GOLDEN TRISECTION NUMBERS AND TWO-PLAYER GAME OF KEEP-OR-EXCHANGE

MINORU SAKAGUCHI

Received February 24, 2005

ABSTRACT. A two-player game of Keep-or-Exchange, in which players aim to get the higher score than the opponent in the game, from one, two or three chances of sampling. The game is investigated as a continuous game on the unit square. It is shown that there exists a common optimal strategy for the players which would be called "golden trisection strategy". Related two other interesting games are also discussed.

1 Two-player Game of "Keep-or-Exchange". Consider the two players I and II (sometimes they are denoted by 1 and 2). I(II) observes the sequence of random variables $X_1, X_2, \cdots, X_n(Y_1, Y_2, \cdots, Y_n)$ one-bye-one sequentially. We assume that X_i's and Y_i' s are i.i.d., each with uniform distribution in $[0, 1]$. I(II) chooses his or her decreasing sequence of decision levels

$$(1.1)\qquad \begin{array}{rcl} 1 & \equiv & a_0^{(n)} > a_1^{(n)} > a_2^{(n)} > \cdots > a_{n-1}^{(n)} > a_n^{(n)} \equiv 0 \\ (1 & \equiv & b_0^{(n)} > b_1^{(n)} > b_2^{(n)} > \cdots > b_{n-1}^{(n)} > b_n^{(n)} \equiv 0) \end{array}$$

so that

$$(1.2)\qquad \begin{array}{rcl} \text{I accepts (rejects) } X_i & = & x, \text{ if } x > (<) a_i^{(n)} \\ \text{II accepts (rejects) } Y_i & = & y, \text{ if } y > (<) b_i^{(n)} \end{array}$$

Note taht each player should accept the last random variable (r.v.) if all of his past $n-1$ r.v.s are rejected, since $a_n^{(n)} = b_n^{(n)} = 0$. Choices of one player's decision levels are made independently of the rival's. The game ends as soon as both of the players accept their r.v.s.

Define the *score* for player I by

$$(1.3)\qquad S^1(X_1, \cdots, X_n) = \begin{cases} X_1, & \\ X_t & \text{if} \end{cases} \begin{cases} X_1 \text{ is accepted,} \\ X_1, X_2, \cdots, X_{t-1} \text{ are rejected} \\ \qquad\text{and } X_t \text{ is accepted.} \end{cases}$$

The score $S^2(Y_1, \cdots, Y_n)$, for player II, is defined similarly, with X_i's replaced by Y_i's. After the play is over (*i.e.*, each player accepts the observed value of his r.v.), the scores are compared, and the player with the higher score than his opponent becomes the *winner*. Each player aims to maximize the probability of his winning. The game is called "Keep-or-Exchange". Here, Keep is, in other words, "Accept" or "Stop". Exchange is "Reject" or "Continue".

2000 *Mathematics Subject Classification.* 60G40, 90C99, 90D40.
Key words and phrases. Two-player continuous games on the unit square, optimal strategy, equlibrium strategy, golden bisection number.

Rejection (or Exchange) in (1.2) entails some extent of disadvantage, since the event for example, $a_i^{(n)} > X_i > X_{i+1} > a_{i+1}^{(n)}$ which occurs with positive probability, decreases I's winning probability. The situation, however, is the same for his rival II.

Let $W_i (i = 1, 2)$ be the event that player i wins. Also let $P(W_i) \equiv M_i(\mathbf{a}^{(n)}, \mathbf{b}^{(n)}), i = 1, 2$, be the winning probability for player i, if I and II choose the strategies $\mathbf{a}^{(n)} \equiv (a_1^{(n)}, a_2^{(n)}, \cdots, a_{n-1}^{(n)})$ and $\mathbf{b}^{(n)} \equiv (b_1^{(n)}, b_2^{(n)}, \cdots, b_{n-1}^{(n)})$, respectively. Since "draw" (*i.e.*, the event that there exist no winner) is impossible, we have $\sum_{i=1,2} P(W_i) = 1$.

When n is 2, we already have Ref.[3 ; Theorem 2].

Theorem 1 *Solution to the two-player game of "Keep-or-Exchange"* (1.1) $\sim$ (1.3) *when* n *is* 2 *is as follows. The game has a unique saddle point* $(a_1^{(2)}, b_1^{(2)}) = (g, g)$, *and the saddle value* $M_1(g, g) = M_2(g, g) = \frac{1}{2}$, *where* $g = \frac{1}{2}(\sqrt{5} - 1) \approx 0.61803$ *is a unique root in* $[0, 1]$ *of the equation*

$$g^2 + g = 1. \tag{1.4}$$

The ratio $\bar{g}/g = g = 1/g^{-1} \approx 1/1.61804$ is called the "golden ratio", a mark of beauty in the history of art.

The main purpose of the present paper is to find the solution to the two-player game of "Keep-or-Exchange" when n is 3. Let denote $\mathbf{a}^{(3)}$ and $\mathbf{b}^{(3)}$, simply by $\mathbf{a}^{(3)} = (a_1, a_2)$ and $\mathbf{b}^{(3)} = (b_1, b_2)$, respectively. It is shown by Theorem 2 in Section 2 that the game has value 1/2 and a unique saddle point (b_1^0, b_2^0), which is the unique root in the unit diagonal $1 > b_1 > b_2 > 0$ of a simultaneous third-order algebraic equations. Considering Theorems 1 and 2 together, we would call (b_1^0, b_2^0) the "golden trisection numbers". We pass over an unrest that whether do people feel the trisection ratio $(b_1^0 - b_2^0) : (1 - b_1^0) : b_2^0$ as beautiful. In Section 3, two remarks are given. One is about another kind of Keep-or-Exchange game which has a different type of optimal strategies. The other is about the game of "Risky Exchange", when n is 3, which is more difficult to solve, since the game has a positive probability of draw and so becomes non-constant-sum. A sketch of deriving the solution is shown.

2 Solution to the Game of "Keep-or-Exchange", when n is 3.

First we note that $P(\text{draw}) = 0$, $\sum_{i=1,2} M_i(a_1, a_2, b_1, b_2) = 1$ and if $a_i = b_i, i = 1, 2$, then

$$M_1(a_1, a_2, , b_1, b_2) = 1/2, \quad \forall 1 \geq a_1 \geq a_2 \geq 0 \tag{2.1}$$

by symmetry of the two players' roles.

Let $p_{RRA-RRA}$ be the winning probability for I, when the play proceeds $X_1 < a_1$ and $X_2 < a_2$ for I, and $Y_1 < b_1, Y_2 < b_2$ for II. Let p_{RA-A} be the winning probability for I, when the play proceeds $X_1 < a_1, X_2 > a_2$ for I and $Y_1 > b_1$ for II. The other seven probabilities p_{RRA-RA}, p_{RRA-A} *etc.*, are defined similarly. Then we find that

$$\begin{aligned} P(W_1) &\equiv M_1(a_1, a_2, b_1, b_2) \\ &= p_{RRA-RRA} + p_{RRA-RA} + (\text{other seven probabilities}) \end{aligned} \tag{2.2}$$

and

$$p_{RRA-RRA} = P\{X_1 < a_1, X_2 < a_2, Y_1 < b_1, Y_2 < b_2, X_3 > Y_3\} = \frac{1}{2} a_1 a_2 b_1 b_2, \tag{2.3}$$

$$p_{RRA-RA} = a_1a_2b_1P\{X_3 > Y_2 > b_2\} = \frac{1}{2}a_1a_2b_1\bar{b}_2{}^2, \tag{2.4}$$

$$p_{RRA-A} = a_1a_2P\{X_3 > Y_1 > b_1\} = \frac{1}{2}a_1a_2\bar{b}_1{}^2, \tag{2.5}$$

$$p_{RA-RRA} = a_1b_1b_2P\{X_2 > a_2 \vee Y_3\} = \frac{1}{2}a_1b_1b_2(1-a_2^2), \tag{2.6}$$

$$\begin{aligned} p_{RA-RA} &= a_1b_1P\{X_2 > a_2, Y_2 > b_2, X_2 > Y_2\} \\ &= \frac{1}{2}a_1b_1\{\bar{b}_2{}^2 - (a_2-b_2)^2 I(a_2 > b_2)\}, \end{aligned} \tag{2.7}$$

$$p_{RA-A} = a_1P\{X_2 > a_2, Y_1 > b_1, X_2 > Y_1\} = \frac{1}{2}a_1\{\bar{b}_1{}^2 - (a_2-b_1)^2 I(a_2 > b_1)\}, \tag{2.8}$$

(where $I(e)$ is the indicator of the event e),

and finally

$$p_{A-RRA} = b_1b_2P\{X_1 > a_1 \vee Y_3\} = \frac{1}{2}b_1b_2(1-a_1^2), \tag{2.9}$$

$$p_{A-RA} = b_1P\{X_1 > a_1, X_1 > Y_2 > b_2\} = \frac{1}{2}b_1\{\bar{b}_2{}^2 - (a_1-b_2)^2 I(a_1 > b_2)\}, \tag{2.10}$$

$$p_{A-A} = P\{X_1 > a_1, X_1 > Y_1 > b_1\} = \frac{1}{2}\{\bar{b}_1{}^2 - (a_1-b_1)^2 I(a_1 > b_1)\}. \tag{2.11}$$

Summing these nine equations (2.3) $\sim$ (2.11), we have from (2.2),

$$\begin{aligned} P(W_1) = \; & \frac{1}{2}a_1a_2(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2) \\ & + \frac{1}{2}a_1\left[b_1b_2(1-a_2^2) + b_1\left\{\bar{b}_2{}^2 - (a_2-b_2)^2 I(a_2 > b_2)\right\}\right. \\ & \quad \left. + \left\{\bar{b}_1{}^2 + (a_2-b_1)^2 I(a_2 > b_1)\right\}\right] \\ & \quad + \frac{1}{2}\left[b_1b_2(1-a_1^2) + b_1\left\{\bar{b}_2{}^2 - (a_1-b_2)^2 I(a_1 > b_2)\right\}\right. \\ & \qquad \left. + \left\{\bar{b}_1{}^2 - (a_1-b_1)^2 I(a_1 > b_1)\right\}\right] \end{aligned} \tag{2.12}$$

We make sure that Eqs (2.3) $\sim$ (2.11) do not involve any error, by showing that, if $a_i = b_i, i = 1, 2$, then Eq.(2.1) holds true. This is easy since (2.12) becomes

$$\begin{aligned} P(W_1) = \; & \frac{1}{2}a_1a_2(a_1a_2 + a_1\bar{a}_2{}^2 + \bar{a}_1{}^2) \\ & + \frac{1}{2}a_1\left\{a_1a_2(1-a_2^2) + a_1\bar{a}_2{}^2 + \bar{a}_1{}^2\right\} \\ & \quad + \frac{1}{2}\left[a_1a_2(1-a_1^2) + a_1\left\{\bar{a}_2{}^2 - (a_1-a_2)^2\right\} + \bar{a}_1{}^2\right] \end{aligned} \tag{2.13}$$

which is found, after some effort of simplification, to be equal to $\frac{1}{2}$, $\forall 1 \geq a_1 \geq a_2 \geq 0$.

Now we prove

Lemma 2.1 *Assume that* $a_1 = b_1$. *Then both of* $\max_{a_2\in[0,a_1]} P(W_1|1 > b_1 > b_2 > 0)$ *and* $\min_{b_1\in[0,b_1]} P(W_1|1 > a_1 > a_2 > 0)$ *are attained at*

$$a_2^* = b_2^* = \frac{1}{2}\left\{\sqrt{4b_1^{-1} - 3 + 4b_1} - 1\right\}, \tag{2.14}$$

if $a_1 = b_1 \in (b_1^*, 1]$, *where* $b_1^*(\approx 0.6825)$ *is a unique root in* $[0,1]$ *of the cubic equation*

$$b^3 + b - 1 = 0. \tag{2.15}$$

From Eq.(2.14), *we see that, if* $b_1 = 1$, *then* $a_2^* = b_2^* = g = \frac{1}{2}(\sqrt{5} - 1) \approx 0.61803$, *the golden bisection number.*

Proof. We try to find I's optimal $a_2 \in [0, a_1)$, when we fix $1 > b_1 > b_2 > 0$.
Since the third term of Eq.(2.12) does not involve a_2, we have

$$\begin{aligned}\frac{\partial}{\partial a_2}P(W_1|1 > b_1 > b_2 > 0) \;=\;& \frac{1}{2}b_1(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2) - b_1^2b_2a_2 \\ &+\begin{cases} 0, & \text{if } 0 < a_2 < b_2 \\ -b_1^2(a_2 - b_2), & \text{if } b_2 < a_2 < b_1. \end{cases}\end{aligned} \tag{2.16}$$

The r.h.s. is decreasing in $0 < a_2 < b_1 = a_1$ and equals zero at $a_2 = b_2$, *i.e.*,

$$\frac{1}{2}b_1(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2) - b_1^2b_2^2 = 0$$

or

$$b_2^2 + b_2 - (b_1^{-1} - 1 + b_1) = 0$$

and hence

$$b_2 = \frac{1}{2}\left\{\sqrt{4b_1^{-1} - 3 + 4b_1} - 1\right\}. \tag{2.17}$$

Therefore the restriction $1 > b_1 > b_2 > 0$ requires that

$$\sqrt{4b_1^{-1} - 3 + 4b_1} < 2b_1 + 1 \qquad (\text{and so } b_1^3 + b_1 - 1 > 0),$$

or equivalently $b_1 \in (b_1^*, 1]$, where $b_1^* \approx 0.6825$ is a unique root in $[0,1]$ of the cubic equation (2.15).

Next we try to find II's optimal $b_2 \in [0, b_1)$ when we fix $1 > a_1 > a_2 > 0$. From (2.12) we find that

$$\begin{aligned}&\frac{\partial}{\partial b_2}P(W_1|1 > a_1 > a_2 > 0) \\ &= \frac{1}{2}a_1a_2(b_1 - 2b_1\bar{b}_2) + \frac{1}{2}a_1b_1\left\{(1 - a_2^2) - 2\bar{b}_2 + 2(a_2 - b_2)I(a_2 > b_2)\right\} \\ &\quad + \frac{1}{2}b_1\left[(1 - a_1^2) - 2\bar{b}_2 + 2(a_1 - b_2)I(a_1 > b_2)\right] \\ &= \frac{1}{2}a_1^2a_2(1 - 2\bar{b}_2) + \frac{1}{2}a_1^2(1 - a_2^2 - 2\bar{b}_2) + \frac{1}{2}a_1(1 - a_1^2 - 2\bar{b}_2) \\ &\quad + \begin{cases} a_1^2(a_2 - b_2) + a_1(a_1 - b_2), & \text{if } 0 < b_2 < a_2 \\ a_1(a_1 - b_2), & \text{if } a_2 < b_2 < a_1 \end{cases}\end{aligned} \tag{2.18}$$

(from our assumption that $a_1 = b_1$)

The r.h.s. is increasing in $0 < b_2 < b_1 (= a_1)$, since the sum of the coefficients of b_2 terms equals $a_1^2 a_2$ if $0 < b_2 < a_2$ and $a_1^2(1+a_2)$ if $a_2 < b_2 < a_1$, and this is equal to zero at $b_2 = a_2$ *i.e.*,

$$\frac{1}{2}b_1^2 b_2(1-2\bar{b}_2) + \frac{1}{2}b_1^2(1-b_2^2-2\bar{b}_2) + \frac{1}{2}b_1(1-b_1^2-2\bar{b}_2) + b_1(b_1-b_2) = 0$$

which, when simplified, becomes Eq.(2.17) again. □

Lemma 2.2 *Assume that* $a_2 = b_2$. *Then both of* $\max_{a_1\in[a_2,1]} P(W_1|1 > b_1 > b_2 > 0)$ *and* $\min_{b_1\in[b_2,1]} P(W_1|1 > a_1 > a_2 > 0)$ *are attained at*

$$a_1^* = b_1^* = \frac{1}{2(1+b_2)}\left\{\sqrt{4b_2^2+8b_2+5}-1\right\}, \tag{2.19}$$

if $a_2 = b_2 \in [0, b_2^*)$, *where* $b_2^*(\approx 0.7546)$ *is a unique root in* $[0,1]$ *of the cubic equation*

$$b^3 + b^2 - 1 = 0. \tag{2.20}$$

Note that Eq.(2.19) *gives* $a_1^* = b_1^* = g = \frac{1}{2}(\sqrt{5}-1)$, *if* $b_2 = 0$.

Proof. We try to find I's optimal choice of $a_1 \in [a_2, 1]$ when we fix $1 > b_1 > b_2 > 0$. From (2.12), we have

$$\begin{aligned}
&\frac{\partial}{\partial a_1} P(W_1|1 > b_1 > b_2 > 0) \qquad (2.21)\\
&= \frac{1}{2}a_2(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2)\\
&\quad + \frac{1}{2}\left[b_1b_2(1-a_2^2) + b_1\left\{\bar{b}_2{}^2 - (a_2-b_2)^2 I(a_2 > b_2)\right\}\right.\\
&\qquad \left. + \left\{\bar{b}_1{}^2 - (a_2-b_1)^2 I(a_2 > b_1)\right\}\right]\\
&\qquad\quad - \left[b_1b_2a_1 + b_1(a_1-b_2)I(a_1 > b_2) + (a_1-b_1)I(a_1 > b_1)\right]\\
&= \frac{1}{2}\left[b_2(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2) + \left\{b_1b_2(1-b_2^2) + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2\right\}\right] - b_1b_2a_1\\
&\qquad - \begin{cases} b_1(a_1-b_2), & \text{if } b_2 < a_1 < b_1\\ b_1(a_1-b_2) + (a_1-b_1), & \text{if } b_1 < a_1 < 1.\end{cases}\\
&\qquad (\text{since } a_2 = b_2 \text{ and } I(a_2 > b_2) = I(a_2 > b_1) = 0)
\end{aligned}$$

The r.h.s. is decreasing in $(a_2 =) b_2 < a_1 < 1$, and equals zero at $a_1 = b_1$ *i.e.*,

$$\frac{1}{2}\left[b_2(b_1b_2 + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2) + \left\{b_1b_2(1-b_2^2) + b_1\bar{b}_2{}^2 + \bar{b}_1{}^2\right\}\right] - b_1^2b_2 - b_1(b_1-b_2) = 0.$$

This equation becomes, after simplification,

$$b_1^2 + (1+b_2)^{-1}b_1 - 1 = 0,$$

or

$$b_1 = \frac{1}{2(1+b_2)}\left\{\sqrt{4b_2^2+8b_2+5}-1\right\}. \tag{2.22}$$

Therefore the restriction $1 > b_1 > b_2 > 0$ requires that

$$2b_2^2 + 2b_2 + 1 < \sqrt{4b_2^2 + 8b_2 + 5} \quad (\text{and so } b_2^3 + b_2^2 - 1 < 0),$$

or equivalently $b_2 \in [0, b_2^*)$, where $b_2^* \approx 0.7546$ is a unique root of Eq.(2.20).

Next we try to find II's optimal $b_1 \in (b_2, 1]$ when we fix $1 > a_1 > a_2 > 0$.

From (2.12) we find that

$$\text{(2.23)} \qquad \frac{\partial}{\partial b_1} P(W_1 | 1 > a_1 > a_2 > 0)$$

$$\begin{aligned}
= \ & \frac{1}{2} a_1 a_2 (b_2 + \bar{b}_2{}^2 - 2\bar{b}_1) \\
& + \frac{1}{2} a_1 \left[b_2(1 - a_2^2) + \left\{ \bar{b}_2{}^2 - (a_2 - b_2)^2 I(a_2 > b_2) \right\} - 2\bar{b}_1 + 2(a_2 - b_1) I(a_2 > b_1) \right] \\
& \quad + \frac{1}{2} \left[b_2(1 - a_1^2) + \left\{ \bar{b}_2{}^2 - (a_1 - b_2)^2 I(a_1 > b_2) \right\} - 2\bar{b}_1 + 2(a_1 - b_1) I(a_1 > b_1) \right] \\
= \ & \frac{1}{2} \left[(a_1 a_2 + a_1 + 1)(1 - b_2 + b_2^2) - a_1 a_2^2 b_2 - a_1^2 b_2 - (a_1 - b_2)^2 \right] \\
& \qquad -\bar{b}_1 (a_1 a_2 + a_1 + 1) + \begin{cases} a_1 - b_1, & \text{if } b_2 < b_1 < a_1 \\ 0, & \text{if } a_1 < b_1 < 1. \end{cases} \\
& (\text{since } a_2 = b_2, I(a_2 > b_2) = I(a_2 > b_1) = 0 \text{ and } I(a_1 > b_2) = 1)
\end{aligned}$$

The r.h.s. is increasing in $(a_2 =) b_2 < b_1 < 1$, since the sum of coefficients of b_1 is $(a_1 a_2 + a_1 + 1) - 1 = a_1(1 + a_2) > 0$. And this is equal to zero at $b_1 = a_1$ *i.e.*,

$$\frac{1}{2} \left[(1 - b_2 + b_2^2)(b_1 b_2 + b_1 + 1) - \left\{ b_1 b_2^3 + b_1^2 b_2 + (b_1 - b_2)^2 \right\} \right] - \bar{b}_1 (b_1 b_2 + b_1 + 1) = 0,$$

which becomes, after some effort of simplification,

$$b_1^2 + (1 + b_2)^{-1} b_1 - 1 = 0,$$

that is, Eq.(2.22) again. □

Considering symmetry for the two players and combining Lemmas 2.1 and 2.2 we obtain

Theorem 2 *Solution to the two-player game of "Keep-or-Exchange"* (1.1) $\sim$ (1.3), *when* n *is* 3, *is as follows. The game has a unique saddle point* $(a_1^{(3)}, a_2^{(3)}, b_1^{(3)}, b_2^{(3)}) = (b_1^0, b_2^0, b_1^0, b_2^0)$, *where* $(b_1^0, b_2^0) \approx (0.743, 0.657)$ *is a unique root in the triangle* $0 < b_2 < b_1 < 1$ *of the simultaneous equation*

$$\text{(2.14)} \qquad b_2 = \frac{1}{2} \left\{ \sqrt{4b_1^{-1} - 3 + 4b_1} - 1 \right\} \qquad \text{(in Lemma 2.1)}$$

$$\text{(2.17)} \qquad b_1 = \frac{1}{2(1 + b_2)} \left\{ \sqrt{4b_2^2 + 8b_2 + 5} - 1 \right\} \qquad \text{(in Lemma 2.2)}$$

The values of the game are $1/2, 1/2$.

Proof. We have to show that Eqs (2.14)-(2.17) has the stated unique root. In the triangle $0 < b_2 < b_1 < 1$, Eq.(2.14) is a convex decreasing function in $b_1 \in (b_1^*, 1]$ connecting the two points (b_1^*, b_1^*) and $(1, g)$. And Eq.(2.17) is a concave increasing function in $b_2 \in [0, b_2^*)$ connecting the two points $(b_1, b_2) = (g, 0)$ and (b_2^*, b_2^*). Here, $b_1^* \approx 0.6825$ and $b_2^* \approx 0.7546$ are given by (2.15) and (2.20), respectively. Therefore a unique root (b_1^0, b_2^0) exists in the triangle $0 < b_2 < b_1 < 1$, and a rough computation gives $b_1^0 \approx 0.743$ and $b_2^0 \approx 0.657$. □

As was mentioned in Section 1, when we compare Theorem 2 with Theorem 1, we would call (b_1^0, b_2^0) the golden trisection numbers. We, however, pass over some unrest, whether do people feel the trisection ratio $(1-b_1^0) : (b_1^0-b_2^0) : b_2^0 \approx 0.257 : 0.086 : 0.657 \approx 1 : 0.335 : 2.556$ is beautiful. A more reasonable understanding may be as follows. In the game of Keep-or-Exchange, an intelligent player would choose his decision levels greater than 1/2 since $EX = EY = 1/2$ (a direct proof will be needed). If we consider the strategy space, the one-fourth of the unit square (*i.e.*, $\frac{1}{2} \le a, b \le 1$), then the player's common decision level(s) has the ratio

$$\bar{g} : (g - \frac{1}{2}) \approx 0.382 : 0.118 \approx 1 : 0.309$$

when n is 2, by Theorem 1; and

$$(1 - b_1^0) : (b_1^0 - b_2^0) : (b_2^0 - \frac{1}{2}) \approx 0.257 : 0.086 : 0.157 \approx 1 : 0.335 : 0.611$$

when n is 3, by Theorem 2.

An ingenious work by Mazalov (Ref.[1]) in 1996 gave the same result by using dynamic programming (DP). The optimality equation is

$$V_i(x|\mathbf{b}) = h(x|\mathbf{b}) \vee EV_{i+1}(X|\mathbf{b}), \quad (i = 1, 2, \cdots, n; V_{n+1}(x|\mathbf{b}) \equiv 0) \tag{2.24}$$

where

$$h(x|\mathbf{b}) = \sum_{j=1}^{n} b_1 b_2 \cdots b_{j-1}(x - b_j) I(x > b_j) \tag{2.25}$$

is I's winning probability if he stops when $X_i = x$. He shows that $a_i^* = b_i^*, \ i = 1, 2, \cdots, n-1$, and these values satisfy the system of equations

$$\sum_{j=1}^{n} \left(\prod_{k=0}^{j-1} b_k \right) \left[1 - 2(b_j \vee b_i) + (b_j \vee b_{i+1})^2\right] = 0, \quad i = 1, 2, \cdots, n-1. \tag{2.26}$$

This system gives

$$b_1^2 + b_1 - 1 = 0$$

i.e., (1.4), where n is 2, and

$$\begin{cases} b_2^2 + b_2 - (b_1^{-1} - 1 + b_1) = 0, \\ b_1^2 + (1 + b_2)^{-1} b_1 - 1 = 0 \end{cases}$$

i.e., (2.17)-(2.22), when n is 3.

Mazalov's work shows a wonderful effect of applying DP to n-stage dynamic games. The present author feels that the routine procedure to derive Theorem 2 without using DP has yet an instructive worth.

3 Remarks.

Remark 1. Consider the one-player version of the Keep-or-Exchange game, when n is 3, where player aims to maximize his expected score. Let $v^{(n)}$ be the optimal expected score. Then the equation

$$v^{(n)} = E\left[X \vee v^{(n-1)}\right], \quad (n = 1, 2, \cdots, v^{(1)} = 1/2)$$

gives the optimal decision levels $(1 >) v^{(3)} \approx 0.695 > v^{(2)} = 5/8 = 0.625 (> 0)$.
Remark 2. If the players are restricted to choosing $a_1^{(3)} = a_2^{(3)} (= a)$ and $b_1^{(3)} = b_2^{(3)} (= b)$. Then the solution to the game becomes different. We prove

Theorem 3 *If the player's choices of decision levels are restricted by* $a_1^{(3)} = a_2^{(3)} = a$ *and* $b_1^{(3)} = b_2^{(3)} = b$, *then the solution is as follows. The game has a unique saddle point* $(a, b) = (b_0, b_0)$, *where* $b_0 \approx 0.728$ *is a unique root in* $[0, 1]$ *of the fourth-order algebraic equation*

$$b^4 + b^3 + 2b^2 - b - 1 = 0. \tag{3.1}$$

The saddle value of the game is $1/2$.

Proof. By substituting $a_1 = a_2$ and $b_1 = b_2$ into Eq.(2.12). collecting terms and simplifying, we get

$$\begin{aligned} 2P(W_1) &= \left[-a^3b^2 + (1 - b - b^2 + b^3)a^2 + (1 + a)(1 - b + b^3)\right] \\ &\qquad -(ab - a + b + 1)(a - b)^2 I(a > b). \end{aligned} \tag{3.2}$$

$$\begin{aligned} 2\frac{\partial}{\partial a} P(W_1) &= -3a^2b^2 + 2a(1 - b)(1 - b^2) + 1 - b + b^3 \\ &\quad + \begin{cases} 0, & \text{if } a < b \\ (3 - b)(a - b)^2 - 2(a - b)(ab + 1), & \text{if } a > b \end{cases} \end{aligned} \tag{3.3}$$

$$\begin{aligned} 2\frac{\partial}{\partial b} P(W_1) &= -2a^3b + (3b^2 - 2b - 1)a^2 + (1 + a)(3b^2 - 1) \\ &\quad + \begin{cases} 0, & \text{if } a < b \\ -(3 + a)(a - b)^2 + 2(a - b)(ab + 1), & \text{if } a > b \end{cases} \end{aligned} \tag{3.4}$$

Both of $\left[\frac{\partial}{\partial a} P(W_1)\right]_{a=b} = 0$ and $\left[\frac{\partial}{\partial b} P(W_1)\right]_{a=b} = 0$ give the same equation (3.1). Therefore, if $b_0 \approx 0.728$ is determined by Eq.(3.1), then we find that

$$P(W_1|0, b_0) = P(W_1|1, b_0) = \frac{1}{2}(1 - b_0 + b_0^3) \approx 0.3289 < 1/2, \tag{3.5}$$

$$\begin{aligned} \left[\frac{\partial}{\partial a} P(W_1|a, b_0)\right]_{a=0} &= \frac{1}{2}(1 - b_0 + b_0^3) > 0 > \left[\frac{\partial}{\partial a} P(W_1|a, b_0)\right]_{a=1} \\ &= b_0^3 + b_0^2 - 5b_0 + 2 \approx -0.7242. \end{aligned} \tag{3.6}$$

$$\frac{\partial^2 P(W_1|a, b_0)}{\partial a^2} = \begin{cases} -3ab_0^2 + (1 - b_0)(1 - b_0^2), & \text{if } a < b_0 \\ -3a(b_0^2 + b_0 - 1) + b_0(b_0^2 + b_0 - 4) < 0, & \text{if } a > b_0 \end{cases} \tag{3.7}$$

Hence $P(W_1|a, b_0)$ is, as a function of $a \in [0,1]$, increasing and convex-concave for $0 < a < b_0$ (with the point of inflexion $a = \frac{(1-b_0)(1-b_0^2)}{3b_0^2} \approx 0.0804$), $P(W_1|b_0, b_0) = \frac{1}{2}$, and decreasing, concave for $b_0 < a < 1$. So we finally find that

$$\max_{a\in[0,1]} P(W_1|a, b_0) = P(W_1|b_0, b_0) = 1/2. \tag{3.8}$$

The proof of the remained part that $\min_{b\in[0,1]} P(W_1|b_0, b) = P(W_1|b_0, b_0) = 1/2$ is almost the same, and so we will not repeat the detail. □

A closely related (and partly more general) game is investigated by the present author in Ref[2 ; Section 4].

Remark 3. The game with the score for player I

$$S^1(X_1, \cdots, X_n) = \begin{cases} X_1, \\ X_t, \\ \\ X_n I(X_n > X_{n-1}), \end{cases} \text{if} \begin{cases} X_1 \text{ is accepted,} \\ X_1, X_2, \cdots, X_{t-1} \text{ are rejected,} \\ \text{and } X_t \text{ is accepted, for } 2 \le t \le n-1 \\ X_1, \cdots, X_{n-1} \text{ are rejected.} \end{cases} \tag{3.9}$$

and the score $S^2(Y_1, \cdots, Y_n)$ for player II, given by similarly with X_i's replaced by Y_i's, is called "Risky Exchange".

Differently from the Keep-or-Exchange game,

$$P(\text{draw}) = \left[\prod_{i=1}^{n-2} a_i^{(n)} b_i^{(n)}\right] P\left\{X_n < X_{n-1} < a_{n-1}^{(n)}, Y_n < Y_{n-1} < b_{n-1}^{(n)}\right\}$$

is positive, and the game is not a constant-sum game.

When n is 2, we already have (Ref.[4 ; Theorems 1 and 2]and Ref.[5 ; Theorem 2])

Theorem 4 *Solution to the two-player game of "Risky Exchange" when n is 2 is as follows. The game has a unique equilibrium point* $(a_1^{(2)}, b_1^{(2)}) = (a^*, a^*)$, *and the equilibrium payoffs*

$$P(\text{draw}) = \frac{1}{4}a^{*4} \approx 0.02184,$$

$$M_1(a^*, a^*) = M_2(a^*, a^*) = \frac{1}{2}(1 - \frac{1}{4}a^{*4}) \approx 0.48908,$$

where $a \approx 0.54368$ *is a unique root in* $[0,1]$ *of the cubic equation*

$$a^3 + a^2 + a = 1. \tag{3.10}$$

Players want to stop(=accept) a little bit earlier than in the Keep-or-Exchange game, in order to avoid his risk in his final(*i.e.*, the n-th) stage (Compare Theorem 4 with Theorem 1, both when n is 2).

It would be interesting to derive the solution of the game of "Risky Exchange" when n is 3, and find how the single characterizing equation (3.10) changes to a simultaneous equation characterizing the equilibrium point $(a_1^{(3)} = b_1^{(3)}, a_2^{(3)} = b_2^{(3)})$. This would not be difficult. First we find

$$P(\text{draw}) = a_1 b_1 P\{X_3 < X_2 < a_2, Y_3 < Y_2 < b_2\} = \frac{1}{4}a_1 a_2^2 b_1 b_2^2.$$

Using again the definitions of probabilities $p_{RRA-RRA}$ *etc*, as the same as in Section 2, we introduce the winning probabilities $q_{RRA-RRA}$ *etc* for player II with similar meanings as in p's. Then we find that

$$\begin{aligned} p_{RRA-RRA} &= a_1 b_1 P\left[(X_2 < a_2 \wedge X_3) \cap \{(Y_2 < b_2 \wedge Y_3, X_3 > Y_3) \cup (Y_3 < Y_2 < b_2)\}\right] \\ &= a_1 b_1 \left[\int\int_{1>t>s>0} (a_2 \wedge t)(b_2 \wedge s) dt ds + \frac{1}{2}(a_2 - \frac{1}{2}a_2^2) b_2^2 \right], \end{aligned}$$

$$p_{RRA-RA} = a_1 b_1 P\left[\{X_2 < a_2 \wedge X_3\} \cap \{X_3 > Y_2 > b_2\}\right] = a_1 b_1 \int\int_{1>t>s>0} (a_2 \wedge t)(s - b_2) dt ds,$$

$$p_{RRA-A} = a_1 P\left[(X_2 < a_2 \wedge X_3) \cap (b_1 < Y_1 < X_3)\right] = a_1 \int_{b_1}^{1} (a_2 \wedge t)(t - b_1) dt$$

and for other six probabilities. Also

$$\begin{aligned} q_{RRA-RRA} &= a_1 b_1 P\left[\{Y_2 < b_2 \wedge Y_3\} \cap \{(X_2 < a_2 \wedge X_3, X_3 < Y_3) \cup (X_3 < X_2 < a_2)\}\right] \\ &= a_1 b_1 \left[\int\int_{1>s>t>0} (a_2 \wedge t)(b_2 \wedge s) dt ds + \frac{1}{2} a_2^2 (b_2 - \frac{1}{2} b_2^2) \right], \end{aligned}$$

$$\begin{aligned} q_{RRA-RA} &= a_1 b_1 P\left[\{Y_2 > b_2\} \cap \{(X_2 < a_2, X_2 < X_3 < Y_2) \cup (X_3 < X_2 < a_2)\}\right] \\ &= a_1 b_1 \left[\int_{b_2}^{1} ds \int_{t < a_2 \wedge s} (s - t) dt + \frac{1}{2} a_2^2 \bar{b}_2 \right], \end{aligned}$$

$$\begin{aligned} q_{RRA-A} &= a_1 P\left[\{Y_1 > b_1\} \cap \{(X_2 < a_2, X_2 < X_3 < Y_1) \cup (X_3 < X_2 < a_2)\}\right] \\ &= a_1 \left[\int_{b_1}^{1} ds \int_{t < a_2 \wedge s} (s - t) dt + \frac{1}{2} a_2^2 \bar{b}_1 \right], \end{aligned}$$

and for other six probalities. In order to compute the above eighteen probabilities p's and q's, Lemmas 1.1 and 1.2 in Ref.[6] are helpful. The game reduces to a non-constant-sum continuous game on the unit square.

Remark 4. Three-player game "Keep-or-Exchange" and "Risky Exchange" both for $n = 2$ are solved in Ref [4] and [6], respectively. Three-player games of these two when n is 3 are left to be solved.

Acknowledgement
I would like to thank Professor Vladimir Mazalov of Karelian Research Center, Petrozadvsk, for some very helpful discussion of this area of problems.

References

[1] Mazalov, V.V., *A game related to optimal stopping of two sequences of independent random variables having different distributions*, Math.Japonica, **43**(1996), 121-128.

[2] Sakaguchi, M. and Szajowski, K., *Single-level strategies for Full-Information best-choice problems II*, Math.Japonica, **46**(1997), 199-208.

[3] Sakaguchi, M., *Two-player games of "Score Showdown"*, Scientiae Math. Japonicae, **60**(2004), 411-421.

[4] Sakaguchi, M., *Three-player game of "Keep-or-Exchange"*, Game Th. Appl. **10**(2005), 148-160.

[5] Sakaguchi, M., *Equilibrium in two-player games of "Showcase Showdown"*, Scientiae Math. Japonicae **61**(2005), 145-151.

[6] Sakaguchi, M., *Equilibrium in the three-player game of "Risky Exchange"*, Scientiae Math. Japonicae **61**(2005), 335-344.

*3-26-4 Midorigaoka, Toyonaka, Osaka, 560-0002, Japan,
Fax: +81-6-6856-2314 E-mail: minorus@tcct.zaq.ne.jp