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FUZZY $\otimes$-SUBALGEBRAS ON FI-ALGEBRAS

KYUNG HO KIM, JONG GEOL LEE, SANG MOON LEE AND EUN HWAN ROH*

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ABSTRACT. We introduce the concept of fuzzy $\otimes$-subalgebras of fuzzy implication algebras, and obtain some related properties.

1. INTRODUCTION

The concept of FI-algebras, which is introduced by W. M. Wu in [10], is the abstract concept of implication connectives of [0,1]-valued logics. In the same paper [10], Wu introduced the notion of the filter in a FI-algebra, and investigated their properties. Recently, many mathematical papers have been written investigating the algebraic properties of FI-algebras(see [1, 2, 3]). In particular, D. Wu [11] introduced the concept of the commutativity in FI-algebras, and studied various properties. T. R. Zou [15] introduced the concept of P-filters and PFI-algebras, and obtained some important results. In this paper, we introduce the concept of fuzzy $\otimes$-sets of FI-algebras, and obtain some related properties.

2. PRELIMINARIES

We recall a few definitions and properties.

Definition 2.1. [10] By a *FI-algebra* we mean an algebra $(X, \rightarrow, 0)$ of type (2,0) satisfying the following axioms: for any $x, y, z \in X$,

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(I3) $x \rightarrow x = 1$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $0 \rightarrow x = 1$,

where $1 = 0 \rightarrow 0$. An FI-algebra X is said to be *regular* if it satisfies $(x')' = x$ for all $x \in X$, where $x' = x \rightarrow 0$. We can define a partial ordering $\leq$ on a FI-algebra X by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a FI-algebra X, the following hold(see [10, 15]): for all $x, y, z \in L$:

(1) $x \rightarrow 1 = 1$,
(2) $1 \rightarrow x = x$,
(3) $(y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(4) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$,
(5) if $x \leq y \rightarrow z$, then $y \leq x \rightarrow z$,
(6) $x \leq y \rightarrow x$,
(7) $x \leq (x \rightarrow y) \rightarrow y$,
(8) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$,

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*Corresponding author: Tel. +82 55 740 1232.

(9) if $x \le y$, then $y = (x \to y) \to y$.

Lemma 2.2. [5] *A Let X be a regular FI-algebra. Then we have $x \to y = y' \to x'$ for all $x, y \in X$.*

Proposition 2.3. [5] *Every filter F of a FI-algebra X has the following property:*

$$x \le y \text{ and } x \in F \text{ imply } y \in F.$$

Lemma 2.4. [1] *Let X be a regular FI-algebra. Then for any $x, y \in X$, the set $\{z \in X | x \le y \to z\}$ has the least element, denoted by $x \otimes y$.*

Lemma 2.5. [1] *Let X be a regular FI-algebra. Then the following hold: for all $a, b, c \in X$,*

(1) $x \otimes y = (x \to y')'$,
(2) $x \le y \to (x \otimes y)$,
(3) *if* $x \le y \to z$, *then* $x \otimes y \le z$.

Definition 2.6. [1] A nonempty subset S of an FI-algebra X is called a *subalgebra* of X if

(i) $0 \in S$,
(ii) $x \to y \in S$ for all $x, y \in S$.

Definition 2.7. [5] Let X be a FI-algebra. Then a nonempty subset A of X is said to be $\otimes$-*closed* if $a \otimes b \in A$ whenever $a, b \in A$.

Note that if S is a subalgebra of a regular FI-algebra X, then S is an $\otimes$-closed subset of X. Indeed, since $x' = x \to 0 \in S$ for all $x \in S$, we have $a \otimes b = (a \to b')' \in S$ for any $a, b \in S$.

We now review some fuzzy concepts. Let X be a set. A function $\mu : X \to [0, 1]$ is called a *fuzzy subset* of X. For any fuzzy subsets μ and ν of a set X, we define

$$\mu \subseteq \nu \Leftrightarrow \mu(x) \le \nu(x) \quad \forall x \in X,$$

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \forall x \in X.$$

Let $f : X \to Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. The fuzzy subset ν of Y defined by

$$\nu(y) := \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\ 0 & \text{otherwise,} \end{cases}$$

is called the *image* of μ under f, denoted by $f[\mu]$. If ν be a fuzzy subset of Y, then the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f and is denoted by $f^{-1}[\nu]$.

A fuzzy subset μ of an FI-algebra X is called a *fuzzy subalgebra* of X if $\mu(x \to y) \ge \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3. Main Results

In this section, we introduce the concept of fuzzy $\otimes$-subalgebras, and we discuss their some properties.

Consider the unit interval $[0, 1]$ and if define $x \to y := \min\{1, 1-x+y\}$ for all $x, y \in [0, 1]$, then $([0, 1], \to, 0)$ is a regular FI-algebra, and we also know that $a \otimes b = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$.

Note that we used the notion of minimum in $[0, 1]$ in defining the concept of fuzzy subalgebras of an FI-algebra. Hereby we will try to introduce the new notion we called fuzzy $\otimes$-subalgebra of an FI-algebra X by using the Lukasiewicz logic.

Since the unit interval $([0,1], \to, 0)$ is a regular FI-algebra where $x \to y := \min\{1, 1 - x + y\}$ for all $x, y \in [0,1]$, we have following definition.

Definition 3.1. A fuzzy subset μ of an FI-algebra X is called a *fuzzy ⊗-subalgebra* of X if

$$(\forall x, y \in X)\ (\mu(x \to y) \geq \mu(x) \otimes \mu(y)).$$

Example 3.2. Let $X := \{0, a, b, c, 1\}$. Define the operation "→" as follows:

→	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

⊗	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	a	b
c	0	0	a	b	c
1	0	a	b	c	1

Then $(X, \to, 0)$ is a regular FI-algebra and we can find the above ⊗-table([5]).
Define fuzzy subsets μ_1, μ_2 and μ_3 of X by

$$\mu_1(1) = 1, \mu_1(a) = \mu_1(b) = \mu_1(c) = 0.7 \text{ and } \mu_1(0) = 0.6.$$

$$\mu_2(1) = 1, \mu_2(a) = \mu_2(b) = 0.7, \mu_2(c) = 0.4 \text{ and } \mu_2(0) = 0.2.$$

$$\mu_3(1) = 1, \mu_3(b) = \mu_3(c) = 0.9 \text{ and } \mu_3(a) = \mu_3(0) = 0.6.$$

Then μ_1, μ_2 and μ_3 are fuzzy ⊗-subalgebras of X. But a fuzzy subset ν of X defined by

$$\nu(1) = 1, \nu(a) = \nu(b) = 0.3, \nu(c) = 0.7 \text{ and } \nu(0) = 0.8$$

is not a fuzzy ⊗-subalgebra of X since

$$\nu(c \to 0) = \nu(a) = 0.3 \ngeq 0.5 = \nu(c) \otimes \nu(0).$$

Note that every fuzzy subalgebra is a fuzzy ⊗-subalgebra, but the converse is not true, because in Example 3.2, a fuzzy subset μ_2 is not fuzzy subalgebra of X since $\mu(b \to a) = \mu(c) = 0.4 \ngeq 0.7 = \min\{\mu(a), \mu(b)\}$.

For every elements x of a regular FI-algebra, we defined

$$x^0 = 1,\ x^n = x^{n-1} \otimes x \text{ and } n(x) \to y = x \to (x \to (\cdots (x \to y) \cdots))$$

in which x occurs n times for $n \in N$.

Proposition 3.3. *If μ is a fuzzy ⊗-subalgebra of an FI-algebra X, then $\mu(1) \geq \mu(x)^2$ and $\mu(n(1) \to x) \geq \mu(x)^{2n+1}$ for all $x \in X$ and $n \in N$.*

Proof. Since $x \to x = 1$ for all $x \in X$, it follows that

$$\mu(1) = \mu(x \to x) \geq \mu(x) \otimes \mu(x) = \mu(x)^2$$

for all $x \in X$.

For the proof of remainder part, we using the induction on n. For $n = 1$, we have $\mu(1 \to x) \geq \mu(1) \otimes \mu(x) \geq \mu(x)^3$ for all $x \in X$. Assume that $\mu(k(1) \to x) \geq \mu(x)^{2k+1}$ for all $x \in X$. Then

$$\begin{aligned} \mu((k+1)1 \to x) &= \mu(1 \to (k(1) \to x)) \\ &\geq \mu(1) \otimes \mu(k(1) \to x) \\ &\geq \mu(x)^2 \otimes \mu(x)^{2k+1} \\ &= \mu(x)^{2(k+1)+1}. \end{aligned}$$

Hence $\mu(n(1) \to x) \geq \mu(x)^{2n+1}$ for all $n \in N$ and $x \in X$. □

Proposition 3.4. *Let μ be a fuzzy $\otimes$-subalgebra of an FI-algebra X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}(\mu(x_n))^2 = 1$, then $\mu(1) = 1$.*

Proof. By Proposition 3.3, we have $\mu(1) \geq \mu(x_n)^2$ for each $n \in N$. Since

$$1 = \lim_{n\to\infty}(\mu(x_n))^2 \leq \mu(1),$$

it follows that $\mu(1) = 1$. □

Note that a fuzzy subset μ of an FI-algebra X is a fuzzy subalgebra of X if and only if a nonempty level subset $U(\mu;t) := \{x \in X|\mu(x) \geq t\}$ is a subalgebra of X for every $t \in [0,1]$. But, we know that if μ is a fuzzy $\otimes$-subalgebra of X, then there exists $t \in [0,1]$ such that $U(\mu;t)$ is not an $\otimes$-closed set of X. In fact, for the fuzzy subset μ_3 of X in Example 3.2, $U(\mu;0.9) = \{b,c,1\}$ is not an $\otimes$-closed set of X.

Proposition 3.5. *If μ is a fuzzy $\otimes$-subalgebra of an FI-algebra X, then*

$$U(\mu;1) := \{x \in X|\mu(x) = 1\}$$

is either empty or an $\otimes$-closed set of X.

Proof. Let $x,y \in X$ be such that x and y belong to $U(\mu;1)$. Then $\mu(x \to y) \geq \mu(x)\otimes\mu(y) = 1$. Hence $\mu(x \to y) = 1$ which implies $x \to y \in U(\mu;1)$. Consequently, $U(\mu;1)$ is an $\otimes$-closed set of X. □

Proposition 3.6. *Let $g : X \to Y$ be a homomorphism of an FI-algebra X and a regular FI-algebra Y. If ν is a fuzzy $\otimes$-subalgebra of Y, then the preimage $g^{-1}[\nu]$ of ν under g is a fuzzy $\otimes$-subalgebra of X.*

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned} g^{-1}[\nu](x_1 \to x_2) &= \nu(g(x_1 \to x_2)) \\ &= \nu(g(x_1) \to g(x_2)) \\ &\geq \nu(g(x_1)) \otimes \nu(g(x_2)) \\ &= g^{-1}[\nu](x_1) \otimes g^{-1}[\nu](x_2). \end{aligned}$$

Thus $g^{-1}[\nu]$ is a fuzzy $\otimes$-subalgebra of X. □

Theorem 3.7. *Let $f : X \to Y$ be a homomorphism of FI-algebras. If μ is a fuzzy $\otimes$-subalgebra of X, then the image $f[\mu]$ of μ under f is a fuzzy $\otimes$-subalgebra of Y.*

Proof. For any $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 \to y_2)$. Consider the set

$$A_1 \to A_2 := \{x \in X|x = a_1 \to a_2 \text{ for some } a_1 \in A_1 \text{ and } a_2 \in A_2\}.$$

If $x \in A_1 \to A_2$, then $x = x_1 \to x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$$f(x) = f(x_1 \to x_2) = f(x_1) \to f(x_2) = y_1 \to y_2,$$

that is, $x \in f^{-1}(y_1 \to y_2) = A_{12}$. Hence $A_1 \to A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f[\mu](y_1 \to y_2) &= \sup_{x\in f^{-1}(y_1\to y_2)} \mu(x) = \sup_{x\in A_{12}} \mu(x) \\ &\geq \sup_{x\in A_1\to A_2} \mu(x) \geq \sup_{x_1\in A_1, x_2\in A_2} \mu(x_1 \to x_2) \\ &\geq \sup_{x_1\in A_1, x_2\in A_2} \mu(x_1) \otimes \mu(x_2). \end{aligned}$$

Since $\otimes : [0,1] \times [0,1] \to [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tilde{x}_1 \geq \sup_{x_1\in A_1} \mu(x_1) - \delta$ and $\tilde{x}_2 \geq \sup_{x_2\in A_2} \mu(x_2) - \delta$, then $\tilde{x}_1 \otimes \tilde{x}_2 \geq \sup_{x_1\in A_1} \mu(x_1) \otimes \sup_{x_2\in A_2} \mu(x_2) - \varepsilon$. Chose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \geq \sup_{x_1\in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \geq \sup_{x_2\in A_2} \mu(x_2) - \delta$. Then

$$\mu(a_1) \otimes \mu(a_2) \geq \sup_{x_1\in A_1} \mu(x_1) \otimes \sup_{x_2\in A_2} \mu(x_2) - \varepsilon.$$

Consequently,

$$\begin{aligned} f[\mu](y_1 \to y_2) &\geq \sup_{x_1\in A_1, x_2\in A_2} \mu(x_1) \otimes \mu(x_2) \\ &\geq \sup_{x_1\in A_1} \mu(x_1) \otimes \sup_{x_2\in A_2} \mu(x_2) \\ &= f[\mu](y_1) \otimes f[\mu](y_2), \end{aligned}$$

and hence $f[\mu]$ is a fuzzy ⊗-subalgebra of Y. □

Let X and Y be FI-algebras and let

$$X \times Y = \{(x, y)|x \in X \text{ and } y \in Y\} .$$

We defined the operation $\to$ on $X \times Y$ by

$$(x_1, y_1) \to (x_2, y_2) := (x_1 \to x_2, y_1 \to y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then we can easily verify that $(X \times Y, \to, (0, 0))$ is an FI-algebra which is called the *direct product algebra* of X and Y.

Proposition 3.8. *For a fuzzy subset σ of an FI-algebra X, let μ_σ be a fuzzy subset of $X \times X$ defined by $\mu_\sigma(x, y) := \sigma(x) \otimes \sigma(y)$ for all $x, y \in X$. Then σ is a fuzzy ⊗-subalgebra of X if and only if μ_σ is a fuzzy ⊗-subalgebra of the direct product algebra $X \times X$.*

Proof. Assume that σ is a fuzzy ⊗-subalgebra of X. For any $x_1, x_2, y_1, y_2 \in X$, we have

$$\begin{aligned} \mu_\sigma((x_1, y_1) \to (x_2, y_2)) &= \mu_\sigma(x_1 \to x_2, y_1 \to y_2) \\ &= \sigma(x_1 \to x_2) \otimes \sigma(y_1 \to y_2) \\ &\geq (\sigma(x_1) \otimes \sigma(x_2)) \otimes (\sigma(y_1) \otimes \sigma(y_2)) \\ &= (\sigma(x_1) \otimes \sigma(y_1)) \otimes (\sigma(x_2) \otimes \sigma(y_2)) \\ &= \mu_\sigma(x_1 \to y_1) \otimes \mu_\sigma(x_2 \to y_2), \end{aligned}$$

and so μ_σ is a fuzzy ⊗-subalgebra of $X \times X$.

Conversely, suppose that μ_σ is a fuzzy ⊗-subalgebra of $X \times X$ and let $x, y \in X$. Then

$$\begin{aligned} (\sigma(x \to y))^2 &= \mu_\sigma(x \to y, x \to y) \\ &= \mu_\sigma((x, x) \to (y, y)) \\ &\geq \mu_\sigma(x, x) \otimes \mu_\sigma(y, y) \\ &= (\sigma(x) \otimes \sigma(y))^2, \end{aligned}$$

and so $\sigma(x \to y) \geq \sigma(x) \otimes \sigma(y)$, that is, σ is a fuzzy ⊗-subalgebra of X. □

A *fuzzy relation* μ on a set X is a fuzzy subset of $X\times X$, that is, a map $\mu : X\times X \to [0, 1]$.

Proposition 3.9. *Let μ be a fuzzy relation on an FI-algebra X satisfying the inequality $\mu(x, y) \leq \mu(x, 1)$ for all $x, y \in X$. Given $z \in X$, let σ_z be a fuzzy subset of X defined by $\sigma_z(x) := \mu(x, z)$ for all $x \in X$. If μ is a fuzzy ⊗-subalgebra of the direct product algebra $X \times X$, then σ_z is a fuzzy ⊗-subalgebra of X for all $z \in X$.*

Proof. Let $x, y, z \in X$. Then

$$\begin{aligned} \sigma_z(x \to y) &= \mu(x \to y, z) = \mu(x \to y, 1 \to z) \\ &= \mu((x, 1) \to (y, z)) \\ &\geq \mu(x, 1) \otimes \mu(y, z) \\ &\geq \mu(x, z) \otimes \mu(y, z) \\ &= \sigma_z(x) \otimes \sigma_z(y), \end{aligned}$$

completing the proof. □

Proposition 3.10. *Let μ be a fuzzy relation on an FI-algebra X and let σ_μ be a fuzzy subset of X given by $\sigma_\mu(x) := \inf_{y\in X}\mu(x, y) \otimes \mu(y, x)$ for all $x \in X$. If μ is a fuzzy ⊗-subalgebra of the direct product algebra $X \times X$ satisfying the equality $\mu(x, 1) = 1 = \mu(1, x)$ for all $x \in X$, then σ_μ is a fuzzy ⊗-subalgebra of X for all $z \in X$.*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned} \mu(x \to y, z) &= \mu(x \to y, 1 \to z) = \mu((x, 1) \to (y, z)) \\ &\geq \mu((x, 1) \otimes \mu(y, z)) = \mu(y, z) \end{aligned}$$

and

$$\begin{aligned} \mu(z, x \to y) &= \mu(1 \to z, x \to y) = \mu((1, x) \to (z, y)) \\ &\geq \mu((z, x) \otimes \mu(z, y)) = \mu(z, y). \end{aligned}$$

It follows that

$$\begin{aligned} \sigma_\mu(x \to y) &= \inf_{z \in X} \mu(x \to y, z) \otimes \mu(z, x \to y) \\ &\geq (\inf_{z \in X} \mu(x, z) \otimes \mu(z, x)) \otimes (\inf_{z \in X} \mu(y, z) \otimes \mu(z, y)) \\ &= \sigma_\mu(x) \otimes \sigma_\mu(y). \end{aligned}$$

This completes the proof. □

Proposition 3.11. *Let μ and ν be fuzzy $\otimes$-subalgebras of FI-algebras X and Y respectively. Then the cross product $\mu \times \nu$ of μ and ν defined by*

$$(\mu \times \nu)(x, y) := \mu(x) \otimes \nu(y)$$

for all $(x, y) \in X \times Y$ is a fuzzy $\otimes$-subalgebra of the direct product algebra $X \times Y$

Proof. The proof is straightforward. □

Proposition 3.12. *Let X and Y be FI-algebras and let μ be a fuzzy $\otimes$-subalgebra of the direct product algebra $X \times Y$. Then the fuzzy subset $P_X[\mu]$ (resp. $P_Y[\mu]$) of X (resp. Y) defined by*

$$P_X[\mu](x) := \mu(x, 1) \text{ (resp.} P_Y[\mu](y) := \mu(1, y))$$

for all $x \in X$ (resp. $y \in Y$) is a fuzzy $\otimes$-subalgebra of X (resp. Y).

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned} P_X[\mu](x_1 \to x_2) &= \mu(x_1 \to x_2, 1) = \mu(x_1 \to x_2, 1 \to 1) \\ &= \mu(x_1, 1) \to \mu(x_2, 1) \\ &\geq \mu(x_1, 1) \otimes \mu(x_2, 1) \\ &= P_X[\mu](x_1) \otimes P_X[\mu](x_2). \end{aligned}$$

Hence $P_X[\mu]$ is a fuzzy $\otimes$-subalgebra of X. Similarly, we can prove that $P_Y[\mu]$ is a fuzzy $\otimes$-subalgebra of Y. □

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K. H. Kim and J. G. Lee, Department of Mathematics, Chungju National University, Chungju 380-702 Korea
E-mail address: ghkim@chungju.ac.kr jglee@chungju.ac.kr

Sang Moon Lee, Department of Computer Science, Chungju National University, Chungju 380-702 Korea
E-mail address: smlee@chungju.ac.kr

E. H. Roh, Department of Mathematics Education, Chinju National University of Education, Jinju 660-756, Korea
E-mail address: ehroh@cue.ac.kr