

SMARANDACHE *BCI*-ALGEBRAS

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ABSTRACT. The notion of Smarandache (positive implicative, commutative, implicative) *BCI*-algebras, Smarandache subalgebras and Smarandache ideals is introduced, examples are given, and related properties are investigated.

1. INTRODUCTION

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure \mathbf{W} on A such that there exists a proper subset B of A which is embedded with a strong structure \mathbf{S} . In [6], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [5]. It will be very interesting to study the Smarandache structure in *BCK/BCI*-algebras. Thus, in this paper, we discuss the Smarandache structure in *BCI*-algebras. We introduce the notion of Smarandache (positive implicative, commutative, implicative) *BCI*-algebras, Smarandache subalgebras and Smarandache ideals, and investigate some related properties.

2. PRELIMINARIES

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$,
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0)$,
- (III) $(\forall x \in X) (x * x = 0)$,
- (IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a *BCI*-algebra X satisfies the following identity:

- (V) $(\forall x \in X) (0 * x = 0)$,

then X is called a *BCK-algebra*. We can define a partial order ' \leq ' on X by $x \leq y$ if and only if $x * y = 0$. Every *BCI*-algebra X has the following properties:

- (a1) $(\forall x \in X) (x * 0 = x)$.
- (a2) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$.
- (a3) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.

A *BCI*-algebra X is called a *medial BCI-algebra* if it satisfies:

$$(\forall x, y, z, u \in X) ((x * y) * (z * u) = (x * z) * (y * u)).$$

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For a *BCI*-algebra X , the set $X_+ := \{x \in X \mid 0 \leq x\}$ is called the *BCK-part* of X . If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra*. Note that X is a medial *BCI*-algebra if and only if X is a *p-semisimple BCI-algebra*. A *BCI*-algebra X is said to be *associative* if it satisfies:

$$(\forall x, y, z \in X) ((x * y) * z = x * (y * z)).$$

Every associative *BCI*-algebra is a *p-semisimple BCI-algebra*. A nonempty subset I of a *BCI*-algebra X is called an *ideal* of X if it satisfies the following conditions:

- (i) $0 \in I$,
- (ii) $(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I)$.

3. SMARANDACHE BCI-ALGEBRAS

A *Smarandache BCI-algebra* is defined to be a *BCI*-algebra X in which there exists a proper subset Q of X such that

- $0 \in Q$ and $|Q| \geq 2$,
- Q is a *BCK*-algebra under the operation of X .

By a *Smarandache positive implicative* (resp. *commutative* and *implicative*) *BCI-algebra*, we mean a *BCI*-algebra X which has a proper subset Q of X such that

- $0 \in Q$ and $|Q| \geq 2$,
- Q is a positive implicative (resp. commutative and implicative) *BCK*-algebra under the operation of X .

Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

| | | | | | | |
|-----|---|---|---|---|---|---|
| $*$ | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 3 | 2 | 3 | 3 |
| 1 | 1 | 0 | 3 | 2 | 3 | 3 |
| 2 | 2 | 2 | 0 | 3 | 0 | 0 |
| 3 | 3 | 3 | 2 | 0 | 2 | 2 |
| 4 | 4 | 2 | 1 | 3 | 0 | 1 |
| 5 | 5 | 2 | 1 | 3 | 1 | 0 |

Table 1

Then $(X; *, 0)$ is a Smarandache *BCI*-algebra. Let $X_1 = \{0, a, b\}$ and $X_2 = \{0, a, b, c\}$ be sets with the following Cayley tables:

| | | | |
|-------|---|---|---|
| $*_1$ | 0 | a | b |
| 0 | 0 | b | a |
| a | a | 0 | b |
| b | b | a | 0 |

Table 2

| | | | | |
|-------|---|---|---|---|
| $*_2$ | 0 | a | b | c |
| 0 | 0 | c | b | a |
| a | a | 0 | c | b |
| b | b | a | 0 | c |
| c | c | b | a | 0 |

Table 3

Then $(X_1; *_1, 0)$ and $(X_2; *_2, 0)$ are not Smarandache *BCI*-algebras. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

| | | | | | | |
|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 3 | 3 | 3 |
| 1 | 1 | 0 | 1 | 3 | 3 | 3 |
| 2 | 2 | 2 | 0 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 | 0 | 0 | 0 |
| 4 | 4 | 3 | 4 | 1 | 0 | 0 |
| 5 | 5 | 3 | 5 | 1 | 1 | 0 |

Table 4

Then $(X; *, 0)$ is a Smarandache implicative *BCI*-algebra, because $(Q := \{0, 1, 2\}; *, 0)$ is an implicative *BCK*-algebra. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

| | | | | | | |
|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 4 | 4 |
| 1 | 1 | 0 | 0 | 1 | 4 | 4 |
| 2 | 2 | 2 | 0 | 2 | 4 | 4 |
| 3 | 3 | 3 | 3 | 0 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 0 | 0 |
| 5 | 5 | 4 | 4 | 5 | 1 | 0 |

Table 5

Then $(X; *, 0)$ is a Smarandache positive implicative *BCI*-algebra, because $(Q := \{0, 1, 2, 3\}; *, 0)$ is a positive implicative *BCK*-algebra. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

| | | | | | | |
|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 0 | 0 | 4 | 4 |
| 1 | 1 | 0 | 0 | 1 | 4 | 4 |
| 2 | 2 | 1 | 0 | 2 | 4 | 4 |
| 3 | 3 | 3 | 3 | 0 | 4 | 4 |
| 4 | 4 | 4 | 4 | 4 | 0 | 0 |
| 5 | 5 | 4 | 4 | 5 | 1 | 0 |

Table 6

Then $(X; *, 0)$ is a Smarandache commutative *BCI*-algebra, because $(Q := \{0, 1, 2, 3\}; *, 0)$ is a commutative *BCK*-algebra. Using the fact that a *BCK*-algebra is implicative if and only if it is both commutative and positive implicative, we know that a *BCI*-algebra X is a Smarandache implicative *BCI*-algebra if and only if X is both a Smarandache commutative *BCI*-algebra and a Smarandache positive implicative *BCI*-algebra. For a *BCI*-algebra X , if $|X_+| = 1$, then there is no non-trivial proper subset Q of X which is a *BCK*-algebra under the operation of X . Hence any *BCI*-algebra X with $|X_+| = 1$ cannot be a Smarandache *BCI*-algebra. Using this result, we know that every p -semisimple or medial *BCI*-algebra cannot be a Smarandache *BCI*-algebra, and every associative *BCI*-algebra cannot be a Smarandache *BCI*-algebra. We also note that if a *BCI*-algebra X satisfies one of the following assertions:

- (i) $(\forall x, y \in X) (x * (x * y) = y)$,
- (ii) $(\forall x, y \in X) (x * y = 0 * (y * x))$,
- (iii) $(\forall x, y, z \in X) (x * (y * z) = z * (y * x))$,
- (iv) $(\forall x \in X) (0 * x = 0 \Rightarrow x = 0)$,

- (v) $(\forall x \in X) (0 * (0 * x) = x)$,
- (vi) $(\forall x, y \in X) (x * (0 * y) = y * (0 * x))$,
- (vii) $(\forall x, y, z \in X) ((x * y) * z = 0 * ((y * (0 * z)) * x))$,
- (viii) $(\forall x, y, z \in X) ((z * y) * (z * x) = x * y)$,
- (ix) $(\forall x, y, z, u \in X) ((x * u) * (z * y) = (y * u) * (z * x))$,
- (x) $(\forall x, y \in X) ((0 * y) * (0 * x) = x * y)$,
- (xi) $(\forall x, y \in X) (0 * (0 * (x * y)) = x * y)$,
- (xii) $(\forall x, y \in X) (z * (z * (x * y)) = x * y)$,
- (xiii) $(\forall x, y \in X) (x * y = 0 \Rightarrow x = y)$,
- (xiv) $(\forall x, y, z \in X) (x * y = x * z \Rightarrow y = z)$,
- (xv) $(\forall x, y, z \in X) (z * x = z * y \Rightarrow x = y)$,
- (xvi) $(\forall x, y, z \in X) ((x * y) * (x * z) = 0 * (y * z))$,
- (xvii) $(\forall x, y, z \in X) (x * y = 0 \Rightarrow (z * x) * (z * y) = 0)$,
- (xviii) $(\forall x, y, z \in X) ((z * x) * (z * y) = 0 \Rightarrow x * y = 0)$,
- (xix) $(\forall x, y, z \in X) (x * y = 0 \Rightarrow (y * z) * (x * z) = 0)$,
- (xx) $(\forall x, y, z \in X) ((x * z) * (y * z) = 0 \Rightarrow x * y = 0)$,
- (xxi) $(\forall x, y, z, w \in X) ((x * y) * (z * w) = (w * z) * (y * x))$,
- (xxii) $(\forall x, y, z \in X) ((x * y) * z = (0 * z) * (y * x))$,
- (xxiii) $(\forall x, y, z \in X) (x * (y * z) = (z * y) * (0 * x))$,

then X cannot be a Smarandache BCI -algebra. Let X be a BCI -algebra that satisfies the identity $0 * x = x$ for all $x \in X$. If X satisfies one of the following assertions:

- (i) $(\forall x, y, z \in X) (x * (y * z) = (x * y) * z)$,
- (ii) $(\forall x, y \in X) ((x * y) * y = x)$,
- (iii) $(\forall x, y, z \in X) ((x * y) * z = (z * y) * x)$,
- (iv) $(\forall x, y, z, u \in X) ((x * y) * (z * u) = (x * z) * (y * u))$,
- (v) $(\forall x, y, z \in X) (x * (x * y) = y)$,

then X cannot be a Smarandache BCI -algebra. Also, an algebra $(X; *, 0)$ of type $(2, 0)$ that satisfies the following conditions:

- (i) $(\forall x, y, z \in X) ((x * 0) * (y * z) = z * (y * x))$,
- (ii) $(\forall x, y \in X) (x * (y * y) = x)$

cannot be a Smarandache BCI -algebra, and an algebra $(X; *, 0)$ of type $(2, 0)$ that satisfies the following conditions:

- (i) $(\forall x \in X) (x * x = 0)$,
- (ii) $(\forall x, y \in X) (x * y = 0 = y * x \Rightarrow x = y)$,
- (iii) $(\forall x, y, z, w \in X) ((x * y) * (w * z) = (x * w) * (y * z))$,
- (v) $(\forall x, y, z \in X) ((x * y) * (x * z) = z * y)$

cannot be a Smarandache BCI -algebra.

A Cayley table of a set X is said to be of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ if the $(1, 1)$, $(1, 2)$ and $(2, 2)$ -entry is 0, and the $(2, 1)$ -entry is 1. For example, tables 4, 5, and 6 are of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We know that every BCI -algebra with a Cayley table of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a Smarandache BCI -algebra, and every BCI -algebra X with $|X_+| \geq 2$ is a Smarandache BCI -algebra. Let X be a nontrivial BCK -algebra and let w be an ideal element which is not contained in X . If we define $x * w = w * x = w$ for any $x \in X$ and $w * w = 0$, then $X \cup \{w\}$ is a BCI -algebra, and so it is a Smarandache BCI -algebra. It is well known that any group G in which the square of every element is the identity e is a BCI -algebra and such a group G

belongs to the variety of quasi-commutative *BCI*-algebras of type $(1, 0; 0, 0)$. The group G is used to make a new *BCI*-algebra together with a *BCK*-algebra. Let X be a nontrivial *BCK*-algebra and let $Y = (G \setminus \{e\}) \cup X$. We define the operation $*$ on Y as follows:

(i) For $x, y \in G \setminus \{e\}$, we put

$$x * y := \begin{cases} xy & \text{in } G \setminus \{e\} \text{ if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

(ii) For $x, y \in X$, we put

$$x * y = x * y \text{ in } X.$$

(iii) For $x \in G \setminus \{e\}$ and $y \in X$, we put

$$x * y = y * x = x.$$

Then Y is a *BCI*-algebra, and so it is a Smarandache *BCI*-algebra.

For a nontrivial *BCK*-algebra $(X_1; *_1, 0)$ and a *BCI*-algebra $(X_2; *_2, 0)$, let $X = X_1 \cup X_2$ and define a binary operation $*$ on X as follows:

$$x * y := \begin{cases} x *_1 y & \text{if } x, y \in X_1, \\ x *_2 y & \text{if } x, y \in X_2, \\ x & \text{if } x \in X_2, y \in X_1, \\ 0 *_2 y & \text{if } x \in X_1, y \in X_2, y \neq 0, \\ x & \text{if } x \in X_1, y \in X_2, y = 0, \end{cases}$$

Then $(X; *, 0)$ is a *BCI*-algebra, and thus it is a Smarandache *BCI*-algebra.

Let $(X; *, 0)$ be a Smarandache *BCI*-algebra and let H be a subset of X such that $0 \in H$ and $|H| \geq 2$. Then H is called a *Smarandache subalgebra* of X if $(H; *, 0)$ is a Smarandache *BCI*-algebra. For example, consider a Smarandache *BCI*-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

| | | | | | | |
|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 0 | 3 | 2 | 3 | 3 |
| 1 | 1 | 0 | 3 | 2 | 3 | 3 |
| 2 | 2 | 2 | 0 | 3 | 0 | 0 |
| 3 | 3 | 3 | 2 | 0 | 2 | 2 |
| 4 | 4 | 2 | 1 | 3 | 0 | 1 |
| 5 | 5 | 2 | 1 | 3 | 1 | 0 |

Table 7

Then $H_1 = \{0, 1, 2, 3\}$ is a Smarandache subalgebra of X . Any subalgebra of a Smarandache *BCI*-algebra X need not in general be a Smarandache subalgebra of X . For example, in the Smarandache *BCI*-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the Table 7, the set $H_2 = \{0, 2, 3\}$ is a subalgebra of X which is not a Smarandache subalgebra of X . For a Smarandache *BCI*-algebra X , let H be a subalgebra of X . If H have a Cayley table of type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then H is a Smarandache subalgebra of X . If a *BCI*-algebra $(X; *, 0)$ contains a Smarandache subalgebra, then X is a Smarandache *BCI*-algebra. In fact, let H be a Smarandache subalgebra of X . Then there is a proper subset Q of H such that $0 \in Q$, $|Q| \geq 2$ and $(Q; *, 0)$ is a *BCK*-algebra. Since $H \subset X$, it follows that X is a Smarandache *BCI*-algebra. Let X be a Smarandache *BCI*-algebra. A nonempty subset I of X is called a *Smarandache ideal* of X related to Q if it satisfies:

- (i) $0 \in I$,
- (ii) $(\forall x \in Q) (\forall y \in I) (x * y \in I \Rightarrow x \in I)$,

where Q is a BCK -algebra contained in X . If I is a Smarandache ideal of X related to every BCK -algebra contained in X , we simply say that I is a *Smarandache ideal* of X . Since X_+ is a maximal BCK -algebra contained in X , every subset I of a Smarandache BCI -algebra X containing X_+ is a Smarandache ideal of X . Let $X = \{0, 1, 2, 3, 4, 5\}$ be a Smarandache BCI -algebra with the table 5. Then subsets $I = \{0, 1, 2\}$ and $J = \{0, 1, 3\}$ of X are Smarandache ideals of X related to a BCK -algebra $Q = \{0, 1, 2, 3\}$ with respect to the operation $*$ on X . Let Q_1 and Q_2 be BCK -algebras contained in a Smarandache BCI -algebra X and $Q_1 \subset Q_2$. Then every Smarandache ideal of X related to Q_2 is a Smarandache ideal of X related to Q_1 , but the converse is not true. For example, consider BCK -algebras $Q_1 = \{0, 1, 2\}$ and $Q_2 = \{0, 1, 2, 3\}$ in a Smarandache BCI -algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the table 5. Then a subset $I = \{0, 2, 3\}$ is a Smarandache ideal of X related to Q_1 , but not a Smarandache ideal of X related to Q_2 since $1 * 2 = 0 \in I$ and $2 \in I$ but $1 \notin I$. Thus we know that there exists a BCK -algebra Q contained in a Smarandache BCI -algebra X such that a Smarandache ideal of X related to Q is not an ideal of X .

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