H-FILTERS OF HILBERT ALGEBRAS

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Abstract. We introduce the concept of Hilbert filter (H-filter, in abbreviation) in Hilbert algebras, and study how to generate an H-filter by a set.

1. Introduction

Following the introduction of Hilbert algebras by A. Diego [5], the algebra and related concepts were developed by D. Busneag [2 - 4]. The present author [7, 8] gave a characterization of a deductive system in a Hilbert algebra, and introduced the notion of commutative Hilbert algebras and gave some characterizations of a commutative Hilbert algebra. In this paper, we introduce the concept of a Hilbert filter (H-filter, in abbreviation) in Hilbert algebras, and study how to generate an H-filter by a set. We also discuss how to generate an H-filter by an H-filter and an element.

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to [2 - 4] and [5].

A Hilbert algebra is a triple \((H, \rightarrow, 1)\), where \(H\) is a nonempty set, \(\rightarrow\) is a binary operation on \(H\), \(1 \in H\) is an element such that the following three axioms are satisfied for every \(x, y, z \in H\):

(i) \(x \rightarrow (y \rightarrow x) = 1\),
(ii) \((x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1\),
(iii) if \(x \rightarrow y = y \rightarrow x = 1\) then \(x = y\).

If \(H\) is a Hilbert algebra, then the relation \(x \leq y\) iff \(x \rightarrow y = 1\) is a partial order on \(H\), which will be called the natural ordering on \(H\). With respect to this ordering 1 is the largest element of \(H\). A bounded Hilbert algebra is a Hilbert algebra with a smallest element 0 relative to the natural ordering. In a bounded Hilbert algebra \(H\) we define a unary operation "G" on \(H\) by \(G(x) := x \rightarrow 0\) for all \(x \in H\).

In a Hilbert algebra \(H\), the following hold:

(1) \(x \leq y \rightarrow x\),
(2) \(x \rightarrow 1 = 1\),
(3) \(x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)\),
(4) \(1 \rightarrow x = x\),
(5) \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)\),
(6) \(x \rightarrow x = 1\),
(7) \(x \leq y\) implies \(z \leq x \rightarrow z\) and \(y \rightarrow z \leq x \rightarrow z\),
(8) \(G(0) = 1\) and \(G(1) = 0\),
(9) \(x \leq y\) implies \(G(y) \leq G(x)\),
(10) \(x \leq G(G(x))\),
(11) \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\).

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(12) \( x \rightarrow y \leq C(y) \rightarrow C(x) \).
(13) \( x \leq (x \rightarrow y) \rightarrow y \) and \( ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y \).

In the sequel, the binary operation “\( \rightarrow \)” will be denoted by juxtaposition.

For any \( x, y \) in a Hilbert algebra \( H \), we define \( x \lor y \) by \((yx)x\). Note that \( x \lor y \) is an upper bound of \( x \) and \( y \).

A Hilbert algebra \( H \) is said to be **commutative** ([8, Definition 2.1]) if for all \( x, y \in H \),
\[
(yx)x = (xy)y, \text{ i.e., } x \lor y = y \lor x.
\]

A subset \( D \) of a Hilbert algebra \( H \) is called a **deductive system** of \( H \) if it satisfies:

(i) \( 1 \in D \),
(ii) \( x \in D \) and \( xy \in D \) imply \( y \in D \).

**2. \( H \)-filters**

In the sequel \( H \) will denote a bounded Hilbert algebra unless otherwise specified. We begin with the following definition.

**Definition 2.1.** A non-empty subset \( F \) of \( H \) is called a **Hilbert filter** (\( H \)-filter, in abbreviation) if

\[
\text{(F1) } 0 \in F,
\text{(F2) } C(C(y)C(x)) \in F \text{ and } y \in F \text{ imply } x \in F,
\text{ for all } x, y \in H.
\]

Under this definition \( \{0\} \) and \( H \) are the simplest examples of \( H \)-filters.

**Example 2.2.** Consider a bounded Hilbert algebra \( H := \{1, x, y, z, 0\} \) with Cayley table as follows:

\[
\begin{array}{cccc}
1 & x & y & z \\
1 & 1 & y & z \\
x & 1 & 1 & y \\
y & 1 & 1 & z \\
z & 1 & 1 & y \\
0 & 1 & 1 & 1 \\
\end{array}
\]

It is easily verified that \( F := \{0, y\} \) and \( G := \{0, z\} \) are \( H \)-filters of \( H \).

**Theorem 2.3.** Let \( F \) be an \( H \)-filter of \( H \) and \( y \in F \). If \( C(y) \leq C(x) \), then \( x \in F \) for all \( x \in H \).

**Proof.** If \( C(y) \leq C(x) \), then \( C(y)C(x) = 1 \) and so \( C(C(y)C(x)) = C(1) = 0 \in F \). It follows from (F2) that \( x \in F \), ending the proof. \( \square \)

Since \( x \leq y \) implies \( C(y) \leq C(x) \) by (9), we have the following corollary.

**Corollary 2.4.** Let \( F \) be an \( H \)-filter of \( H \) and \( y \in F \). If \( x \leq y \), then \( x \in F \) for all \( x \in H \).

**Lemma 2.5.** If \( H \) is commutative, then

(i) \( C(C(x)C(y)) = C(yx) \),
(ii) \( C(C(x)) = x \), for all \( x, y \in H \).
In general, the definition is well-defined. We start with the following.

Then Theorem 2.6.

Assume that C is an H-filter, then 0 ∈ C for all x, y ∈ H.

For a non-empty subset F of H, we define

\[ C(F) := \{ C(x) | x ∈ F \}. \]

In general, C(F) may not be a deductive system even if F is an H-filter of H. In fact, in Example 2.2, C(F) = \{1, z\} is not a deductive system of H since z x = 1 ∈ C(F) and x ∉ C(F).

**Theorem 2.7.** Assume that H is commutative. If F is an H-filter, then C(F) is a deductive system of H.

**Proof.** If F is an H-filter, then 0 ∈ F and so C(0) = 1 ∈ C(F). Let x ∈ C(F) and xy ∈ C(F) for all x, y ∈ H. Then x = C(u) and xy = C(v) for some u, v ∈ F. By using Lemma 2.5(ii), we have

\[ C(C(u)C(C(y))) = C(C(u)y) = C(xy) = C(C(v)) = v ∈ F. \]

It follows from (F2) that C(y) ∈ F so that y = C(C(y)) ∈ C(F). This completes the proof.

**Observation 2.8** Suppose F is a non-empty family of H-filters of H. Then F = ∩F is also an H-filter of H.

Let A be a subset of H. The least H-filter containing A is called the H-filter generated by A, written \( \langle A \rangle \).

Since H is clearly an H-filter containing A, in view of Observation 2.8 we know that the definition is well-defined. We start with the following.

**Observation 2.9.** Let A and B be subsets of H. Then the following hold:

(i) \( \{0\} = \{0\}, \emptyset = \emptyset \).
(ii) \( \langle \emptyset \rangle = H, \langle \{1\} \rangle = H \).
(iii) \( A ⊆ B \) implies \( \langle A \rangle ⊆ \langle B \rangle \).
(iv) \( x ≤ y \) implies \( \langle \{x\} \rangle ⊆ \langle \{y\} \rangle \).
(v) if A is an H-filter of H, then \( \langle A \rangle = A \).

The next statement gives a description of elements of \( \langle A \rangle \).
Theorem 2.8. If $A$ is a non-empty subset of $H$, then

$$\langle A \rangle = \{ x \in H | C(a_n) \ldots (C(a_1)C(x)) \ldots = 1 \text{ for some } a_1, \ldots, a_n \in A \}.$$ 

In order to prove Theorem 2.10 we need the following facts: For any natural number $n$ we define $x^n y$ recursively as follows: $x^1 y = xy$ and $x^{n+1} y = x(x^n y)$. By (5) and induction we know that

$$z(x_n \ldots (x_1 y)) = x_n \ldots (x_1 (zy)) \ldots.$$ 

As a special case of (14) we get

$$z(x^n y) = x^n (zy).$$ 

Now let $a, y, x_1, \ldots, x_n$ be elements of a Hilbert algebra $H$. Then

$$(((x_n \ldots (x_1 (ya)) \ldots))a) = x_n \ldots (x_1 (ya)) \ldots$$

which implies that

$$((x_n \ldots (x_1 (ya)) \ldots))a \leq x_n \ldots (x_1 (ya)) \ldots.$$ 

The reverse inequality follows from (13). Hence we have

$$(x_n \ldots (x_1 (ya)) \ldots)a = x_n \ldots (x_1 (ya)) \ldots.$$ 

Substituting 0 for $a$ and assuming $x_1 = x_2 = \ldots = x_n = x$ in (16), we obtain

$$C(C(x^n y)) = x^n C(y).$$ 

Proof of Theorem 2.10. Denote

$$U = \{ x \in H | C(a_n) \ldots (C(a_1)C(x)) \ldots = 1 \text{ for some } a_1, \ldots, a_n \in A \}.$$ 

We first prove that $U$ is an $H$-filter. Since $A$ is non-empty, there exists $a \in A$. Then $C(a)C(0) = C(a)1 = 1$, whence $0 \in U$. Let $C(C(y)C(x)) \in U$ and $y \in U$. Then there exist $a_i \in A \ (i = 1, \ldots, n)$ and $b_j \in A \ (j = 1, \ldots, m)$ such that

$$C(a_n) \ldots (C(a_1)C(C(y)C(x))) \ldots = 1 \text{ and } C(b_m) \ldots (C(b_1)C(y)) \ldots = 1.$$ 

It follows from (17) that (18) implies

$$C(a_n) \ldots (C(a_1)(C(y)C(x))) \ldots = 1,$$

and so $C(y) \leq C(a_n) \ldots (C(a_1)C(x)) \ldots$. By using (7) we get

$$1 = C(b_m) \ldots (C(b_1)C(y)) \ldots$$

and hence $C(b_m) \ldots (C(b_1)(C(a_n) \ldots (C(a_1)(C(x)) \ldots)) \ldots = 1$. This shows that $x \in U$. Therefore $U$ is an $H$-filter. Now it is clear that $A \subseteq U$. Let $V$ be any $H$-filter containing $A$ and let $x \in U$. Then $C(a_n) \ldots (C(a_1)C(x)) \ldots = 1$ for some $a_1, \ldots, a_n \in A$. Thus

$$1 = C(a_n)(C(a_{n-1}) \ldots (C(a_1)C(x)) \ldots)$$

which implies that

$$C(C(a_n)(C(C(a_n-1)(\ldots (C(a_1)(C(x))))) = C(1) = 0 \in V.$$
Noticing $a_n \in A \subseteq V$ and $V$ to be an $H$-filter, we have $C(C(a_{n-1})\ldots(C(a_1)C(x))\ldots) \in V$. Now
\[ C(C(a_{n-1})\ldots(C(a_1)C(x))\ldots) = C(C(a_{n-1})(C(a_{n-2})\ldots(C(a_1)C(x))\ldots)) = C(C(a_{n-1})(C(C(C(a_{n-2})\ldots(C(a_1)C(x))\ldots)))). \] by (17)

Since $a_{n-1} \in A \subseteq V$, it follows from (F2) that $C(C(a_{n-2})\ldots(C(a_1)C(x))\ldots) \in V$. Repeating the above argument we conclude that $C(C(x)) \in V$. Since $x \leq C(C(x))$, we have $x \in V$ by Corollary 2.4. This proves that $U \subseteq V$, whence $U = (A)$. This completes the proof.

If $A = \{a_1, \ldots, a_n\}$, we will denote $\langle\{a_1, \ldots, a_n\}\rangle = \langle a_1, \ldots, a_n \rangle$ for the sake of convenience. The following corollary is immediate from Theorem 2.10.

**Corollary 2.9.** For any $a \in H$, we have
\[ \langle a \rangle = \{x \in H|C(a)^nC(x) = 1 \text{ for some natural number } n\}. \]

The following theorem shows how to generate an $H$-filter by given an $H$-filter and an element.

**Theorem 2.10.** Let $F$ be an $H$-filter of $H$ and $a \in H$. Then
\[ \langle F \cup \{a\} \rangle = \{x \in H|C(a)^nC(x) \in F \text{ for some natural number } n\}. \]

**Proof.** Denote
\[ U = \{x \in H|C(a)^nC(x) \in F \text{ for some natural number } n\}. \]

Since $C(C(a)^nC(a)) = C(1) = 0 \in F$, therefore $a \in U$. Let $x \in F$. Since $C(x) \leq C(a)C(x) = C(C(C(a)C(x)))$, it follows from Theorem 2.3 that $C(C(a)C(x)) \in F$ so that $x \in U$. Hence $F \cup \{a\} \subseteq U$. In order to prove that $U$ is an $H$-filter, let $C(C(y)C(x)) \in U$ and $y \in U$. Then there are natural numbers $n$ and $m$ such that
\begin{align*}
(19) & \quad C(a)^nC(C(y)C(x))) \in F \\
(20) & \quad C(a)^mC(y)) \in F, \text{ respectively.} \\
(21) & \quad C(a)^n(C(y)C(x)) \in F.
\end{align*}

From (17) it follows that (19) is precisely the following
\[ (20) \quad C(a)^mC(y)) \in F, \text{ and}
(21) \quad C(a)^n(C(y)C(x)) \in F. \]

Using (17) we get
\[ (22) \quad C(a)^nC(C(y)C(x)) = C(C(a)^nC(y)C(x))) = C(u) \quad \text{ and}
(23) \quad C(a)^mC(y) = C(C(a)^mC(y))). = C(v). \]

From (22) we know that $C(y) \leq C(u)(C(a)^nC(x))$, which implies from (5), (7) and (23) that
\[ C(v) = C(a)^mC(y) \leq C(u)(C(a)^{m+n}C(x)). \]

Hence
\[ C(v)(C(u)(C(C(a)^{m+n}C(x)))) = C(v)(C(u)(C(a)^{m+n}C(x))) \quad \text{[by (17)]} \]
\[ = 1. \]

Since $u, v \in F$, it follows from Observation 2.9(v) and Theorem 2.10 that
\[ C(a)^{m+n}C(x)) \in F \]
so that $x \in U$. Clearly, $0 \in U$. Therefore $U$ is an $H$-filter. Finally let $V$ be an $H$-filter containing $F$ and $a$. If $x \in U$, then there exists a natural number $n$ such that $C(C(a)^nC(x)) \in F \subseteq V$. Thus, by (17), we have
\[ C(C(a)(C(C(a)^{n-1}C(x))))) = C(C(a)^nC(x)) \in V. \]
Combining \( a \in V \) and using (F2) we get \( C(C(a)^{-1}C(x)) \in V \). Repeating the procedure above, we conclude that \( C(C(x)) \in V \). It follows from (10) and Corollary 2.4 that \( x \in V \). This proves that \( U \subseteq V \). Therefore \( U \) is the least \( H \)-filter containing \( F \) and \( a \), i.e., \( \langle F \cup \{a\} \rangle = U \). This completes the proof. \( \square \)

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