(α,β)-SEMI CONNECTED IN TOPOLOGICAL SPACES

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ABSTRACT. In this paper we used the definition of (α, β)-semi open sets in order to define (α,β)-semi connected spaces in a topological space (X,τ). Also we study some properties of (α,β)-semi connected spaces and characterize the (α,β)-semi connectedness using $((\alpha, \beta),(\sigma, \theta))$ irresolute maps. Also we can see that this concept generalize the notions of connected, α connected and α semi connected studied before.

1. Introduction

The study of semi open sets was initiated by Levine [1]. C. Carpintero, E. Rosas and J. Vielma, in [6] they introduced the concept of operators associated with a topology τ on the set X and studied some notions of connected using α-semi open sets. E. Rosas, J. Vielma, C. Carpintero and M. Salas, in [9], defined the α-semi T_i spaces for $i = 0, 1/2, 1, 2$, using the operator α and the α-semi open sets. Also in [7], E. Rosas, C. Carpintero and J. Sanabria introduced and studied the notions of (α, β)-semi open sets and some new generalized separation axioms. In this paper, we introduce and study the notions of (α, β)-semi connected sets and $((\alpha, \beta),(\sigma, \theta))$ irresolute maps in order to characterize the (α, β)-semi connected sets and observe that these concepts generalize the semi connected case. Also, we obtain better results in comparison with the results obtained in [8].

2. Preliminaries

In this section, we recall some basic definitions and some important results.

Definition 2.1. *Let (X,τ) be a topological space. We say that α is an ope rator associated to τ, if α:$P(X)\to P(X)$ satisfies $U \subseteq \alpha(U)$, for all $U \in \tau$.*

Definition 2.2. *Let (X,τ) be a topological space and α:$P(X)\to P(X)$ be an operator associated to a topology τ. A subset $A \subseteq X$ is said to be an α-semi open set if there exists $U \in\tau$ such that $U\subseteq A\subseteq\alpha(U)$.*

Definition 2.3. *Let (X,τ) be a topological space and α,β:$P(X)\to P(X)$ be operators associated to a topology τ on X. We say that a subset $A\subseteq X$ is an (α,β)-semi open set if for each $x\in A$, there exists a β-semi open set V such that $x \in V$ and $\alpha(V) \subseteq A$. The complement of an (α, β)-semi open set is an (α, β)-semi closed set.*

We observe that when $\alpha = \beta = id$, we have

$$A \text{ is } (\alpha,\beta)\text{-semi open set} \Leftrightarrow A \text{ is open}$$

If $\beta = id$ and α is arbitrary, then

$$A \text{ is } (\alpha,\beta)\text{-semi open set} \Leftrightarrow A \text{ is } \alpha\text{-open set}$$

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It is interesting to see that if β is any monotone operator (i.e, if $U \subseteq V$ then $\beta(U) \subseteq \beta(V)$), the collection of all β-semi open sets is just the collection of all (id,β)-semi open sets.

The following lemmas give information about some fundamental properties of the (α,β)-semi open sets ((α,β)-semi closed sets, respectively). Also we can see that there is no restriction on the operators when the notions of (α,β) semi closure and (α,β) semi interior are defined.

Lemma 2.1. *Let* (X,τ) *be a topological space and* $\alpha, \beta : P(X) \to P(X)$ *be operators associated to a topology* τ *on* X*. If* $\{A_i : i \in I\}$ *is a collection of (*α*,*β*)-semi open sets, then* $\bigcup_{i \in I} A_i$ *is an (*α*,*β*)-semi open set.*

Proof. Given $x \in \bigcup_i A_i$, then $x \in A_j$ for some $j \in I$. In this case, there exists a β semi open set V_j such that $x \in V_j$ and $\alpha(V_j) \subseteq A_j \subseteq \bigcup_i A_i$. Therefore, given $x \in \bigcup_i A_i$, there exists a β semi open set V_j such that $\alpha(V_j) \subseteq \bigcup_i A_i$. This implies that $\bigcup_i A_i$ is an (α,β)-semi open set. □

Now using the above lemma and the De Morgan laws, we obtain the following corollary.

Corollary 2.2. *Let* (X,τ) *be a topological space and* $\alpha, \beta : P(X) \to P(X)$ *be operators associated with a topology* τ *on* X*. If* $\{A_i : i \in I\}$ *is a collection of (*α*,*β*)-semi closed sets, then* $\bigcap_{i \in I} A_i$ *is an (*α*,*β*)-semi closed set.*

We can observe that from these two properties, it is possible to define, in a natural way, the (α,β)-semi closure and the (α,β)-semi interior of a set $A \subseteq X$. They will be denoted by (α,β)-$sCl(A)$ and (α,β)-$sInt(A)$, respectively.
In a topological space (X,τ) with the associated operators $\alpha,\beta: P(X) \to P(X)$, we have in a natural way some properties that are well known as we can see in the following lemma.

Lemma 2.3. *Let* (X,τ) *be a topological space and* $\alpha, \beta : P(X) \to P(X)$ *be operators associated with a topology* τ *on* X*. Then:*

(a) $(\alpha,\beta) - sInt(A) \subseteq (\alpha,\beta) - sInt(B)$ *if* $A \subseteq B$;
(b) $(\alpha,\beta) - sCl(A) \subseteq (\alpha,\beta) - sCl(B)$ *if* $A \subseteq B$;
(c) A *is* $(\alpha,\beta) -$ *semi open set* $\Leftrightarrow$ $A = (\alpha,\beta) - sInt(A)$;
(d) B *is* $(\alpha,\beta) -$ *semi closed set* $\Leftrightarrow$ $B = (\alpha,\beta) - sCl(B)$;
(e) $x \in (\alpha,\beta) - sInt(A)$ *if and only if there exists an* $(\alpha,\beta) -$ *semi open set* G *such that* $x \in G \subseteq A$;
(f) $x \in (\alpha,\beta) - sCl(B)$ *if and only if for each* $(\alpha,\beta) -$ *semi open set* G *such that* $x \in G$, $G \cap B \neq \emptyset$;
(g) $X \backslash ((\alpha,\beta) - sCl(A))) = (\alpha,\beta) - sInt(X \backslash A)$ *and* $X \backslash ((\alpha,\beta) - sInt(A))) = (\alpha,\beta) - sCl(X \backslash A)$.

Proof. It is a direct consequence of the definitions of (α,β)-semi closure and (α,β)-semi interior.

□

Definition 2.4. *Let* $(X,\tau,\alpha,\beta), (Y,\psi,\sigma,\theta)$ *be two spaces with associated ope rators. A function* $f : (X,\tau) \to (Y,\psi)$ *is said to be* $((\alpha,\beta),(\sigma,\theta))$ *semi open if the direct image of each* (α,β)*-semi open set in* X *is a* (σ,θ)*-semi open set in* Y*.*

Observe that if in the above definition $\alpha = \beta = \sigma = \theta = id$, then we obtain the definition of an open map. In the same way, if $\alpha = \beta = \sigma = id$, then we obtain the definition of θ semi open map given in [9].

Definition 2.5. *Let $(X,\tau,\alpha,\beta),(Y,\psi,\sigma,\theta)$ be two spaces with associated operators. A function $f:(X,\tau)\to(Y,\psi)$ is said to be $((\alpha,\beta),(\sigma,\theta))$ irresolute map if the inverse image of each (σ,θ)-semi open set $U\subseteq Y$ is an (α,β)-semi open set in X*

Observe that if in the above definition $\alpha=\sigma=id$, and $\beta=\theta=cl$ then, we obtain the definition of irresolute maps; but if $\alpha=\sigma=id$, then we obtain the definition of (β,θ) irresolute maps given in [9].

3. (α,β)-semi Connected Spaces

In this section, we first introduce the notion of (α,β)-semi separation sets of a given set, then we introduce the notion of (α,β)-semi connected spaces. Also, we give some characterization of these types of spaces and study the existent relationships between them and other types of well known spaces.

Definition 3.1. *Let (X,τ) be a topological space and $\alpha,\beta:P(X)\to P(X)$ be operators associated to a topology τ on X. A (α,β) semi separation of X is a pair U, V of subset of X such that (α,β)-sCl(U)∩V $=U\cap(\alpha,\beta)-sCl(V)=\emptyset$.*

Definition 3.2. *Let (X,τ) be a topological space and $\alpha,\beta:P(X)\to P(X)$ be operators associated to a topology τ on X. We say that U, V, subsets of X are (α,β)-semi separated, for a given subset A of X, if U, V form a (α,β)-semi separation in X and $U\cup V=A$.*

Definition 3.3. *Let (X,τ) be a topological space and $\alpha,\beta:P(X)\to P(X)$ be operators associated to a topology τ on X. A subset A of X is said to be (α,β)-semi connected if there not exist (α,β)-semi separation U, V of A. in other case, we say that A is (α,β)-semi disconnected.*

We can observe that, when β is the identity, then the notion of (α,β)-semi connected set is equivalent to the notion of α-semi connected sets given in ([8]).

Theorem 3.1. *Let (X,τ) be a topological space and $\alpha,\beta:P(X)\to P(X)$ be operators associated to a topology τ on X. Then X is an (α,β)-semi connected if and only if the only subsets of X that are (α,β)-semi open sets and (α,β)-semi closed sets are $\emptyset$ and X itself.*

Proof. (Sufficiency) If there exist a nonempty proper subset A of X, then A is (α,β)-semi open and (α,β)-semi closed. In these case when take $U=A$ and $V=X\setminus A$, then U and V form an (α,β)-semi separation of X, and therefore X is not (α,β)-semi connected.
(Necessity) Suppose that X is not (α,β)-semi connected, and consider U and V an (α,β)-semi separation of X then, $U\cap V\subseteq(\alpha,\beta)-sCl(U)\cap V=U\cap(\alpha,\beta)-sCl(V)=\emptyset$, but $X=U\cup V$, it follows that U (respectively V) is a proper subset of X which is (α,β)-semi open and (α,β)-semi closed. □

The following theorem characterize the (α,β)-semi connected spaces using the notions of $((\alpha,\beta),(\sigma,\theta))$ irresolute maps.

Theorem 3.2. *Let (X,τ,α,β) and (X,ψ) be a topological space with associa ted operators. X is (α,β) semi connected if and only if for each $f:(X,\tau)\to(Y,\psi)$ is a $((\alpha,\beta),(id,id))$ irresolute map, where Y is a topological space with the discrete topology and contains at least two points, is the constant map.*

Proof. (Sufficiency) Consider $f:(X,\tau)\to(Y,\psi)$ a $((\alpha,\beta),(id,id))$ irresolute map, where Y is a topological space with the discrete topology and contains at least two points. Then, X can be cover by a collection of (α,β)-semi open and (α,β)-semi closed sets of the form $\{f^{-1}(y):y\in Y\}$, from these, we conclude that there exists a y_0 such that $f^{-1}(y_0)=X$

and f is a constant map.
(Necessity)Let W be a subset of X that is (α, β)-semi open and (α, β)-semi closed. Suppose that $W \neq \emptyset$ and let $f : (X, \tau) \to (Y, \psi)$ be a $((\alpha, \beta), (id, id))$ irresolute map defined by $f(W) = \{y\}$ and $f(X \setminus W) = \{y_1\}$ for $y \neq y_1$. Since f is a constant map, it follows that $X = W$. □

Observe that, if in the above theorem we choose the operators $\alpha, \beta, \sigma, \theta$, we obtain the usual characterization of connected and the α-semi connected described in [8]. Other important properties of (α, β)-semi connected sets are described in the following theorems.

Theorem 3.3. *Let (X,τ) be a topological space and $\alpha,\beta{:}P(X)\to P(X)$ be operators associated to a topology τ on X. If $A \subseteq X$ is (α,β)-semi connected and $A \subseteq C \cup D$, with C and D (α, β)-semi separated, then $A \subseteq C$ or $A \subseteq D$.*

Proof. Observe

$$(A \cap C) \cap ((\alpha, \beta) - sCl(A \cap D)) \subseteq [A \cap ((\alpha, \beta) - sCl(A))] \cap [C \cap ((\alpha, \beta) - sCl(D))].$$

It follows

$$((\alpha, \beta) - sCl(A \cap D)) \subseteq ((\alpha, \beta) - sCl(A)) \cap ((\alpha, \beta) - sCl(D)),$$

but C and D are (α, β)-semi separated; therefore, $C \cap ((\alpha, \beta) - sCl(D)) = \emptyset$. In consequence, $(A \cap C) \cap ((\alpha, \beta) - sCl(A \cap D)) = \emptyset$. In similar way, we find $(A \cap D) \cap ((\alpha, \beta) - sCl(A \cap C)) = \emptyset$. But $A = (A \cap C) \cup (A \cap D)$ and A is (α, β)-semi connected, we conclude $(A \cap C) = \emptyset$ or $(A \cap D) = \emptyset$. Thus, $A \subseteq C$ or $A \subseteq D$. □

Theorem 3.4. *The union of a family $\{C_i : i \in I\}$ of (α, β)-semi connected sets with nonempty intersection is (α, β)-semi connected.*

Proof. Denote by $E = \bigcup_{i \in I} C_i$. Suppose that E is (α, β)-semi disconnected and let U, V be an (α, β)-semi separation of E. By hypothesis, there exists a point $x \in \bigcap_{i \in I} C_i$. Then, $x \in U$ or $x \in V$, but U and V are disjoint sets. It follows that $C_i \subseteq U$ $(\forall i \in I)$ and $E \subseteq U$. In consequence, $V = \emptyset$, contradiction. In the case that $x \in V$, we obtain similar result. □

From the above theorem, we have the following consequence.

Theorem 3.5. *Let $\{C_i : i \in N\}$ be a collection of (α, β)-semi connected sets such that $C_i \cap C_{i+1} \neq \emptyset$ then $\cup C_i$ is an (α, β)-semi connected set.*

Theorem 3.6. *If C is an (α, β)-semi connected set and $C \subseteq B \subseteq (\alpha, \beta) - sCl(C)$, then the B is an (α, β)-semi connected set.*

Proof. Let us suppose that B is not an (α, β)-semi connected set and let U, V be an (α, β)-semi separation of B. Using hypothesis and the same argument as in the proof of theorem 4.3, $C \subseteq U$ or $C \subseteq V$. If $C \subseteq U$ then, by lemma 3.3 (part b), $(\alpha, \beta) - sCl(C) \subseteq (\alpha, \beta) - sCl(U)$. It follows that

$$((\alpha, \beta) - sCl(C)) \cap V \subseteq ((\alpha, \beta) - sCl(U)) \cap V = \emptyset.$$

Therefore

$$V = V \cap B \subseteq ((\alpha, \beta) - sCl(C)) \cap V = \emptyset.$$

From these, we conclude that $V = \emptyset$.
In the same way, if $C \subseteq V$, we obtain $U = \emptyset$. But this is impossible, therefore B is an (α, β)-semi connected. □

An immediate consequence of the above theorem is the following lemma.

Lemma 3.7. *If B is an (α,β)-semi connected set, then the $((\alpha,\beta)-sCl(B))$ is an (α,β)-semi connected set.*

Proof. Observe that $B \subseteq (\alpha,\beta)-sCl(B) \subseteq (\alpha,\beta)-sCl(B)$. □

Definition 3.4. *Let (X,τ) be a topological space, $\alpha,\beta{:}P(X)\to P(X)$ be operators associated to a topology τ on X and $x\in X$. The (α,β)-semi component of x denoted by $(\alpha,\beta)-SC(x)$, is the union of all subsets of X that are (α,β)-semi connected in X and contain x.*

Theorem 3.8. *Let (X,τ) be a topological space and $\alpha,\beta{:}P(X)\to P(X)$ be operators associated to a topology τ on X. Then*
(a) Each component $(\alpha,\beta)-SC(x)$ is the maximal (α,β)-semi connected set of X;
(b) The set of all distinct (α,β)-semi components of points of X forms a partition of X;
(c) Each $(\alpha,\beta)-SC(x)$ is (α,β)-semi closed.

Proof. (a) Follows directly from the definition of $(\alpha,\beta)-SC(x)$.
(b) Define the relation $\sim$ on X in the following form: $x\sim y$ if and only if there exists a subset (α,β)-semi connected that contain x and y. Observe that $\sim$ is a equivalence relation on X and $[x]=(\alpha,\beta)-SC(x)$, where $[x]$ denote the equivalence class of x by $\sim$.
(c) Given a point $x\in X$, the $(\alpha,\beta)-sCl((\alpha,\beta)-SC(x))$ is a (α,β)-semi connected of X and contain the point x. Since the $(\alpha,\beta)-SC(x)$ is the maximal (α,β)-semi connected set of X that contains the point x. It follows that $(\alpha,\beta)-sCl((\alpha,\beta)-SC(x))\subseteq(\alpha,\beta)-SC(x)$. From the other side, $(\alpha,\beta)-SC(x)\subseteq(\alpha,\beta)-sCl((\alpha,\beta)-SC(x))$; in consequence, the $(\alpha,\beta)-SC(x)=(\alpha,\beta)-sCl((\alpha,\beta)-SC(x))$ and, therefore, $(\alpha,\beta)-SC(x)$ is (α,β)-semi closed set. □

Now, we are going to study under what condition is preserved the direct image of an (α,β)-semi connected space.

Theorem 3.9. *Let $(X,\tau,\alpha,\beta),(Y,\psi,\sigma,\theta)$ be two spaces with associated ope rators and $f:(X,\tau)\to(Y,\psi)$ be a $((\alpha,\beta),(\sigma,\theta))$ irresolute map and surjective. If X is an (α,β)-semi connected space, then Y is a (σ,θ)-semi connected space.*

Using the (α,β)-semi open sets, it is possible to generalize the notions of compactness, α-semi compactness, etc, in a natural way. Let us see the following definition.

Definition 3.5. *Let (X,τ) be a topological space, $\alpha,\beta:P(X)\to P(X)$ be operators associated with a topology τ on X. X is said to be (α,β)-semi compact if given any cover of X by (α,β)-semi open sets, there is a finite subcollection that also cover X.*

The following theorem shows that the (α,β)- semi connected is preserved by direct image of $((\alpha,\beta),(\sigma,\theta))$ irresolute functions.

Theorem 3.10. *Let $(X,\tau,\alpha,\beta),(Y,\psi,\sigma,\theta)$ be two spaces with associated operators and $f:(X,\tau)\to(Y,\psi)$ be an $((\alpha,\beta),(\sigma,\theta))$ irresolute map and surjective. If X is an (α,β)-semi compact space, then Y is a (σ,θ)-semi compact space.*

Proof. The proof is analogous to the classical case. □

The following theorem shows that the (α,β)- semi connected (semi compactness respectively) is preserved by inverse image of $((\alpha,\beta),(\sigma,\theta))$ semi open functions.

Theorem 3.11. *Let $(X,\tau,\alpha,\beta),(Y,\psi,\sigma,\theta)$ two topological spaces with associated operators and $f:(X,\tau)\to(Y,\psi)$ be an $((\alpha,\beta),(\sigma,\theta))$ semi open and bijective function. If Y is a (σ,θ)-semi connected (semi compact respectively)space, then X is an (α,β)-semi connected (semi compact)space.*

Proof. Let $\{U_i : i \in I\}$ be a cover of X by (α, β)-semi open sets. Then, using the fact that f is $((\alpha, \beta), (\sigma, \theta))$ semi open function, the collection $\{f(U_i) : i \in I\}$ is a cover of Y by (σ, θ)-semi open sets. Since Y is (σ, θ)- semi compact space, we obtain a finite subcover $\{f(U_1), f(U_2), ..., f(U_n)\}$ such that $Y = \bigcup_{i=1}^{n} f(U_i)$. It follows that

$$f^{-1}(Y) = f^{-1}(\bigcup_{i=1}^{n} f(U_i)) = \bigcup_{i=1}^{n} f^{-1}(f(U_i)).$$

But f is a bijective function, then

$$X = \bigcup_{i=1}^{n} U_i.$$

Thus X is (α, β)-semi compact.
In the case that Y is a (σ, θ)-semi connected space, use Theorem 4.1 and Definition 3.4, in order to prove that X is an (α, β)-semi connected space. □

Remark The above theorem is false if the condition of bijectivity is changed by surjectivity.

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