A NOTE ON MARTINDALE QUOTIENT RINGS OF DUBROVIN VALUATION RINGS

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Abstract. Let R be a Dubrovin valutaion ring of a simple Artinian ring Q and let $Q_s(R)(Q_l(R))$ be the symmetric (left) Martindale ring of quotients of R. It is shown that either $Q_s(R) = R_P = Q_l(R)$ for the minimal Goldie prime ideal P of R such that the prime segment $P \supset (0)$ is simple or $Q_s(R) = Q = Q_l(R)$.

1. Introduction.

Throughout this note, R will be a Dubrovin valuation ring of a simple Artinian ring Q, and $Q_s(R)(Q_l(R))$ will be the symmetric (left) Martindale ring of quotients of R, respectively. We will prove that either $Q_s(R) = R_P = Q_l(R)$ for the minimal prime ideal P of R such that the prime segment $P \supset (0)$ is simple or $Q_s(R) = Q = Q_l(R)$ by using the prime segments, where R_P is a localization of R at P. Since Q is an injective hull of a left R-module R, it follows that $Q_l(R) = \{q \in Q \mid Aq \subseteq R$ for some non-zero ideal A of $R\}$and $Q_s(R) = \{q \in Q_l(R) \mid qB \subseteq R$ for some non-zero ideal B of $R\}$(see [7,(10.6)]). For any ideal A we write $(R : A)_l = \{q \in Q \mid qA \subseteq R\}$ and $(R : A)_r = \{q \in Q \mid Aq \subseteq R\}$. With these notation, we have $Q_l(R) = \cup\{(R : A)_l \mid A$ runs over all non-zero ideals of $R\}$. A prime ideal P of R is called *Goldie prime* if R/P is a prime Goldie ring. Let P_1 and P_2 be Goldie prime ideals of R with $P_1 \supset P_2$. The pair $P_1 \supset P_2$ is said to be a *prime segment* of R if there are no Goldie prime ideals properly between P_1 and P_2. Recall some properties of a Dubrovin valuation ring R as follows :

(1) A prime ideal P of R is Goldie prime if and only if it is localizable, i.e., $C(P) = \{c \in R \mid c$ is regular mod $P\}$ is a regular Ore set ([1,§1]).

(2) There is a one-to-one correspondence between the set of all overrings of R and the set of all Glodie prime ideals of R, which is given by

$$S \longrightarrow J(S); P \longrightarrow R_P$$

, where S is an overring of R, P is a Goldie prime ideal of R and $J(S)$ is the Jacobson radical of S. In particular, the mapping is anti-inclusion-preserving ([6, (6.7) and (14.5)]).

(3) Let S be a proper overring of R. Then $(R : S)_l = J(S) = (R : S)_r$ and $(R : J(S))_l = S = (R : J(S))_r$ ([5 ,(1.1)]).

2. The symmetiric (left)Martindale ring of quotients of R.

In order to characterize the symmetric (left)Martindale ring of quotients of R, we will

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consider the following two cases :
Case 1.There is the minimal Goldie prime ideal P of R:
(a)The prime segment $P \supset (0)$ is Archimedean, i.e., for any $a \in P \setminus (0)$, there is an ideal A such that $a \in A$ and $\cap A^n = (0)$.
(b)The prime segment $P \supset (0)$ is exceptional, i.e., there is a non-Goldie prime ideal C such that $P \supset C \supset (0)$, there are no ideals properly between P and C and $\cap C^n = (0)$.
(c)The prime segment $P \supset (0)$ is simple, i.e., there are no ideals properly between P and (0).
Case 2. There are no minimal Goldie prime ideals, i.e., $\cap P_\alpha = (0)$, where P_α runs over all non-zero Goldie prime ideals.

With these notation, we have the following theorem.

Theorem. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then*
(1)$Q_s(R) = Q = Q_l(R)$ in the case where Case 1 (a) or (b), or Case 2.
(2)$Q_s(R) = R_P = Q_l(R)$ in the case where Case 1 (c).

Proof. Suppose that Case 2 occurs. Then for any Goldie prime ideal P_α, we have $R \supseteq P_\alpha = P_\alpha R_{P_\alpha} = R_{P_\alpha} P_\alpha$, because $J(R_{P_\alpha}) = P_\alpha$ and so $R_{P_\alpha} \subseteq Q_s(R)$. Thus $Q_s(R) \supseteq S = \cup R_{P_\alpha}$, an overring of R, where P_α runs over all non-zero Goldie prime ideals of R. Assume that $Q \supset S$. Then $J(S)$ is a non-zero Goldie prime ideal by (2) and $S \supseteq R_{P_\alpha}$. So $J(S) \subseteq P_\alpha$ for any P_α by (2),i.e., $J(S) \subseteq \cap P_\alpha = (0)$, a contradiction. Hence $Q_s(R) = Q$ and so $Q_s(R) = Q = Q_l(R)$.

Suppose that Case 1 occurs. Then since $R \supseteq P = R_P P = P R_P$, we have $R_P \subseteq Q_s(R) \subseteq Q_l(R) \subseteq Q$. Thus either $R_P = Q_s(R)$ or $Q_s(R) = Q$ by (2).

Suppose that Case 1 (a) occurs. Then there is an element b in R_P such that $I = bR_P = R_P b \subseteq P = J(R_P)$ and $\mathcal{C} = \{b^n \mid n = 1, 2, ...\}$ is a regular Ore set of R_P with $Q = (R_P)_{\mathcal{C}} = \{\alpha b^{-n} \mid \alpha \in R_P, n = 1, 2, ...\}$ by the proof of [2,(2.3)]. So, for any regular element c in R, there is a natural number n such that $c^{-1} b^n \in R_P$ and $b^n c^{-1} \in R_P$. Hence $c^{-1} I^n P \subseteq R_P P = P \subseteq R$ and $P I^n c^{-1} \subseteq R$, which show $c^{-1} \in Q_s(R)$. Therefore $Q_s(R) = Q$ follows.

Suppose that Case 1 (b) occurs. Then we claim that $P \supset C \supset (0)$ is an exceptional prime segment of R_P. By [2, (2.2)], C is an ideal of R_P. In fact, it is a prime ideal of R_P, because there are no ideals of R properly between P and C. Suppose that C is a Goldie prime ideal of R_P. Then $(R_P)_C$ is a proper overring of R_P by (2) which contradicts the fact that R_P is the maximal over ring of R by (2). Hence C is a non-Goldie prime ideal of R_P. This means the prime segment $P \supset C \supset (0)$ is exceptional as the prime ideals of R_P. Since R_P is of rank one, i.e., $P = J(R_P)$ is the only non-zero Goldie prime ideal of R_P, we can use the method in [2,(2.2)]. For any regular element c in R, there is a natural number n such that $c^{-1} C^n \subseteq R_P$ and $C^n c^{-1} \subseteq R_P$. So as in Case 1 (a), $c^{-1} C^n P \subseteq R_P P \subseteq R$ and $R C^n c^{-1} \subseteq P R_P \subseteq R$. Hence $c^{-1} \in Q_s(R)$ and thus $Q_s(R) = Q = Q_l(R)$ follows.

Suppose that Case 1 (c) occurs. Assume that $Q_l(R) = Q$. Then for any $q \in Q \setminus R_P$, there is a non-zero ideal A of R with $Aq \subseteq R$. If $A \supset P$, then $R_P A = R_P$ so that $q \in R_P q = R_P A q \subseteq R_P$, a contradiction. Thus $A \subseteq P$ and so $A = P$, because the prime segment is simple. Hence $q \in (R : A)_r = (R : P)_r = R_P$ by (3), because $P = J(R_P)$, a contradiction. Hence $Q_l(R) \subset Q$. Since R_P is the maximal overring of R by (2), It follows that $Q_s(R) = R_P = Q_l(R)$, because $R_P \subseteq Q_s(R) \subseteq Q_l(R)$.

We end this note with remarks on examples : All prime segments of any commutative valuation rings are Archimedean. Moreoveer, if Q is of finite dimensional over its center, then all prime segments of R is Archimedean. Let $G = \oplus Z_i$ be the direct sums of Z_i with $Z_i = Z$, where Z is the ring of integers, which is a totally ordered abelian group by lexicographical ordering and let V be the valuation ring with G as its value group (see[4]). Then V does not have the minimal prime ideals. Examples of Dubrovin(in fact, total) valuation rings satisfying Case 1 (b) and Case (c) are given in [1, Examples 11 and 12]. We will give another example of Dubrovin valuation rings satisfying Case 1 (c) which is taken from [8,(2.4)]: Let F_0 be any field and let $F = F_0(\{y_i \mid i \in Z\})$ be the rational function field over F_0 in indeterminates y_i. We define an automorphism σ of F as follows;$\sigma(a) = a$ for any $a \in F_0$ and $\sigma(y_i) = y_{i-1}$ for all $i \in Z$ (note that we defined $\sigma(y_i) = y_{i+1}$ in [8]). Also define a valuation v of F as follows; $v(a) = 0$ for any $a \in F_0$ and $v(Y_i) = g_i = (\cdots, 0, 1, 0, \cdots) \in G = \oplus Z_i(i \in Z$ and $Z_i = Z)$, the ith component is one and the other components are all zeros. Let V be the valuation ring of F determined by v. Then σ induces an automorphism of V. Furthermore, we defin $P_i = \bigcap_{j=1}^{\infty} y_{i+1}^{j} V$ for any $i \in Z$. Then $P's$ are prime ideals by [4,(17.1)] such that $\sigma(P_i) = P_{i-1} \subset P_i, J(V) = \cup P_i$ and $\cap_i P_i = (0)$. Let $V[x, \sigma]$ be the skew polynomial ring over V in an indeterminate x and let $R = V[x, \sigma]_{J(V)[x,\sigma]}$ is a total valuation ring of rank one and the prime segment $J(R) \supset (0)$ is simple by [3, Theorem 1 and Lemma 5].

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