ON SUBTRACTION SEMIGROUPS

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Abstract. In this paper, we define an ideal of a subtraction semigroup and a strong subtraction semigroup and characterizations of ideals is given. We introduce the notion of a relation on subtraction semigroup, called a *SS-relation*, which is a generalization of a subtraction semigroup homomorphism, and then we discuss the fundamental properties related to sub-subtraction semigroups.

1 Introduction B. M. Schein [4] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [5] discussed a problem proposed by B. M. Schein [4] concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we define an ideal of a subtraction semigroup and a strong subtraction semigroup and characterizations of ideals is given. We introduce the notion of a relation on subtraction semigroup, called a *SS-relation*, which is a generalization of a subtraction semigroup homomorphism, and then we discuss the fundamental properties related to sub-subtraction semigroups.

2 Preliminaries By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(SA1) $x - (y - x) = x$;

(SA2) $x - (x - y) = y - (y - x)$;

(SA3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

A subset I of a subtraction algebra X is called a *subalgebra* of X if $x - y \in I$ for all $x, y \in I$.

In a subtraction algebra, the following hold:

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(S1) $x - 0 = x$ and $0 - x = 0$.

(S2) $x - (x - y) \leq y$.

(S3) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(S4) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(S5) $x - (x - (x - y)) = x - y$.

(S6) $(x - y) - x = 0$.

(p7) $(x - y) - y = x - y$.

Lemma 2.1. [3] Let X be a subtraction algebra. Then $(X; \leq)$ is a poset, where $x \leq y \Leftrightarrow x - y = 0$ for any $x, y \in X$.

By a *subtraction semigroup* we mean an algebra $(X; \cdot, -)$ with two binary operations "$-$" and "$\cdot$"that satisfies the following axioms: for any $x, y, z \in X$,

(SS1) $(X; \cdot)$ is a semigroup;

(SS2) $(X; -)$ is a subtraction algebra;

(SS3) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$.

Example 2.2. [3] Let $X = \{0, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1
0	0	0
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

It is easy to check that X is a subtraction semigroup.

Lemma 2.3. [3] Let X be a subtraction semigroup. Then the following hold.

(1) $x0 = 0$ and $0x = 0$

(2) $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$.

(3) $x(y \wedge z) = xy \wedge xz$ and $(x \wedge y)z = xz \wedge yz$

3 Ideals of subtraction semigroups In what follows, let X denote a subtraction semigroup unless otherwise specified.

Definition 3.1. Let $(X, -, \cdot, 0)$ be a subtraction semigroup. A non-empty subset S of X is called a *sub-subtraction semigroup* of X if $x - y \in S$ and $xy \in S$ for all $x, y \in S$

Definition 3.2. A nonempty subset A of a subtraction semigroup X is called to be *left* (resp. *right*) *stable* if $x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$.

Definition 3.3. A non-empty subset I of a subtraction semigroup X is called a *left* (resp. *right*) *ideal* of X if

(1) I is a stable subset of X.

(2) $y \in I$ and $x - y \in I$ imply $x \in I$ for all $x, y \in X$.

If I is a both left and right ideal, it is called a *two-sided ideal* of X. Note that $\{0\}$ and X are ideals. If A is a left (resp. right) ideal of a subtraction semigroup X, then $0 \in A$.

Example 3.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	3	4	3	1
2	2	5	0	2	5	4
3	3	0	3	0	3	3
4	4	0	0	4	0	4
5	5	5	0	5	5	0

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	0
2	0	4	2	0	4	5
3	0	3	0	3	0	0
4	0	4	4	0	4	0
5	0	0	5	0	0	5

It is easy to check that $(X; -, \cdot)$ is a subtraction semigroup. Let $I = \{0, 1, 3, 4\}$. Then I is an ideal of X.

Theorem 3.5. *Suppose I is an ideal of subtraction semigroup X and $x \in I$. If $y \leq x$, then $y \in I$.*

Proof. $y \leq x$ implies $y - x = 0 \in I$. Combining $x \in I$ and using Definition 3.3, (2), we obtain $y \in I$, proving the theorem. □

Theorem 3.6. *Every ideal of a subtraction semigroup X is a sub-subtraction semigroup of X, but the converse is not true.*

Proof. Suppose I is an ideal of X and $x, y \in I$. Since $(x - y) \leq x$ by S(6), it follows from Theorem 3.5, that $x - y \in I$. Hence I is a subalgebra of X. In the following Example 3.10, $\{0, a, b\}$ is a sub-subtraction semigroup of X but not an ideal of X because $1 - b = a \in \{0, a, b\}$ but $1 \notin \{0, a, b\}$. The proof is complete. □

Theorem 3.7. *Let $\{A_i\}$ be an arbitrary collection of ideals of the subtraction semigroup X, where i ranges over some index set. Then $\cap A_i$ is also an ideal of X.*

Proof. Note that each ideal of X contains the zero element of X. Let $x - y, y \in \cap A_i$. Then $x - y, y \in A_i$ for every i. Since each A_i is a ideal of X, it follows that $x \in A_i$ for all i. Hence $x \in \cap A_i$. Next let $x \in \cap A_i$ and $a \in X$. Then $x \in A_i$ for every i, and so $ax, xa \in \cap A_i$ for all i. Thus $ax, xa \in \cap A_i$. Therefore $\cap A_i$ is an ideal of X. □

Let us define the center of a subtraction semigroup X, denoted by *cent(X)*, to be the set

$$\textit{cent}\ (X) = \{x \in X \mid ax = xa \text{ for all } a \in X\}.$$

Let $x, y \in$ *cent* (X). Then $xa = ax$ and $yb = by$ for all $a, b \in X$. Thus $(x - y)a = xa - ya = ax - ay = a(x - y)$ for all $a \in X$. This implies that $x - y \in$ *cent* (X). showing that *cent* (X) is a subalgebra of a subtraction algebra X. Next since $x, y \in$ *cent*(X), we have $xa = ax$ and $ya = ay$. Thus $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$ for all $a \in X$. The following theorems are obvious.

Theorem 3.8. *For any subtraction semigroup X, cent (X) is a sub-subtraction semigroup of X.*

Theorem 3.9. *Let X be a subtraction semigroup X and $a \in X$. Then the set $C(a) = \{x \in X \mid ax = xa\}$ is a sub-subtraction semigroup, and cent $(X) = \bigcap_{a \in X} C(a)$.*

The element 1 is called a *unity* in a subtraction semigroup X if $1x = x1 = x$ for all $x \in X$. A *strong subtraction semigroup* is a subtraction semigroup X that satisfies the following condition : for each $x, y \in X$,

$$x - y = x - xy.$$

If X ia a strong subtraction semigroup with a unity 1, then 1 is the greatest element in X since $x - 1 = x - x1 = x - x = 0$ for all $x \in X$.

Example 3.10. Let $X = \{0, a, b, 1\}$ in which "$-$" and "$\cdot$" are defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

It is easy to check that $(X; -, \cdot)$ is a strong subtraction semigroup with unity 1.

Lemma 3.11. [3] Let X be a strong subtraction semigroup. Then

(1) $xy \leq y$ for all $x, y \in X$,

(2) $x \leq y$, $x, y \in X$ if and only if $x \leq xy$.

Theorem 3.12. Let $(X, -, \cdot)$ be a strong subtraction semigroup and I a subalgebra of $(X, -)$. If I ia an ideal of X, then $y \in I$ and $x \leq y$ imply $x \in I$.

Proof. Suppose that I is an ideal in X, and let $y \in I$ and $x \leq y$. Then $x = y - w$ for some $w \in X$ from (S3), and so $x = y - w = y - yw \in I$. □

Theorem 3.13. Let X be a strong subtraction semigroup and A a subset of X. If $y \in A$ and $x \leq y$ imply $x \in A$, then A is a stable subset of X.

Proof. Suppose that $y \in A$ and $x \leq y$ imply $x \in A$. If $s \in X$ and $a \in A$, then by the Lemma 3.11, $sa \leq a \in A$, hence $sa \in A$. Since $s \leq s$ and $s \leq sa$ from Lemma 3.11, we have

$$as - a = as - (as)a = as - a(sa) = a(s - sa) = a0 = 0,$$

and $as \leq a \in A$, and hence $as \in A$. This completes the proof. □

Lemma 3.14. Let X be a subtraction semigroup. Then we have

$$(x - z) - (y - z) = (x - y) - z$$

for all x, y and $z \in X$.

Proof. Let x, y and z in X. Then we have

$$\begin{aligned}
&((x - z) - (y - z)) - ((x - y) - z) \\
&= (((x - z) - z) - (y - z)) - ((x - y) - z) \quad \text{(p1)} \\
&\leq ((x - z) - y) - ((x - y) - z) \quad \text{(p1, p9)} \\
&= ((x - y) - z) - ((x - y) - z) \quad \text{(S3)} \\
&= 0.
\end{aligned}$$

Thus

$$(x - z) - (y - z) \leq (x - y) - z.$$

The converse inequality is clear. This completes the proof. □

Definition 3.15. Let X be a subtraction semigroup and $a \in X$. Set $A(a) = \{x \in X \mid x \leq a\}$. Then we call $A(a)$ the *initial section* of the element a.

Lemma 3.16. Let X be a subtraction semigroup, $x \leq y$ and $y \in A(a)$. Then $x \in A(a)$.

Proof. Since $y \in A(a)$, we have $y \leq a$. Hence $x \leq y \leq a$, that is, $x \leq a$. This implies $x \in A(a)$. □

Proposition 3.17. Let X be a strong subtraction semigroup and $a \in X$. Then $A(a)$ is an left ideal of X.

Proof. Let $y \in A(a)$ and $x - y \in A(a)$. Then we have $y \leq a$ and $x - y \leq a$. So, by(p4) $x - a \leq x - y$. Hence $x - a \leq x - y \leq a$. $x - a \leq a$ implies $(x - a) - a = x - a = 0$ by (S7), and so $x - a = 0$, that is, $x \leq a$. This implies $x \in A(a)$. Next let $b \in A(a)$ and $x \in X$. Then we have $xb \leq b$ by Lemma 3.11 and $b \leq a$. Thus $xb \leq a$, and so $xb \in A(a)$. This completes the proof. □

Theorem 3.18. Let X be a subtraction semigroup, I an ideal and $x \in I$. Then $A(x) \subset I$.

Proof. If $y \in A(x)$, then we have $y \leq x$. Hence $y - x = 0 \in I$. Since I is an ideal of X and $x \in X$, we obtain $y \in I$. Therefore $A(x) \subset I$. □

Definition 3.19. An element x in a subtraction semigroup X with unity 1 is said to be *left* (resp. *right*) *invertible* if there exists $y \in X$ (resp. $z \in X$) such that $yx = 1_X$ (resp. $xz = 1_X$). The element y (resp. z) is called a *left* (resp. *right*) *inverse* of x. An element $x \in X$ that is both left and right invertible is said to be *invertible* or to be an *unit*.

Theorem 3.20. Let X be a strong subtraction semigroup with unity 1. If y in X is a right invertible element of $x \in X$, then $x \leq y$.

Proof. Let $y \in X$ be a right invertible element of x. Then we have $x - y = x - xy = x - 1 = x - x = 0$, and so $x \leq y$. □

Let X and X' be subtraction semigroups. A mapping $f : X \to X'$ is called a *subtraction semigroup homomorphism* (briefly, *homomorphism*) if $f(x-y) = f(x) - f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Let $f : X \to Y$ be a homomorphism of subtraction semigroup. Then the set $\{x \in X \mid f(x) = 0\}$ is called the *kernel* of f, and denote by $kerf$. Moreover, the set $\{f(x) \in Y \mid x \in X\}$ is called the *image* of f, and denote by $im\ f$.

Lemma 3.21. [3] Let $f : X \to X'$ be a subtraction semigroup homomorphism. Then

(1) $f(0) = 0$,

(2) $x \leq y$ imply $f(x) \leq f(y)$.

(3) $f(x \wedge y) = f(x) \wedge f(y)$.

Proposition 3.22. [3] Let $f : X \to X'$ be a subtraction semigroup homomorphism and $J = f^{-1}(0) = \{0\}$. Then $f(x) \leq f(y)$ imply $x \leq y$.

Proposition 3.23. Let $f : X \to X'$ be a subtraction homomorphism. Then $Kerf$ is a sub-subtraction semigroup of X and Imf a sub-subtraction semigroup of X'.

Proof. The proof is routine and easy, and so omitted. □

4 subtraction semigroup relation We introduce the notion of a relation on subtraction semigroups, called *SS-relation*, which is a generalization of a subtraction semigroup homomorphism.

Definition 4.1. Let X and Y be subtraction semigroups. A nonempty relation $\mathcal{H} \subseteq X \times Y$ is called a *SS-relation* if

(R1) for every $x \in X$ there exists $y \in Y$ such that $x\mathcal{H}y$,

(R2) $x\mathcal{H}a$ and $y\mathcal{H}b$ imply $(x - y)\mathcal{H}(a - b)$,

(R3) $x\mathcal{H}a$ and $y\mathcal{H}b$ imply $(x \cdot y)\mathcal{H}(a \cdot b)$.

We usually denote such relation by $\mathcal{H} : X \to Y$. It is clear from (R1) and (R2) that $0_X\mathcal{H}0_Y$.

Example 4.2. Consider a proper subtraction semigroup $X = \{0, a, b, 1\}$ having the following Cayley table:

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Define a relation $\mathcal{H} : X \to X$ by $0\mathcal{H}0, a\mathcal{H}a, b\mathcal{H}b, 1\mathcal{D}1$. It is easy to verify that $\mathcal{H}$ is a SS-relation. A relation $\mathcal{D} : X \to X$ given by $0\mathcal{D}0, 0\mathcal{D}a, a\mathcal{D}0, a\mathcal{D}a, b\mathcal{D}0, b\mathcal{D}a, 1\mathcal{D}0$ is a SS-relation.

Theorem 4.3. Every subtraction semigroup homomorphism is a SS-relation.

Proof. Let $\mathcal{H} : X \to X$ be a subtraction semigroup homomorphism. Clearly, $\mathcal{H}$ satisfies conditions (R1), R(2) and R(3). ☐

Note that every diagonal SS-relation on a subtraction semigroup X (i.e., a SS-relation satisfying $x\mathcal{H}x$ for all $x \in X$ in which $x\mathcal{D}y$ is false whenever $x \neq y$) is a clearly a subtraction semigroup homomorphism. But in general, the converse of Theorem 4.3 need not be true as seen in the following example.

Example 4.4. The SS-relation $\mathcal{D}$ in Example 4.2 is not a subtraction semigroup homomorphism.

Let $\mathcal{H} : X \to Y$ be a relation. For any $x \in X$ and $y \in Y$, let

$$\mathcal{H}[x] := \{y \in Y \mid x\mathcal{H}y\} \text{ and } \mathcal{H}^{-1}[y] := \{x \in X \mid x\mathcal{H}y\}.$$

Note that $\mathcal{H}[x]$ and $\mathcal{H}^{-1}[y]$ are not subalgebras of X and Y, respectively, as seen in the following example.

Example 4.5. Let $\mathcal{H}$ be a SS-relation in Example 4.2. Then $\mathcal{H}^{-1}[b] = \{b\}$ (resp. $\mathcal{H}[a] = \{a\}$) is not a subtraction semigroup subalgebra of X (resp. Y).

Theorem 4.6. For any SS-relation $\mathcal{H} : X \to Y$, we have

(1) $\mathcal{H}[0_X]$, called the zero image of $\mathcal{H}$, is a sub-subtraction semigroup of Y.

(2) $\mathcal{H}^{-1}[0_Y]$, called the kernel of $\mathcal{H}$ and denoted by Ker$\mathcal{H}$, is a sub-subtraction semigroup of X.

Proof. (1) Let $y_1, y_2 \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}y_1$ and $0_X\mathcal{H}y_2$. It follows from (R2) and R(3) that $0_X\mathcal{H}(y_1 - y_2)$ and $0_X\mathcal{H}(y_1 \cdot y_2)$, that is, $y_1 - y_2 \in \mathcal{H}[0_X]$ and $y_1 \cdot y_2 \in \mathcal{H}[0_X]$.

(2) Let $x_1, X_2 \in Ker\mathcal{H}$. Then $x_1\mathcal{H}0_Y$ and $x_2\mathcal{H}0_Y$. By using (R2) and (R3), we get $(x_1 - x_2)\mathcal{H}0_Y$ and $(x_1 \cdot x_2)\mathcal{H}0_Y$, and so $x_1 - x_2 \in Ker\mathcal{H}, x_1 \cdot x_2 \in Ker\mathcal{H}$. This completes the proof. □

Proposition 4.7. Let $\mathcal{H} : X \to Y$ be a SS-relation.Then we have

(1) If $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$ where $a, b \in X$, then $a - b \in Ker\mathcal{H}$.

(2) If $\mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v] \neq \emptyset$ where $u, v \in Y$, then $u - v \in Ker\mathcal{H}[0_X]$.

Proof. (1) Let $a, b \in X$ be such that $\mathcal{H}[a] \cap \mathcal{H}[b] \neq \emptyset$. Taking $y \in \mathcal{H}[a] \cap \mathcal{H}[b]$, we have $a\mathcal{H}y$ and $b\mathcal{H}y$. It follows from (R2) that $(a-b)\mathcal{H}(y-y) = (a-b)\mathcal{H}(y-y) = (a-b)\mathcal{H}0_Y$ so that $a - b \in Ker\mathcal{H}$.

(2) Let $x \in \mathcal{H}^{-1}[u] \cap \mathcal{H}^{-1}[v]$. Then $x\mathcal{H}u$ and $x\mathcal{H}v$. Using (R2), we obtain $x - x)\mathcal{H}(u-v) = 0_X\mathcal{H}(u - v)$, i.e., $u - v \in \mathcal{H}[0_X]$. This completes the proof. □

Theorem 4.8. Let $\mathcal{H} : X \to Y$ be a SS-relation and let S be a sub-subtraction semigroup of X. Then

$$\mathcal{H}[S] := \{y \mid x\mathcal{H}y \text{ for some } x \in S\}$$

is a sub-subtraction semigroup of Y.

Proof. Clearly, $\mathcal{H}[S] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $y_1, y_2 \in \mathcal{H}[S]$. Then $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$ for some $x_1, x_2 \in S$. Using (R2) and (R3), we obtain $(x_1 - x_2)\mathcal{H}(y_1 - y_2)$ and $(x_1 \cdot x_2)\mathcal{H}(y_1 \cdot y_2)$ which implies that $y_1 - y_2 \in \mathcal{H}[S]$ and $y_1 \cdot y_2 \in \mathcal{H}[S]$ since $x_1 - x_2$ and $x_1 \cdot x_2 \in S$. Therefore $\mathcal{H}[S]$ is a sub-subtraction semigroup of Y. This completes the proof. □

Corollary 4.9. Let $\mathcal{H} : X \to Y$ be a SS-relation. Then we have

(1) $\mathcal{H}[X]$ is a sub-subtraction semigroup of Y,

(2) $\mathcal{H}[X] = \bigcup_{x \in X} \mathcal{H}[x]$,

(3) The zero image of $\mathcal{H}$ is a sub-subtraction semigroup of $\mathcal{H}[X]$.

Proof. (1) and (2) are straightforward. (3) Let $a, b \in \mathcal{H}[0_X]$. Then $0_X\mathcal{H}a$ and $0_X\mathcal{H}b$, and so $0_X\mathcal{H}(a-b)$ and $0_X\mathcal{H}(a \cdot b)$, i.e., $a - b$ and $a \cdot b \in \mathcal{H}[0_X]$. Therefore $\mathcal{H}[0_X]$ is a sub-subtraction semigroup of $\mathcal{H}[X]$. □

Theorem 4.10. Let $\mathcal{H} : X \to Y$ be a SS-relation and let T be a sub-subtraction semigroup of Y. Then

$$\mathcal{H}^{-1}[T] := \{x \in X \mid x\mathcal{H}y \text{ for some } y \in T\}$$

is a sub-subtraction semigroup of X.

Proof. Obviously, $\mathcal{H}^{-1}[T] \neq \emptyset$ since $0_X\mathcal{H}0_Y$. Let $x_1, x_2 \in \mathcal{H}^{-1}[T]$. Then there exist $y_1, y_2 \in T$ such that $x_1\mathcal{H}y_1$ and $x_2\mathcal{H}y_2$. Note that $y_1 - y_2 \in T$ and $y_1 \cdot y_2 \in T$ since T is a sub-subtraction semigroup of Y. It follows from (R2) and (R3) that $(x_1 - x_2)\mathcal{H}(y_1 - y_2)$ and $(x_1 \cdot x_2)\mathcal{H}(y_1 \cdot y_2)$ so that $x_1 - x_2 \in \mathcal{H}^{-1}[T]$ and $x_1 \cdot x_2 \in \mathcal{H}^{-1}[T]$. Hence $\mathcal{H}^{-1}[T]$ is a sub-subtraction semigroup of X. □

Corollary 4.11. Let $\mathcal{H} : X \to Y$ be a SS-relation. Then

(1) $\mathcal{H}^{-1}[Y]$ is a sub-subtraction semigroup of X,

(2) $\mathcal{H}^{-1}[Y] = \bigcup_{y \in Y} \mathcal{H}[y]$,

(3) The kernel of $\mathcal{H}$ is a sub-subtraction semigroup of $\mathcal{H}^{-1}[Y]$.

Proof. (1) and (2) are straightforward. (3) Let $x, y \in Ker\mathcal{H}$. Then $x\mathcal{H}0_Y$ and $y\mathcal{H}0_Y$. It follows from (R2) and (R3) that

$$(x - y)\mathcal{H}(0_Y - 0_Y) = (x - y)\mathcal{H}0_Y \text{ and } (x \cdot y)\mathcal{H}(0_Y \cdot 0_Y) = (x \cdot y)\mathcal{H}0_Y$$

so that $x - y \in Ker\mathcal{H}$. Hence $Ker\mathcal{H}$ is a sub-subtraction semigroup of $\mathcal{H}^{[}Y]$. This completes the proof. □

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