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ON COMPARISONS OF NORMS AND THE NUMERICAL RANGES OF AN OPERATOR WITH ITS GENERALIZED ALUTHGE TRANSFORMATION

Sharandas M. Patel and Takeaki Yamazaki

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Abstract. In the present article, we have attempted to reveal some relationships between a bounded linear operator T acting on a Hilbert space and its generalized Aluthge transformation $T(s,t)$ in terms of their numerical ranges and norms. In fact, we have shown the following relations:

(i) $\overline{W(f(T(t,1-t)))} \subseteq \overline{W(T)}$ for $t \in [0,1]$ and any rational function f.

(ii) For an $n\times n$ matrix T, T is convexoid iff $W(T) = W(T(t,1-t,))$ for all $t \in [0,1]$.

(ii) is an extension of Ando's result in [2].

1 Introduction Let T be a bounded linear operator acting on a complex Hilbert space $\mathcal{H}$, and let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$. By the polar decomposition of $T \in B(\mathcal{H})$, we mean the expression $T = U|T|$, where U is a partial isometry and $|T|$ is the positive square root of T^*T such that $\ker U = \ker |T|$. In [1], Aluthge introduced the class of p-hyponormal operators that generalizes the widely studied class of hyponormal operators. In order to reveal some important features of p-hyponormal operators, he exploited the operator $\widetilde{T}$ which is now popularly known as the *Aluthge Transformation* and which is defined as

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Motivated by this article [1], several authors explored and studied new classes of operators closely connected to p-hyponormal operators with the help of the Aluthge transformation and its generalization, known as the generalized Aluthge transformation. By the generalized Aluthge transformation of $T \in B(\mathcal{H})$, we mean the bounded operator $T(s,t)$ on $\mathcal{H}$ for which

$$T(s,t) = |T|^sU|T|^t, \quad \text{where } s \geq 0 \text{ and } t \geq 0.$$

Especially, $T(1,0) = |T|^1U|T|^0 = |T|UU^*U = |T|U$ and $T(0,1) = |T|^0U|T|^1 = U^*UU|T|$. In recent years, one can find number of articles in which various relations among T, $\widetilde{T}$ and $T(s,t)$ are obtained. It is obvious that $\|\widetilde{T}\| \leq \|T\|$. Okubo [13] gave a non-obvious extension of this inequality by deriving $\|f(\widetilde{T})\| \leq \|f(T)\|$ for any polynomial $f(t)$ by proving more general result. As a corollary to this inequality, he showed that $\overline{W(f(\widetilde{T}))} \subseteq \overline{W(f(T))}$, extending some results known to be true [10] or in case either $f(t) = t$ [15, 18]. According to [11, 19], the iterated Aluthge transformation, called the n th Aluthge transformation and written as $\widetilde{T_n}$ is defined by

$$\widetilde{T_1} = \widetilde{T} \quad \text{and} \quad \widetilde{T_n} = (\widetilde{\widetilde{T_{n-1}}}) \text{ for } n > 1.$$

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In [17], the second author proved that the norm of the iterated Aluthge transformations converges to the spectral radius $r(T)$ of T. Ando [2] proved that for a square matrix T of order n, the sequence of the closure of the numerical ranges of iterated Aluthge transformations of T converges to the convex hull of the spectrum $\sigma(T)$ of T. In [3], it was shown that every iterated Aluthge transformation of a matrix of order 2 converges to a normal matrix. Our main object of the present paper is to compare the numerical range of T with that of $T(s,t)$ for some restricted values of s and t.

A bounded operator T is said to satisfy the growth condition (G_1) or called a (G_1) operator if

$$\|(T-zI)^{-1}\| = \frac{1}{\operatorname{dist}(z,\sigma(T))} \quad \text{for all } z \notin \sigma(T).$$

It is well known that the hyponormal operators satisfy the growth condition (G_1), a result still not known for p-hyponormal operators. Moreover, (G_1) operators are convexoid operators. The corresponding fact for p-hyponormal operators remains as an open problem.

Let C_ρ $(\rho > 0)$ be the class of all bounded operators with unitary ρ-dilations in the sense of B.Sz.-Nagy and C. Foiaş [12]. According to Holbrook [9], an operator radius $w_\rho(T)$ of T is defined as $w_\rho(T) = \inf\{a > 0 \text{ and } a^{-1}T \in C_\rho\}$. For further properties of operator radii, we refer to [9].

In section 2, some results are given that will be of use in the succeeding sections. Section 3 is devoted to establishing inclusion relations among the numerical ranges of rational functions of operators $T(0,1)$, $T(1,0)$ and T. The inequality that says $\|f(\widetilde{T})\| \le \|f(T)\|$ for every polynomial f is extended further in section 4 by proving $\|f(T(s,t))\| \le \|f(T)\|$ with $s+t=1$ for every rational function f for which $f(T)$ exists. Finally, in section 5, we introduce a numerical range value function on $[0,1]$ and obtain an improvement over a characterization of convexoid matrices due to Ando [2].

In what follows, we assume, unless it is stated otherwise, that f will be a rational function with poles off $\sigma(T)$.

2 Fundamental properties

Lemma 2.2.1 *Let $T = U|T|$ be the polar decomposition of T. Then $\dim\ker T \le \dim\ker T^*$ if and only if there exists an isometry V such that $V|T| = U|T|$.*

Although our first lemma is well known [6, p. 75], [16, p. 4], we would like to present it with a proof.

Proof. Let $\mathcal{H} = \overline{R(|T|)} \oplus R(|T|)^\perp = \overline{R(T)} \oplus R(T)^\perp$. Then U is an isometry from $\overline{R(|T|)}$ to $\overline{R(T)}$. On the other hand, there exists an isometry $U_1 : R(|T|)^\perp \longrightarrow R(T)^\perp$ if and only if $\dim(R(|T|)^\perp) \le \dim(R(T)^\perp)$. By $R(|T|)^\perp = \ker|T| = \ker T$ and $R(T)^\perp = \ker T^*$, it is equivalent to $\dim\ker T \le \dim\ker T^*$. So the underlying kernel condition ensures the existence of an isometry $U_1 : R(|T|)^\perp \longrightarrow R(T)^\perp$. Let

$$V = UU^*U + U_1(I-U^*U) = U + U_1(I-U^*U).$$

The facts that $I - U^*U$ is the projection onto $\ker U = \ker T = \ker|T| = R(|T|)^\perp = (\ker U_1)^\perp$, $U_1^*U_1x = x$ on $R(|T|)^\perp$ and $R(U_1) \subseteq R(T)^\perp = \ker T^* = \ker U^*$ will give

$$\begin{aligned} V^*V &= \{U + U_1(I-U^*U)\}^*\{U + U_1(I-U^*U)\} \\ &= U^*U + (I-U^*U)U_1^*U + U^*U_1(I-U^*U) + (I-U^*U)U_1^*U_1(I-U^*U) \\ &= U^*U + \{U^*U_1(I-U^*U)\}^* + U^*U_1(I-U^*U) + I - U^*U \\ &= U^*U + I - U^*U \\ &= I. \end{aligned}$$

Thus V is an isometry. Moreover,

$$V|T| = \{U + U_1(I - U^*U)\}|T| = U|T|.$$

Lemma 2.2.2 *Let $A \in B(\mathcal{H})$. Then the following assertions hold:*

(i) If P is a projection with $PAP = AP$, then

$$f(AP) = Pf(A)P + f(0)(I - P).$$

(ii) If V is an isometry, then

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

Proof. *(i).* Let $\mathcal{H} = (\ker P)^{\perp} \oplus \ker P$. Then by the assumption $PAP = AP$, A can be expressed as follows:

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \quad \text{on } \mathcal{H} = (\ker P)^{\perp} \oplus \ker P.$$

Hence

$$f(AP) = f(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f(X) & 0 \\ 0 & f(0)I \end{pmatrix}.$$

On the other hand,

$$f(A) = f(\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}) = \begin{pmatrix} f(X) & Y' \\ 0 & f(Z) \end{pmatrix}.$$

Hence we have

$$Pf(A)P + f(0)(I - P) = \begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(0)I \end{pmatrix} = f(AP).$$

(ii). For an isometry V, note that $\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$ is unitary. Then we have

$$\begin{aligned}
\begin{pmatrix} f(VAV^*) & 0 \\ 0 & f(0)I \end{pmatrix} &= f(\begin{pmatrix} VAV^* & 0 \\ 0 & 0 \end{pmatrix}) \\
&= f(\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix}) \\
&= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} f(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\
&= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} f(A) & 0 \\ 0 & f(0)I \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\
&= \begin{pmatrix} Vf(A)V^* + f(0)(I - VV^*) & 0 \\ 0 & f(0)I \end{pmatrix}.
\end{aligned}$$

Hence

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

The following result is a modification of [10, Proposition 4.5].

Proposition 2.2.3 *Let $A, B \in B(\mathcal{H})$. Then the following assertions are mutually equivalent:*

(i) $\overline{W(f(A))} \subseteq \overline{W(f(B))}$ *for all* f.

(ii) $w(f(A)) \leq w(f(B))$ *for all* f.

(iii) $\|f(A)\| \leq \|f(B)\|$ *for all* f.

The proof is almost identical to the one given for Proposition 4.5 of [10].

3 Numerical ranges of $T(0,1)$ and $T(1,0)$ The primary object of the present section is to establish the connection among the numerical ranges of T, $T(0,1)$ and $T(1,0)$.

Theorem 3.3.1 *Let $T \in B(\mathcal{H})$. Then the following assertions hold:*

(i) $W(f(T(0,1))) \subseteq W(f(T))$.

(ii) $W(f(T(1,0))) \subseteq W(f(T))$.

Proof. *(i).* Let $T = U|T|$ be the polar decomposition of T, and $\mathcal{H} = (\ker T)^{\perp} \oplus \ker T$. Then

$$T(0,1) = U^*UU|T| = U^*UT = U^*UTU^*U.$$

Since U^*U is a projection, (i) of Lemma 2.2.2 yields

$$f(T(0,1)) = U^*Uf(T)U^*U + f(0)(I - U^*U). \tag{3.1}$$

In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.1), $f(T(0,1)) = f(T)$, and hence

$$W(f(T(0,1))) = W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.1), we obtain

$$W(f(T(0,1))) \subseteq \text{conv } \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(0,1))) \subseteq \text{conv } \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

(ii). Step 1. We shall show the following equality:

$$f(T(1,0)) = U^*f(T)U + f(0)(I - U^*U). \tag{3.2}$$

We shall establish this equality separately for each of the cases when $\dim \ker T \leq \dim \ker T^*$ and $\dim \ker T \geq \dim \ker T^*$.

(a) The case $\dim \ker T \leq \dim \ker T^*$. By the Lemma 2.2.1, there is an isometry V satisfying $U|T| = V|T|$. Note that in the proof of Lemma 2.2.1,

$$V = U + U_1(I - U^*U), \text{ where } U_1 \text{ is isometry with } R(U_1) \subseteq \ker T^*. \tag{3.3}$$

Then by (3.3), we have

$$UU^*TUU^* = TUU^* = U|T|UU^* = V|T|UV^*.$$

Hence by (ii) Lemma 2.2.2, we obtain

$$f(TUU^*) = f(V|T|UV^*) = Vf(|T|U)V^* + f(0)(I - VV^*).$$

Moreover since V is isometry, we have

$$f(|T|U) = V^*f(TUU^*)V.$$

Therefore

$$\begin{aligned} f(|T|U) &= V^*f(TUU^*)V \\ &= V^*\{UU^*f(T)UU^* + f(0)(I - UU^*)\}V \quad \text{by (i) of Lemma 2.2.2} \\ &= U^*f(T)U + f(0)(I - U^*U) \quad \text{by (3.3).} \end{aligned}$$

On the other hand, by (3.3),

$$U|T^*| = UU^*UU|T|U^* = VU^*UTV^*.$$

Then by Lemma 2.2.2, we obtain

$$\begin{aligned} f(U|T^*|) &= f(VU^*UTV^*) \\ &= Vf(U^*UT)V^* + f(0)(I - VV^*) \quad \text{by (ii) of Lemma 2.2.2} \\ &= V\{U^*Uf(T)U^*U + f(0)(I - U^*U)\}V^* + f(0)(I - VV^*) \\ &\qquad\qquad\qquad\qquad \text{by (i) of Lemma 2.2.2} \\ &= Uf(T)U^* + f(0)(I - UU^*) \quad \text{by (3.3).} \end{aligned} \tag{3.4}$$

(b) The case $\dim\ker T \geq \dim\ker T^*$. Replacing T by T^* in (3.4), we have

$$\begin{aligned} &f(U^*|T|) = U^*f(T^*)U + f(0)(I - U^*U) \\ &\iff f(|T|U) = U^*f(T)U + f(0)(I - U^*U). \end{aligned}$$

Step 2. In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.2), $f(T(1,0)) = f(|T|U) = U^*f(T)U$, and hence

$$W(f(T(1,0))) \subseteq W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.2), we obtain

$$W(f(T(1,0))) = W(f(|T|U)) \subseteq \text{conv}\,\{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(1,0))) \subseteq \text{conv}\,\{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

Hence the proof is complete.

Corollary 3.3.2 *Let $T = U|T|$. Then*

(i) $W(T(1,0)) = W(T)$ *if* $\ker T^* \subseteq \ker T$.

(ii) $W(T(0,1)) = W(T)$ *if* $\ker T \subseteq \ker T^*$.

(iii) $W(T(0,1)) \subseteq W(T(1,0))$ *if* $\ker T^* \subseteq \ker T$.

(iv) $W(T(1,0)) \subseteq W(T(0,1))$ *if* $\ker T \subseteq \ker T^*$.

Proof. (i). In view of Theorem 3.3.1, only we have to prove $W(T) \subseteq W(T(1,0))$.

If $\ker T^* \subseteq \ker T$, then $T = U|T| = U|T|UU^* = UT(1,0)U^*$, and we have

$$W(T) \subseteq \text{conv}\{W(T(1,0)) \cup \{0\}\}.$$

If $\ker T \neq \{0\}$, then $0 \in W(T(1,0))$ and we have

$$W(T) \subseteq \text{conv}\{W(T(1,0)) \cup \{0\}\} = W(T(1,0)).$$

If $\ker T = \{0\}$, then $\{0\} = \ker T \supset \ker T^*$, and U^* must be an isometry. Hence we have $W(T) \subseteq W(T(1,0))$.

(ii). If $\ker T \subseteq \ker T^*$, then $U^*UU = U$ holds. Hence $T(0,1) = U^*UU|T| = U|T| = T$, and $W(T(0,1)) = W(T)$.

(iii). If $\ker T^* \subseteq \ker T$, then by (i) and Theorem 3.3.1, we have

$$W(T(0,1)) \subseteq W(T) = W(T(1,0)).$$

(iv). If $\ker T \subseteq \ker T^*$, then by (ii) and Theorem 3.3.1, we have

$$W(T(1,0)) \subseteq W(T) = W(T(0,1)).$$

Remark 3.3.3 *If we drop the kernel condition from the statements of Corollary 3.3.2, then we may not get the same conclusions as following indicate.*

Example 3.3.4 *Let* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $U = T$. *Clearly* $W(T(1,0)) = \{0\} \neq W(T)$.

Example 3.3.5 *For* $\alpha > 0$, *let*

$$T = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(0,1) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(1,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also $W(T(0,1)) = \{z \in \mathbb{C} : \ |z| \leq \frac{\alpha}{2}\}$ *and* $W(T(1,0)) = \{z \in \mathbb{C} : \ |z| \leq \frac{1}{2}\}$. *Then*

(i) for $\alpha \in (0,1)$, $W(T(0,1)) \subsetneqq W(T(1,0))$,

(ii) for $\alpha > 1$, $W(T(1,0)) \subsetneqq W(T(0,1))$.

4 Norm inequality involving a rational function of T and $T(s,t)$.

Theorem 4.4.1 *Let $T \in B(\mathcal{H})$. Then*

$$\|f(T(s,t))\| \leq \|f(T)\|$$

holds for $s,t \geq 0$ with $s+t=1$.

Proof. Let $T = U|T|$ be the polar decomposition of T. Let $|T_\varepsilon| = |T| + \varepsilon I > 0$. Note that

$$\lim_{\varepsilon \to +0} |T_\varepsilon|^{-1}|T| = \lim_{\varepsilon \to +0} (|T| + \varepsilon I)^{-1}|T| = U^*U.$$

We prepare the important inequality due to [7]. For $X \in B(\mathcal{H})$ and positive operators A and B,

$$\|A^s X B^s\| \leq \|AXB\|^s \|X\|^{1-s} \tag{4.1}$$

holds for $s \in [0,1]$. Then we have

$$\begin{aligned}
\|f(T(s,t))\| &= \|f(|T|^s U|T|^t)\| \\
&= \|f(|T_\varepsilon|^s |T_\varepsilon|^{-s} |T|^s U|T|^t |T_\varepsilon|^s |T_\varepsilon|^{-s})\| \\
&= \||T_\varepsilon|^s f(|T_\varepsilon|^{-s}|T|^s U|T|^t |T_\varepsilon|^s)|T_\varepsilon|^{-s}\| \\
&\leq \||T_\varepsilon| f(|T_\varepsilon|^{-s}|T|^s U|T|^t |T_\varepsilon|^s)|T_\varepsilon|^{-1}\|^s \|f(|T_\varepsilon|^{-s}|T|^s U|T|^t |T_\varepsilon|^s)\|^t \quad \text{by (4.1)} \\
&= \|f(|T_\varepsilon|^{1-s}|T|^s U|T|^t |T_\varepsilon|^{s-1})\|^s \|f(|T_\varepsilon|^{-s}|T|^s U|T|^t |T_\varepsilon|^s)\|^t \\
&\longrightarrow \|f(|T|UU^*U)\|^s \|f(U^*UU|T|)\|^t \quad \text{as } \varepsilon \to +0 \\
&= \|f(T(1,0))\|^s \|f(T(0,1))\|^t \\
&\leq \|f(T)\| \quad \text{by Theorem 3.3.1 and Proposition 2.2.3.}
\end{aligned}$$

Hence the proof is complete.

Remark 4.4.2

(i) Above theorem is not true if $s+t \neq 1$ as can be illustrated with the following Example 4.4.3.

(ii) Notice that a simple consequence of Theorem 4.4.1 shows that the sequence $\{\|f(\widetilde{T_n})\|\}$ is a non-increasing sequence in positive numbers and therefore is convergent to a positive number. In particular, when $f(t) = t$, the second author [17] proved that the limit of the sequence is $r(T)$, the spectral radius of T. However, in general situation, whether the sequence $\{\|f(\widetilde{T_n})\|\}$ converges to $r(f(T))$ remains as an open problem to us.

Example 4.4.3 *Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$. Then $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $|T| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Also $T(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It easy to find that $\overline{W(T(2,1))} = [0,2]$. Moreover $\overline{W(T)}$ is a closed elliptic disc with foci at 0 and 1, and the major axis $\sqrt{2}$ and the minor axis 1. This fact shows that $\overline{W(T)}$ excludes 2 and therefore $\overline{W(T(2,1))}$ is not a subset of $\overline{W(T)}$. If the theorem were true for $s+t \neq 1$, then we would have in particular, $\|T(s,t) - zI\| \leq \|T - zI\|$ for all z. Then $\overline{W(T(s,t))} \subseteq \overline{W(T)}$, which is not correct.*

As simple consequence of Proposition 2.2.3 and Theorem 4.4.1, we obtain the following corollary.

Corollary 4.4.4 *Let $T \in B(\mathcal{H})$. Then*

$$\overline{W(f(T(s,t)))} \subseteq \overline{W(f(T))}$$

holds for $s, t \geq 0$ with $s + t = 1$.

The following corollary is an extension of [13, Corollary 3].

Corollary 4.4.5 *Let $T \in B(\mathcal{H})$. Then*

$$w_\rho\left(f(T(s,t))\right) \leq w_\rho\left(f(T)\right)$$

holds for each $s, t \geq 0$ with $s + t = 1$ and $\rho > 0$.

Proof. First, we note that

(i) for $0 < \rho < 2$ and $\rho \neq 1$, $w_\rho \leq 1$ if and only if

$$\|T - zI\| \leq \frac{|z|}{|\rho - 1|} \quad \text{with } \left|\frac{\rho - 1}{\rho - 2}\right| \leq |z| < \infty$$

in [12],

(ii) for $\rho = 2$, $w(T) = w_2(T) \leq 1$ if and only if

$$\|T - zI\| \leq 1 + (1 + |z|^2)^{\frac{1}{2}}$$

for each complex number z in [4],

(iii) for $\rho > 2$, $w_\rho(T) \leq 1$ if and only if $r(T) \leq 1$ and

$$\|(T - zI)^{-1}\| \leq \frac{1}{|z| - 1}$$

for $1 < |z| < \dfrac{\rho - 1}{\rho - 2}$ in [12].

Now it is not difficult to prove the corollary by applying Theorem 4.4.1.

Corollary 4.4.6 *If T satisfies the growth condition (G_1), then so does $T(s,t)$ for $s, t \geq 0$ with $s + t = 1$.*

Proof. Since $\sigma(T) = \sigma(T(s,t))$, the result follows from Theorem 4.4.1.

5 The convexity of $F(x) = \overline{W(f(T(x, 1-x)))}$.

Theorem 5.5.1 *For an operator T, let*

$$F(x) = \overline{W(f(T(x, 1-x)))} \quad \text{for } x \in [0,1].$$

Then

$$F(\alpha x + (1-\alpha)y) \subseteq \alpha F(x) + (1-\alpha)F(y) \tag{5.1}$$

holds for all $x, y \in [0,1]$ and $\alpha \in [0,1]$.

As a consequence of Theorem 5.5.1, the function $\Phi(x) = w(T(x, 1-x))$ turns out to be a convex function on $[0,1]$.

Proof. Let $T = U|T|$ be the polar decomposition. Firstly, we shall prove

$$F\left(\frac{x+y}{2}\right) \subseteq \frac{1}{2}\{F(x) + F(y)\}. \tag{5.2}$$

Note that for a positive invertible operator S and $A \in B(\mathcal{H})$,

$$\|A\| \le \frac{1}{2}\|SAS^{-1} + S^{-1}AS\|.$$

in [5]. Let $\varepsilon > 0$ and $|T_\varepsilon| = (|T| + \varepsilon I) > 0$. By the above inequality, we obtain

$$\begin{aligned}
&\|f(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right))\| \\
&= \|f(|T|^{\frac{x+y}{2}}U|T|^{1-\frac{x+y}{2}})\| \\
&\le \frac{1}{2}\||T_\varepsilon|^{\frac{x-y}{2}} f(|T|^{\frac{x+y}{2}}U|T|^{1-\frac{x+y}{2}})|T_\varepsilon|^{\frac{y-x}{2}} + |T_\varepsilon|^{\frac{y-x}{2}} f(|T|^{\frac{x+y}{2}}U|T|^{1-\frac{x+y}{2}})|T_\varepsilon|^{\frac{x-y}{2}}\| \\
&= \frac{1}{2}\|f(|T_\varepsilon|^{\frac{x-y}{2}}|T|^{\frac{x+y}{2}}U|T|^{1-\frac{x+y}{2}}|T_\varepsilon|^{\frac{y-x}{2}}) + f(|T_\varepsilon|^{\frac{y-x}{2}}|T|^{\frac{x+y}{2}}U|T|^{1-\frac{x+y}{2}}|T_\varepsilon|^{\frac{x-y}{2}})\| \\
&\longrightarrow \frac{1}{2}\|f(|T|^x U|T|^{1-x}) + f(|T|^y U|T|^{1-y})\| \quad \text{as } \varepsilon \to +0 \\
&= \frac{1}{2}\|f(T(x, 1-x)) + f(T(y, 1-y))\|.
\end{aligned}$$

Hence for any complex number λ,

$$\|f(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)) - \lambda I\| \le \|\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2} - \lambda I\|.$$

Since

$$\overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : \ |z - \lambda| \le \|T - \lambda I\|\}$$

in [8, 14], we have

$$\begin{aligned}
F\left(\frac{x+y}{2}\right) &= \overline{W\left(f\left(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)\right)\right)} \\
&\subseteq \overline{W\left(\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2}\right)} \\
&\subseteq \frac{1}{2}\{\overline{W(f(T(x, 1-x)))} + \overline{W(f(T(y, 1-y)))}\} \\
&= \frac{1}{2}\{F(x) + F(y)\}.
\end{aligned}$$

Next, we will extend (5.2) to (5.1) ¿From (5.2), one can easily derive

$$F\left(\frac{x_1+x_2+\cdots+x_{2^n}}{2^n}\right) \subseteq \frac{1}{2^n}\{F(x_1)+F(x_2)+\cdots+F(x_{2^n})\}$$

for all $x_i \in [0,1]$ $(i=1,2,\cdots)$. Hence for any rational number $\alpha \in [0,1]$, we have (5.1). Since F is continuous, we have (5.1) for any real number $\alpha \in [0,1]$.

This completes the proof.

Remark 5.5.2 *The conclusion of Theorem 5.5.1 cannot be strengthened further to*

$$F(\alpha x+(1-\alpha)y) = \alpha F(x)+(1-\alpha)F(y)$$

as Example 5.5.3 will show. However, whether the range of F is convex remains as an open problem.

Example 5.5.3 *For $\alpha > 0$, let*

$$T = \begin{pmatrix} 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(s,t) = \begin{pmatrix} 0 & 16^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $F(x) = W(T(x,1-x)) = \{z : |z| \leq \frac{16^{1-x}}{2}\}$. Let $x=\frac{1}{4}$, $y=\frac{3}{4}$ and $\alpha=\frac{1}{2}$. Then

(i) $F(\frac{1}{2}) = F(\alpha x+(1-\alpha)y) = \{z : |z| \leq 2\}$.

(ii) $F(x) = \{z : |z| \leq 4\}$.

(iii) $F(y) = \{z : |z| \leq 1\}$.

Hence

$$\begin{aligned} F\left(\frac{1}{2}\right) &= F(\alpha x+(1-\alpha)y) = \{z : |z| \leq 2\} \\ &\subsetneqq \{z : |z| \leq \frac{5}{2}\} = \alpha F(x)+(1-\alpha)F(y). \end{aligned}$$

Corollary 5.5.4 *Let T be an operator. Then*

$$\begin{aligned} \overline{W(f(\widetilde{T}))} = F\left(\frac{1}{2}\right) &\subseteq \frac{1}{2}\{F(s)+F(1-s)\} \\ &\subseteq \frac{1}{2}\{F(t)+F(1-t)\} \subseteq \overline{W(f(T))} \end{aligned}$$

holds for all $\frac{1}{2} \leq s \leq t \leq 1$.

Proof. Since $\frac{1}{2} = \frac{s+1-s}{2}$ and $F(x) = \overline{W(f(T(x,1-x)))} \subseteq \overline{W(f(T))}$ for $x \in [0,1]$, we have

$$\overline{W(f(\widetilde{T}))} = F\left(\frac{1}{2}\right) \subseteq \frac{1}{2}\big\{F(s) + F(1-s)\big\} \quad \text{by Theorem 5.5.1}$$
$$\subseteq \overline{W(f(T))} \quad \text{by Corollary 4.4.4.}$$

Next, let $\frac{1}{2} \le s \le t \le 1$. Then we have $[1-s, s] \subseteq [1-t, t]$. Then there exist $\alpha_1, \alpha_2 \in [0,1]$ such that

$$s = \alpha_1 t + (1-\alpha_1)(1-t) \quad \text{and} \quad 1-s = \alpha_2 t + (1-\alpha_2)(1-t).$$

By an easy calculation, we have $\alpha_1 + \alpha_2 = 1$, and by Theorem 5.5.1, we have

$$\frac{1}{2}\{F(s) + F(1-s)\} \subseteq \frac{1}{2}\{\alpha_1 F(t) + (1-\alpha_1)F(1-t) + \alpha_2 F(t) + (1-\alpha_2)F(1-t)\}$$
$$= \frac{1}{2}\{F(t) + F(1-t)\}.$$

As a simple consequence of Corollary 4.4.4, one can see that if T is covexoid then so is $T(s,t)$ with $\overline{W(T(s,t))} = \overline{W(T)}$. The converse is obvious. However, if we do not assume $\overline{W(T(s,t))} = \overline{W(T)}$, then mere convexoidity of $T(s,t)$ does not guarantee that T is convexoid even if $\mathcal{H}$ is finite dimensional. To see this, we refer to Example 4.4.3. That convexoidity of $\widetilde{T} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is clear from the fact that it is selfadjoint. On the other hand as conv $\sigma(T) = [0,1] \neq \overline{W(T)}$, T is not convexoid. However, if $\mathcal{H}$ is finite-dimensional, then our next result will show that the condition $W(T(s,t)) = W(T)$ is just equivalent to the convexoidity of T. In case $\mathcal{H}$ is infinite dimensional, we do not know the validity of this result.

Corollary 5.5.5 *For an $n \times n$ matrix T, the following assertions are mutually equivalent:*

(i) *T is convexoid.*

(ii) $W(\widetilde{T}) = W(T)$.

(iii) $W(T(s_0, 1-s_0)) = W(T)$ *for a fixed* $s_0 \in (0,1)$.

(iv) $W(T(s, 1-s)) = W(T)$ *for all* $s \in [0,1]$.

In order to prove Corollary 5.5.5, we shall need the following theorem, a remarkable result due to Ando [2].

Theorem 5.A ([2]) *Let T be an $n \times n$ matrix. Then T is convexoid if and only if $W(\widetilde{T}) = W(T)$.*

Proof. (i) $\Longleftrightarrow$ (ii) has been shown in Theorem A. (iv) $\Longrightarrow$ (ii), (iii) are obvious. So only we have to show (ii) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii).

Proof of (ii) $\Longrightarrow$ (iv). Since $W(T(s,1-s)) \subseteq W(T)$ and $W(T(1-s,s)) \subseteq W(T)$ for all $s \in [0,1]$ hold and Corollary 5.5.4, we have

$$W(T) = W(\widetilde{T}) \subseteq \frac{1}{2}\big\{W(T(s,1-s)) + W(T(1-s,s))\big\}$$
$$\subseteq \frac{1}{2}\big\{W(T(s,1-s)) + W(T)\big\} \subseteq W(T).$$

Then we have

$$\frac{1}{2}\{W(T(s,1-s))+W(T)\}=W(T). \tag{5.3}$$

For any $\theta \in [0,2\pi)$, let λ be an extreme point of Re $e^{i\theta}W(T)$. Then by (5.3), there exist $\lambda_1 \in$ Re $e^{i\theta}W(T)$ and $\mu_1 \in$ Re $e^{i\theta}W(T(s,1-s))$ such that

$$\lambda = \frac{\lambda_1+\mu_1}{2}.$$

Since Re $e^{i\theta}W(T)$ is a line segment, and λ is a extreme point of Re $e^{i\theta}W(T)$, it must be $\lambda=\lambda_1=\mu_1 \in$ Re $e^{i\theta}W(T(s,1-s))$, i.e., Re $e^{i\theta}W(T) \subseteq$ Re $e^{i\theta}W(T(s,1-s))$ for any $\theta \in [0,2\pi)$. Since $W(T)$ is convex, and $W(T(s,1-s)) \subseteq W(T)$ always holds, we have $W(T)=W(T(s,1-s))$ for all $s \in [0,1]$.

Proof of (iii) $\Longrightarrow$ (ii). We may assume $s_0 > \frac{1}{2}$. For each $s_0 \in (\frac{1}{2},1)$, there exists $\alpha \in (0,1)$ such that

$$s_0 = \alpha\frac{1}{2}+(1-\alpha)\cdot 1.$$

Then by Theorem 5.5.1,

$$W(T)=W(T(s_0,1-s_0)) \subseteq \alpha W(\widetilde{T})+(1-\alpha)W(T(1,0)) \subseteq W(T).$$

By the same argument of the above one, we have $W(\widetilde{T})=W(T)$.

Remark 5.5.6

(i) In (iii) of Corollary 5.5.5, s_0 must not be 0 or 1, because if T is invertible, then U is unitary and $W(T)=W(T(0,1))=W(T(1,0))$. But in general, $W(T)\neq W(\widetilde{T})$.

(ii) If T is spectraloid (i.e., $w(T)=r(T)$), then an applcation of Corollary 5.5.4 shows that $w(T)=w(T(s,1-s))$ for all $s\in[0,1]$. This along with Corollary 5.5.5 raises the following as conjecture:

Conjecture. For an $n\times n$ matrix T, the following assertions are equivalent:

(i) T is spectraloid.

(ii) $w(T)=w(\widetilde{T})$.

(iii) $w(T)=w(T(s_0,1-s_0))$ for a fixed $s_0\in(0,1)$.

(iv) $w(T)=w(T(s,1-s))$ for all $s\in[0,1]$.

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Department of Mathematics, Kanagawa University, 221-8686 Yokohama, Japan.
E-mail : yamazt26@kanagawa-u.ac.jp