INTUITIONISTIC Ω-FUZZY IDEALS OF BCK-ALGEBRAS

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Abstract. Given a set Ω, the notion of intuitionistic Ω-fuzzy ideals of BCK-algebras is introduced, and some related properties are investigated. Relations between intuitionistic Ω-fuzzy subalgebras in BCK-algebras are given. Finally, we study the properties of homomorphism of BCK-algebras.

1.Introduction and Preliminaries After the introduction of the concept of fuzzy sets by Zadeh ([7]), many researches were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy sets" was first published by Atanassov (see [1,2]), as a generalization of the notion of fuzzy sets. In this paper, using Atanassov's idea, we establish the intuitionistic fuzzification of the concept of Ω-subalgebras and Ω-ideals in BCK-algebras, and investigate some of their properties.

By a BCK-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$

(II) $(x*(x*y))*y=0$

(III) $x*x=0$

(IV) $0*x=0$

(V) $x*y=0$ and $y*x=0$ imply $x=y$

for all $x,y,z\in X$.

A partial ordering " $\leq$ " on X can be defined by $x\leq y$ if and only if $x*y=0$.

A nonempty subset S of a BCK-algebra X is called a subalgebra of X if $x*y\in S$ for all $x,y\in S$. A nonempty subset I of a BCK-algebra X is called an ideal of X if

(i) $0\in I$

(ii) $x*y\in I$ and $y\in I$ imply that $x\in I$ for all $x,y\in X$.

By a fuzzy set μ in a nonempty set X we mean a function $\mu: X\to[0,1]$, and the complement of μ, denoted by $\overline{\mu}$, is the fuzzy set in X given by $\overline{\mu}(x)=1-\mu(x)$ for all $x\in X$. We will use the symbol $a\wedge b$ for $\min\{a,b\}$ and $a\vee b$ for $\max\{a,b\}$, where a and b are any real numbers. A fuzzy set μ in a BCK-algebra X is called a fuzzy subalgebra of X if $\mu(x*y)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in X$. A fuzzy set μ in a BCK-algebra X is called a fuzzy ideal of X if (i) $\mu(0)\geq\mu(x)$, (ii) $\mu(x)\geq\mu(x*y)\wedge\mu(y)$ for all $x,y\in X$. In what follows, let Ω denote a set unless otherwise specified. A mapping $H: X\times\Omega\to[0,1]$ is called an Ω-fuzzy set in X. An intuitionistic Ω-fuzzy set (briefly, $I\Omega FS)A$ in a nonempty set X is an object having the form

$$A=\{(x,\alpha_A(x,q),\beta_A(x,q))\mid x\in X,q\in\Omega\}$$

where the functions $\alpha_A: X\times\Omega\to[0,1]$ and $\beta_A: X\times\Omega\to[0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and $0\leq\alpha_A(x,q)+\beta_A(x,q)\leq$

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$1, \forall x \in X, q \in \Omega$. An intuitionistic Ω-fuzzy set $A = \{(x, \alpha_A(x,q), \beta_A(x,q)) \mid x \in X, q \in \Omega\}$ in X can be identified to an ordered pair (α_A, β_A) in $I^{X\times\Omega} \times I^{X\times\Omega}$. For the sake of simplicity, we shall use the symbol $A = (\alpha_A, \beta_A)$ for the $I\Omega FSA = \{(x, \alpha_A(x,q), \beta_A(x,q)) \mid x \in X, q \in \Omega\}$.

2. Intuitionistic Ω-fuzzy Ideals

Definition 2.1. An $I\Omega FSA$=(α_A, β_A) in X is called an intuitionistic fuzzy subalgebra of X over Ω (briefly, Intuitionistic Ω-fuzzy subalgebra of X) if it satisfies

(i) $\alpha_A(x * y, q) \geq \alpha_A(x,q) \wedge \alpha_A(y,q)$

(ii) $\beta_A(x * y, q) \leq \beta_A(x,q) \vee \beta_A(y,q)$

for all $x, y \in X$ and $q \in \Omega$.

Example 2.2. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $A = (\alpha_A, \beta_A)$ be an $I\Omega FS$ in X defined by $\alpha_A(0,q) = \alpha_A(a,q) = \alpha_A(c,q) = 0.7 > 0.3 = \alpha_A(b,q), \beta_A(0,q) = \beta_A(a,q) = \beta_A(c,q) = 0.2 < 0.5 = \beta_A(b,q)$ for all $q \in \Omega$. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic Ω-fuzzy subalgebra of X.

Proposition 2.3. Every intuitionistic Ω-fuzzy subalgebra $A = (\alpha_A, \beta_A)$ of X satisfies the inequalities $\alpha_A(0,q) \geq \alpha_A(x,q)$ and $\beta_A(0,q) \leq \beta_A(x,q)$ for all $x \in X$ and $q \in \Omega$.

Proof. For any $x \in X$ and $q \in \Omega$, we have $\alpha_A(0,q) = \alpha_A(x * x, q) \geq \alpha_A(x,q) \wedge \alpha_A(x,q) = \alpha_A(x,q), \beta_A(0,q) = \beta_A(x * x, q) \leq \beta_A(x,q) \vee \beta_A(x,q) = \beta_A(x,q)$. This completes the proof.

Definition 2.4. An $I\Omega FSA$=(α_A, β_A) in X is called an intuitionistic fuzzy ideal of X over Ω (briefly, intuitionistic Ω-fuzzy ideal of X) if

(i) $\alpha_A(0,q) \geq \alpha_A(x,q)$ and $\beta_A(0,q) \leq \beta_A(x,q)$

(ii) $\alpha_A(x,q) \geq \alpha_A(x * y, q) \wedge \alpha_A(y,q)$

(iii) $\beta_A(x,q) \leq \beta_A(x * y, q) \vee \beta_A(y,q)$

for all $x, y \in X$ and $q \in \Omega$.

Example 2.5. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	0	0	0
4	4	3	4	1	0

Define an $I\Omega FSA$=(α_A, β_A) in X as follows: for every $q \in \Omega, \alpha_A(0,q) = \alpha_A(2,q) = 1, \alpha_A(1,q) = \alpha_A(3,q) = \alpha_A(4,q) = t, \beta_A(0,q) = \beta_A(2,q) = 0, \beta_A(1,q) = \beta_A(3,q) = \beta_A(4,q) = s$, where $t, s \in [0,1]$ and $s + t \leq 1$. By routine calculation we know that $A = (\alpha_A, \beta_A)$ is an intuitionistic Ω-fuzzy ideal of X.

Lemma 2.6. Let an $I\Omega FSA$=(α_A, β_A) in X be an intuitionistic Ω-fuzzy ideal of X. If the inequality $x*y \leq z$ holds in X, then for any $q \in \Omega, \alpha_A(x,q) \geq \alpha_A(y,q) \wedge \alpha_A(z,q), \beta_A(x,q) \leq \beta_A(y,q) \vee \beta_A(z,q)$.

Proof. Let $x, y, z \in X$ by such that $x * y \leq z$. Then $(x * y) * z = 0$, and thus for any $q \in \Omega, \alpha_A(x,q) \geq \alpha_A(x*y,q) \wedge \alpha_A(y,q) \geq (\alpha_A((x*y)*z,q) \wedge \alpha_A(z,q)) \wedge \alpha_A(y,q) = (\alpha_A(0,q)) \wedge \alpha_A(z,q)) \wedge \alpha_A(y,q) = \alpha_A(z,q) \wedge \alpha_A(y,q), \beta_A(x,q) \leq \beta_A(x*y,q) \vee \beta_A(y,q) \leq (\beta_A((x*y)*z,q) \vee \beta_A(z,q)) \vee \beta_A(y,q) = (\beta_A(0,q) \vee \beta_A(z,q)) \vee \beta_A(y,q) = \beta_A(z,q) \vee \beta_A(y,q)$. This completes the proof.

Lemma 2.7. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic Ω-fuzzy ideal of X. If $x \leq y$ in X, then for any $q \in \Omega, \alpha_A(x,q) \geq \alpha_A(y,q), \beta_A(x,q) \leq \beta_A(y,q)$, that is, α_A is order-reserving, and β_A is order-preserving.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x*y = 0, \alpha_A(x,q) \geq \alpha_A(x*y,q) \wedge \alpha_A(y,q) = \alpha_A(0,q) \wedge \alpha_A(y,q) = \alpha_A(y,q), \beta_A(x,q) \leq \beta_A(x*y,q) \vee \beta_A(y,q) = \beta_A(0,q) \vee \beta_A(y,q) = \beta_A(y,q)$. This completes the proof.

Theorem 2.8. If $A = (\alpha_A, \beta_A)$ is an intuitionistic Ω-fuzzy ideal of X, then for any $x, a_1, \cdots, a_n \in X$ and $q \in \Omega, (\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$ implies $\alpha_A(x,q) \geq \alpha_A(a_1,q) \wedge \alpha_A(a_2,q) \wedge \cdots \wedge \alpha_A(a_n,q), \beta_A(x,q) \leq \beta_A(a_1,q) \vee \beta_A(a_2,q) \vee \cdots \vee \beta_A(a_n,q)$.

Proof. Using induction on n and Lemma 2.6 and lemma 2.7.

Theorem 2.9. Every intuitionistic Ω-fuzzy ideal of X is an intuitionistic Ω-fuzzy subalgebra of X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic Ω-fuzzy ideal of X. Since $x * y \leq x$ for all $x, y \in X$, it follows that $\alpha_A(x*y,q) \geq \alpha_A(x), \beta_A(x*y,q) \leq \beta_A(x,q)$ for all $q \in \Omega$. Hence $\alpha_A(x*y,q) \geq (x*y,q) \geq \alpha_A(y,q) \geq \alpha_A(x,q) \wedge \alpha_A(y,q) \geq \alpha_A(x,q) \wedge \alpha_A(y,q), \beta_A(x*y,q) \leq \beta_A(x,q) \leq \beta_A(x*y,q) \vee \beta_A(y,q) \leq \beta_A(x,q) \vee \beta_A(y,q)$. This shows that $A = (\alpha_A, \beta_A)$ is an intaitionistic Ω-fuzzy subalgebra of X.

The converse of Theorem 2.9 may not be true. For example, the intuitionistic Ω-fuzzy subalgebra $A = (\alpha_A, \beta_A)$ in Example 2.2 is not an intuitionistic Ω-fuzzy ideal of X since $\beta_A(b,q) = 0.5 > 0.2 = \beta_A(b*a,q) \wedge \beta_A(a,q)$. We now give a condition for an intuitionistic Ω-fuzzy subalgebra to be an intuitionistic Ω-fuzzy ideal.

Theorem 2.10. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic Ω-fuzzy subalgebra of X such that $\alpha_A(x,q) \geq \alpha_A(y,q) \wedge \alpha_A(z,q), \beta_A(x,q) \leq \beta_A(y,q) \vee \beta_A(z,q)$ for all $x, y, z \in X$ satisfying the inequality $x * y \leq z$ and $q \in \Omega$. Then $A = (\alpha_A, \beta_A)$ is an intuitionistic Ω-fuzzy ideal of X.

Proof. Let $A = (\alpha_A, \beta_A)$ be an intuitionistic Ω-fuzzy subalgebra of X. Recall that $\alpha_A(0,q) \geq \alpha_A(x,q)$ and $\beta_A(0,q) \leq \beta_A(x,q)$ for all $x \in X$ and $q \in \Omega$. Since $x*(x*y) \leq y$, it follows from the hypothesis that $\alpha_A(x,q) \geq \alpha_A(x*y,q) \wedge \alpha_A(y,q), \beta_A(x,q) \leq \beta_A(x*y,q) \vee \beta_A(y,q)$. Hence $A = (\alpha_A, \beta_A)$ is an intuitionistic Ω-fuzzy ideal of X.

Lemma 2.11. An $I\Omega FSA$=(α_A, β_A) is an intuitionistic Ω-fuzzy ideal of X if and only if the Ω-fuzzy sets α_A and $\overline{\beta}_A$ are Ω-fuzzy ideals of X.

Proof. Let $A=(\alpha_A,\beta_A)$ be an intuitionistic Ω-fuzzy ideal of X. Clearly α_A is an Ω-fuzzy ideal of X. For every $x,y\in X$ and $q\in\Omega$, we have $\overline{\beta}_A(0,q)=1-\beta_A(0,q)\leq 1-\beta_A(x,q)=\overline{\beta}_A(x,q),\overline{\beta}_A(x,q)=1-\beta_A(x,q)\geq 1-\beta_A(x*y,q)=1-\beta_A(x*y,q)\vee\beta_A(x,q)=(1-\beta_A(x*y,q))\wedge(1-\beta_A(y,q))=\overline{\beta}_A(x*y,q)\wedge\beta_A(y,q)$. Hence β_A is an Ω-fuzzy ideal of X.

Conversely, assume that α_A and $\overline{\beta}_A$ are Ω-fuzzy ideals of X. For every $x,y\in X$ and $q\in\Omega$, we get $\alpha_A(0,q)\geq\alpha_A(x,q)1-\beta_A(0,q)=\overline{\beta}_A(0,q)\geq\overline{\beta}_A(x,q)=1-\beta_A(x,q)$, that is, $\beta_A(0,q)\leq\beta_A(x,q),\alpha_A(x,q)\geq\alpha_A(x*y,q)\wedge\alpha_A(y,q)$ and $1-\beta_A(x,q)=\overline{\beta}_A(x,q)\leq\overline{\beta}_A(x*y,q)\wedge\overline{\beta}_A(y,q)=(1-\beta_A(x*y,q)\wedge(1-\beta_A(y,q))=1-\beta_A(x*y,q)\vee 1-\beta_A(y,q)$, that is, $\beta_A(x,q)\leq\beta_A(x*y,q)\vee\beta_A(y,q)$. Hence $A=(\alpha_A,\beta_A)$ is an intuitionistic Ω-fuzzy ideal of X.

Theorem 2.12. Let $A=(\alpha_A,\beta_A)$ be an $I\Omega FS$ in X. Then $A=(\alpha_A,\beta_A)$ is an intuitionstic Ω-fuzzy ideal of X if and only if $\Box A=(\alpha_A,\overline{\alpha}_A)$ and $\Diamond A=(\overline{\beta}_A,\beta_A)$ are intuitionistic Ω-fuzzy ideals of X.

Proof. If $A=(\alpha_A,\beta_A)$ is an intuitionistic Ω-fuzzy ideal of X, then $\overline{\overline{\alpha}}_A=\alpha_A$ and $\overline{\beta}_A$ are Ω-fuzzy ideals of X from lemma 2.11, thence $\Box A=(\alpha_A,\overline{\alpha}_A)$ and $\Diamond A=(\overline{\beta}_A,\beta_A)$ are intuitionistic Ω-fuzzy ideals of X.

Conversely, if $\Box A=(\alpha_A,\overline{\alpha}_A)$ and $\Diamond A=(\overline{\beta}_A,\beta_A)$ are intuitionistic Ω-fuzzy ideals of X, then the Ω-fuzzy sets α_A and $\overline{\beta}_A$ are Ω-fuzzy ideals of X, hence $A=(\alpha_A,\beta_A)$ is an intuionistic Ω-fuzzy ideal of X.

A mapping $f:X\to Y$ of BCK-algebras is called a homomorphism if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. Note that if $f:X\to Y$ is a homomorphism of BCK-algebras, then $f(0)=0$. Let $f:X\to Y$ be a homomorphism of BCK-algebras. For any $I\Omega FSA$=(α_A,β_A) in Y, we define a new $I\Omega FSA^f=(\alpha_A^f,\beta_A^f)$ in X by $\alpha_A^f(x,q)=\alpha_A(f(x),q),\beta_A^f(x,q)=\beta_A(f(x),q),\forall x\in X$ and $q\in\Omega$.

Theorem 2.13. Let $f:X\to Y$ be a homomorphism of BCK-algebras. If an $I\Omega FSA=(\alpha_A,\ \beta_A)$ in Y is an intuitionistic Ω-fuzzy ideal of Y, then an $I\Omega FSA^f=(\alpha_A^f,\beta_A^f)$ in X is an intuitionistic Ω-fuzzy ideal of X.

Proof. We first have that $\alpha_A^f(x,q)=\alpha_A(f(x),q)\geq\alpha_A(0,q)=\alpha_A(f(0),q)=\alpha_A(0,q),\beta_A^f(x,q)=\beta_A(f(x),q)\leq\beta_A(0,q)=\beta_A(f(0),q)=\beta_A^f(0,q)$ for all $x\in X$ and $q\in\Omega$. Let $x,y\in X$ and $q\in\Omega$. Then $\alpha_A^f(x,q)=\alpha_A(f(x),q)\geq\alpha_A(f(x)*f(y),q)\wedge\alpha_A(f(y),q)=\alpha_A(f(x*y),q)\wedge\alpha_A(f(y),q)=\alpha_A^f(x*y,q)\wedge\alpha_A^f(y,q),\beta_A^f(x,q)=\beta_A(f(x),q)\leq\beta_A(f(x)*f(y),q)\vee\beta_A(f(y),q)=\beta_A(f(x*y),q)\vee\beta_A(f(y),q)=\beta_A^f(x*y,q)\vee\beta_A^f(y,q)$. Hence $\alpha_A^f=(\alpha_A^f,\beta_A^f)$ is an intuitionistic Ω-fuzzy ideal of X.

If we strengthen the condition of f, then we can construct the converse of Theorem 2.13 as follows.

Theorem 2.14. Let $f:X\to Y$ be an epimorphism of BCK-algebras and let $A=(\alpha_A,\beta_A)$ be an $I\Omega FSA$ in Y. If $A^f=(\alpha_A^f,\beta_A^f)$ is an intuitionistic Ω-fuzzy ideal of X, then $A=(\alpha_A,\beta_A)$ is an intuitionistic Ω-fuzzy ideal of Y.

Proof. For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then for any $q \in \Omega$, $\alpha_A(x,q) = \alpha_A(f(a),q) = \alpha_A^f(a,q) \geq \alpha_A^f(0,q) = \alpha_A(f(0),q) = \alpha_A(0,q), \beta_A(x,q) = \beta_A(f(a),q) = \beta_A^f(a,q) \leq \beta_A^f(0,q) = \beta_A(f(0),q) = \beta_A(0,q)$. Let $x, y \in Y$ and $q \in \Omega$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. It follows that $\alpha_A(x,q) = \alpha_A(f(a),q) = \alpha_A^f(a,q) \geq \alpha_A^f(a*b,q) \wedge \alpha_A^f(b,q) = \alpha_A(f(a*b),q) \wedge \alpha_A(f(b),q) = \alpha_A(f(a)*f(b),q) \wedge \alpha_A(f(b),q) = \alpha_A(x*y,q) \wedge \alpha_A(y,q), \beta_A(x,q) = \beta_A(f(a),q) = \beta_A^f(a,q) \leq \beta_A^f(a*b,q) \vee \beta_A^f(b,q) = \beta_A(f(a*b),q) \vee \beta_A(f(b),q) = \beta_A(f(a)*f(b),q) \vee \beta_A(f(b),q) = \beta_A(x*y,q) \vee \beta_A(y,q)$. This completes the proof.

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