta b)^2}{\alpha c+\beta d} &\le \alpha\frac{a^2}{c}+\beta\frac{b^2}{d} \quad (\forall \alpha,\beta\ge 0:\alpha^2+\beta^2\ne 0)\\ &\iff 2abcd\le (ad)^2+(bc)^2 \end{aligned}$$

and hence $m_2(Lf_1, Lf_2) \le L(m_2 \circ (f_1, f_2))$ holds, that is, $m_2$ is subaveraging on $D$.

The above examples are all homogeneous, but we can give non-homogeneous examples. Let $D = \{(x,y) \in \mathbf{R}^2 : x^2+y^2 \le 1\}$, $S = \mathbf{R}^2$ and $S_0 = \{(x,y) \in \mathbf{R}^2 : |x|,|y| \le 1/\sqrt{2}\}$. Note that $(D, S_0)$ satisfies the condition (0). In this case, we have

(iii) Let $A, B, C \in \mathbf{R}$ and put $\varphi(x,y) = Ax + By + C$ for each $(x,y) \in D$. Then $\varphi$ is subaveraging (superaveraging) function on $D$ with respect to $(S, S_0)$ if and only if $C \le 0$ (resp. $C \ge 0$).

Also let $A, B \ge 0$. Then we have

(iv) $Ax^2 + By^2 + C$ is subaveraging with respect to $(S, S_0) \iff C \le 0$.

(iv') $C - Ax^2 - By^2$ is superaveraging with respect to $(S, S_0) \iff C \ge 0$.

In fact, note that an arbitrary real function $\varphi$ on $D$ satisfies that $\varphi \circ (f_1, f_2) \in S$ for all $f_1, f_2 \in S_0$. Now let $L$ be an arbitrary positive linear functional from $S$ into $\mathbf{R}$ such that $(Lf_1, Lf_2) \in D$ for all $f_1, f_2 \in S_0$. Then we can write $L(x,y) = \alpha x + \beta y$ for all $(x,y) \in S$ and some $\alpha, \beta \in \mathbf{R}$. In this case, $\alpha, \beta \ge 0$ by the positivity of $L$. Now let $f_1 = (a,b)$, $f_2 = (c,d) \in S_0$. Then

$$\begin{aligned} & (Lf_1, Lf_1) \in D \quad (\forall f_1, f_2 \in S_0)\\ \iff\ & (\alpha a+\beta b, \alpha c+\beta d) \in D \quad (\forall (a,b),(c,d) \in S_0)\\ \iff\ & (\alpha a+\beta b)^2 + (\alpha c+\beta d)^2 \le 1 \quad (\forall a,b,c,d \in \mathbf{R} : |a|,|b|,|c|,|d| \le 1/\sqrt{2})\\ \iff\ & (a^2+c^2)\alpha^2 + (b^2+d^2)\beta^2 + 2(ab+cd)\alpha\beta \le 1\\ & \qquad (\forall a,b,c,d \in \mathbf{R} : |a|,|b|,|c|,|d| \le 1/\sqrt{2})\\ \iff\ & \alpha^2+\beta^2+2\alpha\beta \le 1\\ \iff\ & \alpha+\beta \le 1. \end{aligned}$$

Hence the set of all positive linear functionals from $S$ into $\mathbf{R}$ such that $(Lf_1, Lf_2) \in D$ for all $f_1, f_2 \in S_0$ must be $\{(\alpha,\beta) \in \mathbf{R}^2 : \alpha,\beta \ge 0, \alpha+\beta \le 1\}$. Therefore we can conclude that a real function $\varphi$ on $D$ is subaveraging (superaveraging) with respect to $(S, S_0)$ if and only if

$$\begin{aligned} &\varphi(\alpha a+\beta b, \alpha c+\beta d) \le \alpha\varphi(a,c) + \beta\varphi(b,d)\\ &(\text{resp. } \varphi(\alpha a+\beta b, \alpha c+\beta d) \ge \alpha\varphi(a,c) + \beta\varphi(b,d)) \end{aligned}$$

for all $a,b,c,d,\alpha,\beta \in \mathbf{R}$ such that $|a|,|b|,|c|,|d| \le 1/\sqrt{2}$, $\alpha,\beta \ge 0$ and $\alpha+\beta \le 1$. This implies immediately (iii). Also this implies that $Ax^2+By^2+C$ is subaveraging with respect to $(S, S_0)$ if and only if

$$\begin{aligned} (3)\quad & 2\alpha\beta(Aab+Bcd)+C\\ &\le A(\alpha-\alpha^2)a^2 + B(\alpha-\alpha^2)c^2 + A(\beta-\beta^2)b^2 + B(\beta-\beta^2)d^2 + (\alpha+\beta)C \end{aligned}$$

for all $a,b,c,d,\alpha,\beta \in \mathbf{R}$ such that $|a|,|b|,|c|,|d| \le 1/\sqrt{2}$, $\alpha,\beta \ge 0$ and $\alpha+\beta \le 1$. Set $\vec{x} = (\sqrt{A}a, \sqrt{B}c)$ and $\vec{y} = (\sqrt{A}b, \sqrt{B}d)$. Then (3) can be transposed into

$$(4) \qquad 2\alpha\beta < \vec{x}, \vec{y} > +C \le (\alpha-\alpha^2)\|\vec{x}\|^2 + (\beta-\beta^2)\|\vec{y}\|^2 + (\alpha+\beta)C.$$

But we can see easily that $2\alpha\beta < \vec{x}, \vec{y} > \le (\alpha-\alpha^2)\|\vec{x}\|^2 + (\beta-\beta^2)\|\vec{y}\|^2$ for all $0 \le \alpha, \beta \le 1$. Therefore we have that $C \le 0$ if and only if (4) is true for all $a,b,c,d,\alpha,\beta \in \mathbf{R}$ such that

$|a|, |b|, |c|, |d| \le 1/\sqrt{2}$, $\alpha, \beta \ge 0$ and $\alpha + \beta \le 1$, and then (iv) holds. Similarly we can see that (iv') holds.

## 2. A construction of subaveraging functions and superaveraging functions

We construct more general subaveraging functions and superaveraging functions. Let $D$, $S$ and $S_0$ be as in Definition 1. Let $T$ be a set and $\alpha_1, \cdots, \alpha_n$, $\beta_1, \cdots, \beta_n$ real functions on $T$ such that

$$\sup_{t\in T}\{\alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n\} < \infty \quad \text{and} \quad -\infty < \inf_{t\in T}\{\beta_1(t)a_1 + \cdots + \beta_n(t)a_n\}$$

for each $(a_1, \cdots, a_n) \in D$. In this case, we define

$$m_\alpha(a_1, \cdots, a_n) = \sup_{t\in T}(\alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n)$$

and

$$M_\beta(a_1, \cdots, a_n) = \inf_{t\in T}(\beta_1(t)a_1 + \cdots + \beta_n(t)a_n)$$

for each $(a_1, \cdots, a_n) \in D$. Then we have the following

**Proposition 1.** *Suppose that $m_\alpha \circ (f_1, \cdots, f_n) \in S$ and $M_\beta \circ (f_1, \cdots, f_n) \in S$ for each $f_1, \cdots, f_n \in S_0$. Then $m_\alpha$ is a subaveraging function on $D$ and $M_\beta$ is a superaveraging function on $D$ with respect to $(S, S_0)$.*

*Proof.* Let $f_1, \cdots, f_n \in S_0$ and $L$ a positive linear functional from $S$ into $\mathbf{R}$ such that $(Lf_1, \cdots, Lf_n) \in D$ for all $f_1, \cdots, f_n \in S_0$. Note that

$$\alpha_1(t)f_1 + \cdots + \alpha_n(t)f_n \le m_\alpha \circ (f_1, \cdots, f_n) \quad \text{and} \quad \beta_1(t)f_1 + \cdots + \beta_n(t)f_n \ge M_\beta \circ (f_1, \cdots, f_n)$$

for all $t \in T$. Then

$$\alpha_1(t)Lf_1 + \cdots + \alpha_n(t)Lf_n = L(\alpha_1(t)f_1 + \cdots + \alpha_n(t)f_n) \le L(m_\alpha \circ (f_1, \cdots, f_n))$$

and

$$\beta_1(t)Lf_1 + \cdots + \beta_n(t)Lf_n = L(\beta_1(t)f_1 + \cdots + \beta_n(t)f_n) \ge L(M_\beta \circ (f_1, \cdots, f_n))$$

for all $t \in T$. Therefore

$$m_\alpha(Lf_1, \cdots, Lf_n) \le L(m_\alpha \circ (f_1, \cdots, f_n)) \quad \text{and} \quad M_\beta(Lf_1, \cdots, Lf_n) \ge L(M_\beta \circ (f_1, \cdots, f_n)),$$

so that $m_\alpha$ is subaveraging on $D$ and $M_\beta$ is superaveraging on $D$. □

## 3. Hölder type functions

Let $S$ and $S_0$ be as in Definition 1. Let $D = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+$ and $p_1, \cdots, p_n \in \mathbf{R}$ with $p_1 + \cdots + p_n = 1$. Set

$$\text{Höl}(a_1, \cdots, a_n) \equiv \text{Höl}_{p_1,\cdots,p_n}(a_1, \cdots, a_n) = \prod_{i=1}^n a_i^{p_i}$$

for each $(a_1, \cdots, a_n) \in D$. In this case, we have the following

**Proposition 2.** *Suppose that* Höl $\circ$ $(f_1, \cdots, f_n) \in S$ *for all* $f_1, \cdots, f_n \in S_0$. *Then*

(i) *If all $p_i$ are positive, then* $\text{Höl}_{p_1,\cdots,p_n}$ *is a superaveraging function on $D$ with respect to $(S, S_0)$.*

(ii) *If the only one of $\{p_1, \cdots, p_n\}$ is positive, then* $\text{Höl}_{p_1,\cdots,p_n}$ *is a subaveraging function on $D$ with respect to $(S, S_0)$.*

*Proof.* Let $T = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+$.

(i) Suppose that all $p_i$ are positive and let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have

$$\sum_{i=1}^{n} p_i t_i a_i \geq \prod_{i=1}^{n} (t_i a_i)^{p_i} = \prod_{j=1}^{n} {t_j}^{p_j} \prod_{i=1}^{n} {a_i}^{p_i}$$

and hence

$$\sum_{i=1}^{n} \left( p_i t_i \prod_{j=1}^{n} {t_j}^{-p_j} \right) a_i \geq \prod_{i=1}^{n} {a_i}^{p_i} = \text{Höl}(a_1, \cdots, a_n).$$

Set

$$\beta_1(t) = p_1 t_1 \prod_{j=1}^{n} {t_j}^{-p_j}, \cdots, \beta_n(t) = p_n t_n \prod_{j=1}^{n} {t_j}^{-p_j}$$

and

$$h^*(t, a_1, \cdots, a_n) = \beta_1(t) a_1 + \cdots + \beta_n(t) a_n$$

for each $t = (t_1, \cdots, t_n) \in T$. Then we have

$$\inf_{t \in T} h^*(t, a_1, \cdots, a_n) \geq \text{Höl}(a_1, \cdots, a_n).$$

Also since $h^*(t_*, a_1, \cdots, a_n) = \text{Höl}(a_1, \cdots, a_n)$ for $t_* = ({a_1}^{-1}, \cdots, {a_n}^{-1}) \in T$, it follows that $\inf_{t \in T} h^*(t, a_1, \cdots, a_n) = \text{Höl}(a_1, \cdots, a_n)$. Therefore the desired result follows from Proposition 1.

(ii) Suppose that the only one of $\{p_1, \cdots, p_n\}$ is positive and let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have

$$\sum_{i=1}^{n} p_i t_i a_i \leq \prod_{i=1}^{n} (t_i a_i)^{p_i} = \prod_{j=1}^{n} {t_j}^{p_j} \prod_{i=1}^{n} {a_i}^{p_i}. \tag{5}$$

In fact we can assume that $p_1 > 0,\ p_2, \cdots, p_n < 0$ without loss of generality. Set $q_1 = p_1, q_i = -p_i\ (i = 2, \cdots, n)$. Then $q_i > 0\ (i = 1, \cdots, n)$ and $q_1 = 1 + q_2 + \cdots + q_n$. Also set $x_i = t_i a_i\ (i = 1, \cdots, n)$. Then we have from the usual arithmetic-geometric mean inequality that

$$\begin{aligned}
&\frac{{x_1}^{q_1} + q_2 x_2 \prod_{i=2}^{n} {x_i}^{q_i} + \cdots + q_n x_n \prod_{i=2}^{n} {x_i}^{q_i}}{q_1} \\
&\geq ({x_1}^{q_1})^{1/q_1} \left( x_2 \prod_{i=2}^{n} {x_i}^{q_i} \right)^{q_2/q_1} \cdots \left( x_n \prod_{i=2}^{n} {x_i}^{q_i} \right)^{q_n/q_1} \\
&= x_1 (x_2)^{\frac{(1+q_2)q_2}{q_1} + \frac{q_2 q_3}{q_1} + \cdots + \frac{q_2 q_n}{q_1}} \cdots (x_n)^{\frac{q_n q_2}{q_1} + \cdots + \frac{q_n q_{n-1}}{q_1} + \frac{(1+q_n)q_n}{q_1}} \\
&= x_1 \prod_{i=2}^{n} {x_i}^{q_i}
\end{aligned}$$

and hence ${x_1}^{q_1} \prod_{i=2}^{n} {x_i}^{-q_i} + \sum_{i=2}^{n} q_i x_i \geq q_1 x_1$. But this can be transposed into $\sum_{i=1}^{n} p_i t_i a_i \leq \prod_{i=1}^{n} (t_i a_i)^{p_i}$ as required. By (5) we have

$$\sum_{i=1}^{n} \left( p_i t_i \prod_{j=1}^{n} {t_j}^{-p_j} \right) a_i \leq \prod_{i=1}^{n} {a_i}^{p_i} = \text{Höl}(a_1, \cdots, a_n).$$

Set

$$\alpha_1(t) = p_1 t_1 \prod_{j=1}^{n} {t_j}^{-p_j}, \cdots, \alpha_n(t) = p_n t_n \prod_{j=1}^{n} {t_j}^{-p_j}$$

and

$$h_*(t, a_1, \cdots, a_n) = \alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n$$

for each $t = (t_1, \cdots, t_n) \in T$. Then we have

$$\sup_{t \in T} h_*(t, a_1, \cdots, a_n) \leq \text{Höl}(a_1, \cdots, a_n).$$

Also since $h_*(t_*, a_1, \cdots, a_n) = \text{Höl}(a_1, \cdots, a_n)$ for $t_* = ({a_1}^{-1}, \cdots, {a_n}^{-1}) \in T$, it follows that $\sup_{t \in T} h_*(t, a_1, \cdots, a_n) = \text{Höl}(a_1, \cdots, a_n)$. Therefore the desired result also follows from Proposition 1. □

## 4. Minkowski type functions

Let $D = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+ \subset \mathbf{R}^n$, $\rho : \mathbf{R}^+ \to \mathbf{R}^+$ and $f : \mathbf{R}^+ \to \mathbf{R}$ a concave (convex) function. We define

$$f_\rho(a_1, \cdots, a_n) = f\left(\sum_{i=1}^{n} \rho(a_i)\right)$$

for each $(a_1, \cdots, a_n) \in D$. Also suppose that

$$-\infty < \inf_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) \quad \left(\text{resp. } \sup_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) < \infty\right)$$

for each $0 < \tau < 1$. In this case, we define

$$\mu_f^-(\tau) = \inf_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) \quad \left(\text{resp. } \mu_f^+(\tau) = \sup_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right)\right)$$

for each $0 < \tau < 1$. Moreover set

$$T = \{t = (t_1, \cdots, t_n) : t_1 + \cdots + t_n = 1,\ t_1, \cdots, t_n > 0\}$$

and

$$(6) \quad \alpha_1(t) = \mu_f^-(t_1), \cdots, \alpha_n(t) = \mu_f^-(t_n) \qquad \left(\text{resp. } \beta_1(t) = \mu_f^+(t_1), \cdots, \beta_n(t) = \mu_f^+(t_n)\right)$$

for each $t = (t_1, \cdots, t_n) \in T$. Then we have the following

**Lemma 3.** (i) *If $f$ is concave, then* $\sup_{t \in T} (\alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n) \leq f_\rho(a_1, \cdots, a_n)$ *for each* $(a_1, \cdots, a_n) \in D$.

(ii) *If $f$ is convex, then* $\inf_{t \in T} (\beta_1(t)a_1 + \cdots + \beta_n(t)a_n) \geq f_\rho(a_1, \cdots, a_n)$ *for each* $(a_1, \cdots, a_n) \in D$.

*Proof.* (i) Suppose that $f$ is concave and let $(a_1, \cdots, a_n) \in D$. For each $t = (t_1, \cdots, t_n) \in T$, we have

$$\sum_{i=1}^{n} t_i f(b_i) \leq f\left(\sum_{i=1}^{n} t_i b_i\right)$$

for each $(b_1, \cdots, b_n) \in D$ and hence by putting $b_1 = \rho(a_1)/t_1, \cdots, b_n = \rho(a_n)/t_n$ in the above inequality,

$$\sum_{i=1}^{n} t_i f\left(\frac{\rho(a_i)}{t_i}\right) \leq f\left(\sum_{i=1}^{n} \rho(a_i)\right)$$

holds. Note that

$$\sum_{i=1}^{n} \mu_f^-(t_i)a_i \leq \sum_{i=1}^{n} t_i f\left(\frac{\rho(a_i)}{t_i}\right)$$

for each $t = (t_1, \cdots, t_n) \in T$. Then we have

$$\begin{aligned} \alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n &= \sum_{i=1}^{n} \mu_f^-(t_i)a_i \\ &\leq f\left(\sum_{i=1}^{n} \rho(a_i)\right) = f_\rho(a_1, \cdots, a_n) \end{aligned}$$

for each $t = (t_1, \cdots, t_n) \in T$, so that we have the desired result.

(ii) Similarly, we can treat the case where $f$ is convex. □

The above lemma suggests to us the following

**Definition 2.** We say that $f_\rho$ is of Minkowski type when

$$f_\rho(a_1, \cdots, a_n) = \begin{cases} \sup\limits_{t \in T} (\alpha_1(t)a_1 + \cdots + \alpha_n(t)a_n) & \text{if } f \text{ is concave} \\ \inf\limits_{t \in T} (\beta_1(t)a_1 + \cdots + \beta_n(t)a_n) & \text{if } f \text{ is convex} \end{cases}$$

for each $(a_1, \cdots, a_n) \in D$, where $\alpha_k$ and $\beta_k$ are as in (6).

We will give an example of Minkowski type function. Let $0 < p < 1$ ($p > 1$ or $p < 0$) and $f(t) = t^p$, $\rho(t) = t^{1/p}$ ($t > 0$). Then $f$ is a concave (resp. convex) function on $\mathbf{R}^+$. Put $\mathrm{Mink}_p = f_\rho$ and then

$$\mathrm{Mink}_p(a_1, \cdots, a_n) = \left(\sum_{i=1}^{n} {a_i}^{1/p}\right)^p$$

for each $(a_1, \cdots, a_n) \in D$. Note that

$$\frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) = \frac{\tau}{s}\frac{s}{\tau^p} = \tau^{1-p}$$

for all $s > 0$ and $0 < \tau < 1$. Then

$$\mu_f^-(\tau) = \inf_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) = \tau^{1-p} \qquad \left(\text{resp. } \mu_f^+(\tau) = \sup_{s>0} \frac{\tau}{s} f\left(\frac{\rho(s)}{\tau}\right) = \tau^{1-p}\right)$$

for all $0 < \tau < 1$. Therefore the functions $\alpha_i$ (resp. $\beta_i$) defined in (6) are such that

$$\alpha_i(t) = {t_i}^{1-p} \ (i = 1, \cdots, n) \qquad (\text{resp. } \beta_i(t) = {t_i}^{1-p} \ (i = 1, \cdots, n))$$

for all $t = (t_1, \cdots, t_n) \in T$. Fix $(a_1, \cdots, a_n) \in D$ arbitrarily and set

$${t_i}^* = \frac{{a_i}^{1/p}}{{a_1}^{1/p} + \cdots + {a_n}^{1/p}}$$

for each $(i = 1, \cdots, n)$. Put $t^* = ({t_1}^*, \cdots, {t_n}^*)$ and then $t^* \in T$. In this case, we can see that

$$\mathrm{Mink}_p(a_1, \cdots, a_n) = \begin{cases} \alpha_1(t^*)a_1 + \cdots + \alpha_n(t^*)a_n & \text{if } 0 < p < 1 \\ \beta_1(t^*)a_1 + \cdots + \beta_n(t^*)a_n & \text{if } p > 1 \text{ or } p < 0 \end{cases}$$

from an easy computation. Then by Lemma 3, we have that $\mathrm{Mink}_p$ is a Minkowski type function on $D$. Therefore we have the following

**Lemma 4.** *Let $p \neq 0$. Then* $\mathrm{Mink}_p$ *is a Minkowski type function on $D$.*

Let $S$ and $S_0$ be as in Definition 1. Then we have the following

**Proposition 5.** *Suppose that $f_\rho \circ (f_1, \cdots, f_n) \in S$ for all $f_1, \cdots, f_n \in S_0$. Then*

(i) *If $f$ is concave and $f_\rho$ is of Minkowski type, then $f_\rho$ is a subaveraging function on $D$ with respect to $(S, S_0)$.*

(ii) *If $f$ is convex and $f_\rho$ is of Minkowski type, then $f_\rho$ is a superaveraging function on $D$ with respect to $(S, S_0)$.*

*Proof.* This follows directly from Proposition 1. □

## 5. Hölder-Rogers inequality and Minkowski inequality

Let $D = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+ \subset \mathbf{R}^n$, $S = \{f\colon X \to \mathbf{R} : f \in L^1(X, \mu)\}$ and $S_0 = \{f \in S : f(x) > 0\,(\forall x \in X)\}$. Then we have $(f_1(x), \cdots, f_n(x)) \in D$ for all $f_1, \cdots, f_n \in S_0$ and $x \in X$. Now let $p \neq 0$ and $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$. Then $\text{Höl} \circ (f_1, \cdots, f_n) \in S$ and $\text{Mink}_p \circ (f_1, \cdots, f_n) \in S$ for each $f_1, \cdots, f_n \in S_0$. In fact, let $f_1, \cdots, f_n \in S_0$. Then we have

$$\text{Höl} \circ (f_1, \cdots, f_n) = {f_1}^{p_1} \cdots {f_n}^{p_n} \leq p_1 f_1 + \cdots + p_n f_n \in S$$

and

$$\text{Mink}_p \circ (f_1, \cdots, f_n) = \left(\sum_{i=1}^{n} {f_i}^{1/p}\right)^p \leq n^p \max\{f_1, \cdots, f_n\} \in S.$$

Therefore we have $\text{Höl} \circ (f_1, \cdots, f_n)$, $\text{Mink}_p \circ (f_1, \cdots, f_n) \in S$. Set $L(f) = \int f \, d\mu$ for each $f \in S$. Then $L$ is a positive linear functional from $S$ into $\mathbf{R}$ such that $(Lf_1, \cdots, Lf_n) \in D$ for all $f_1, \cdots, f_n \in S_0$. Also we have

$$\text{Höl}(Lf_1, \cdots, Lf_n) = \left(\int f_1 \, d\mu\right)^{p_1} \cdots \left(\int f_n \, d\mu\right)^{p_n}$$

and

$$L\left(\text{Höl}(f_1, \cdots, f_n)\right) = \int {f_1}^{p_1} \cdots {f_n}^{p_n} \, d\mu$$

for all $f_1, \cdots, f_n \in S_0$. Then by Proposition 2, (i), we have the Hölder-Rogers inequality:

$$\int |f_1|^{p_1} \cdots |f_n|^{p_n} \, d\mu \leq \left(\int |f_1| \, d\mu\right)^{p_1} \cdots \left(\int |f_n| \, d\mu\right)^{p_n}$$
$$(f_1, \cdots, f_n \in L^1(X, \mu),\ p_1, \cdots, p_n > 0 : p_1 + \cdots + p_n = 1).$$

Moreover, we have

$$\text{Mink}_p(Lf_1, \cdots, Lf_n) = \left(\left(\int f_1 \, d\mu\right)^{1/p} + \cdots + \left(\int f_n \, d\mu\right)^{1/p}\right)^p$$

and

$$L\left(\text{Mink}_p(f_1, \cdots, f_n)\right) = \int \left({f_1}^{1/p} + \cdots + {f_n}^{1/p}\right)^p d\mu$$

for all $f_1, \cdots, f_n \in S_0$. Then by Lemma 4 and Proposition 5 we have the Minkowski inequalities:

$$\int \left(|f_1|^{1/p} + \cdots + |f_n|^{1/p}\right)^p d\mu \leq \left(\left(\int |f_1| \, d\mu\right)^{1/p} + \cdots + \left(\int |f_n| \, d\mu\right)^{1/p}\right)^p$$
$$(f_1, \cdots, f_n \in L^1(X, \mu),\ p \geq 1 \ \text{or} \ p < 0)$$

and

$$\int \left(|f_1|^{1/p} + \cdots + |f_n|^{1/p}\right)^p d\mu \geq \left(\left(\int |f_1|\, d\mu\right)^{1/p} + \cdots + \left(\int |f_n|\, d\mu\right)^{1/p}\right)^p$$
$$(f_1, \cdots, f_n \in L^1(X, \mu),\ 0 < p < 1).$$

In case that $p_1, \cdots, p_n$ are in $\mathbf{R}$ with $p_1 + \cdots + p_n = 1$ such that the only one of $\{p_1, \cdots, p_n\}$ is positive,

$$f_1, \cdots, f_n \in S_0 \Rightarrow \text{Höl} \circ (f_1, \cdots, f_n) \in S$$

doesn't hold in general. Then we consider a discrete version of the Hölder-Rogers inequality. Let $n, m \in \mathbf{N}$, $D = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+ \subset \mathbf{R}^n$, $S = \mathbf{R}^m$ and $S_0 = \mathbf{R}^+ \times \cdots \times \mathbf{R}^+ \subset \mathbf{R}^m$. Of course, we regard $S$ as a real linear space consisting of all real functions on the set $\{1, 2, \cdots, n\}$. Note that $(D, S_0)$ satisfies the condition (0). Also it is clear that $\text{Höl} \circ (f_1, \cdots, f_n) \in S$ for all $f_1, \cdots, f_n \in S_0$.

Now set

$$L(x_1, \cdots, x_n) = \sum_{j=1}^{m} x_j$$

for each $(x_1, \cdots, x_m) \in S$. Then $L$ is a positive linear functional from $S$ into $\mathbf{R}$ such that $(Lf_1, \cdots, Lf_n) \in D$ for all $f_1, \cdots, f_n \in S_0$. Moreover we have

$$\text{Höl}(Lf_1, \cdots, Lf_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{m} f_{ij}\right)^{p_i} \quad \text{and} \quad L\left(\text{Höl}(f_1, \cdots, f_n)\right) = \sum_{j=1}^{m} \prod_{i=1}^{n} {f_{ij}}^{p_i}$$

for all $f_1 = (f_{11}, \cdots, f_{1m}), \cdots, f_n = (f_{n1}, \cdots, f_{nm}) \in S_0$. Then by Proposition 2, (ii), we have the Hölder-Rogers inequality:

$$\sum_{j=1}^{m} \prod_{i=1}^{n} {f_{ij}}^{p_i} \geq \prod_{i=1}^{n} \left(\sum_{j=1}^{m} f_{ij}\right)^{p_i}$$

when the only one of $\{p_1, \cdots, p_n\}$ is positive and $p_1 + \cdots + p_n = 1$.

**Remark 2.** The so-called Hölder's inequality :

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} {a_k}^p\right)^{1/p} \left(\sum_{k=1}^{n} {b_k}^q\right)^{1/q}$$
$$\left(p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ a_k > 0,\ b_k > 0,\ k = 1, \cdots, n\right)$$

was discovered by L. J. Rogers in 1888. However, O. Hölder discovered independently this inequality (cf. [1, 4]). Therefore, following L. Maligranda, we will call it the Hölder-Rogers inequality (cf. [3]).

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