CORRECTIONS AND ADDITION TO THE PAPER "KELLERER-STRASSEN TYPE MARGINAL MEASURE PROBLEM"

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Abstract. A report on an error in the paper [5] is given. The error is in **Theorem 2.4** of [5]. The condition imposed on *the dominant measure* ρ in the theorem of [5] is insufficient in order to deduce the conclusion of the theorem. In this note a corrected theorem is given. See **Theorem A** in §2 of this note. In this note a stronger condition on ρ is imposed. Corrections for the two corollaries of the theorem in [5] are also given. See **Corollaries A1** and **A2** in this note.

1. Introduction. The fact that Theorem 2.4 in [5] is not true is seen by giving a counter example for a corollary to the theorem. See **Example** below in this note. Since Theorem 2.4 in [5] is not true, there must be one or more errors in the proof in [5] of the theorem. In fact the author found an error in the Step 4 of the proof of the theorem. The statement

> *For any $f \in C(T)$ there is a monotone increasing sequence $\{h_n\}$ of step functions such that $\lim_{n\to\infty} h_n(t) = f(t)$ for all $t \in T$ and $T[h_n \neq 0] = \{t \in T; h_n(t) \neq 0\} \subset T_n$ for any n*

in Step 4 of the proof of the theorem in [5] is not true. It follows that the statement

> *$\widetilde{\eta}$ defined in Step 2 belongs to $\Lambda(T)$. Therefore one has $\widetilde{\eta}(T) = 1$.*

in Step 5 cannot be proved. Our aim in this note is to give a stronger condition for the dominant measure ρ in order for the conclusions of the theorem and its corollaries in [5] to remain true. In the case where the component spaces are Polish spaces the new condition for the dominant measure ρ is the condition that it is σ-finite on the algebra consisting of all finite unions of measurable rectangles of the product space. In the cases where the component spaces are separable metric spaces or separable measurable spaces, the condition that ρ is rectangle-normal is imposed.

2. Results. Let X be a Hausdorff space. We denote by $\beta(X)$ the set of all Borel subsets of X. Let $B^b(X)$ denote the set of all bounded real valued Borel measurable functions on X.

In what follows, unless the contrary is explicitly stated, $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ denote separable measurable spaces. We denote by $(Z, \underline{\underline{A}} \otimes \underline{\underline{B}})$ the product space of these measurable spaces. Let π_X and π_Y be the canonical projections from Z onto X and Y, respectively. We denote by $R(Z)$ the algebra consisting of all finite unions of measurable rectangles of Z. $P(Z)$ denotes the set of all probability measures on Z. When ρ is a σ-finite measure on Z, we put

$$\Lambda(Z) = \{\lambda \in P(Z); \lambda \leq \rho\}.$$

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For any set A we denote by χ_A the characteristic function of A.

First we shall give a counterexample for Corollary 2.7 in [5].

Example. Put $X = Y = (0,\ 1]$ and $Z = (0,\ 1] \times (0,\ 1]$. Let λ be the Lebesgue measure on $(0,\ 1]$. Let $\lambda \otimes \lambda$ be the product measure of λ. Let $0 < \epsilon < \frac{2}{3}$. Put

$$S_\epsilon = \{(x, y) \in Z; 0 < x < y \leq \min\{x + \epsilon, 1\}\}.$$

Put

$$\varphi_\epsilon(x, y) = \frac{2}{\epsilon(2-\epsilon)} \chi_{S_\epsilon}(x, y)$$

We denote by θ_ϵ the measure on Z having φ_ϵ as the density function concerning $\lambda \otimes \lambda$. Put

$$h(x, y) = \begin{cases} \dfrac{1}{(y-x)^2} & \text{if } 0 < x < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We denote by ρ the σ-finite measure on Z having h as the density function concerning $\lambda \otimes \lambda$.

Lemma 2.1. *If* $0 < \epsilon < \frac{2}{3}$, *then* $\theta_\epsilon \in \Lambda(Z) = \{\theta \in P(Z); \theta \leq \rho\}$.

Proof. Since the area of S_ϵ is $\frac{\epsilon(2-\epsilon)}{2}$, we have $\theta_\epsilon \in P(Z)$. If $x < y \leq x + \epsilon$ and $0 < x \leq 1 - \epsilon$, then we have $0 < y - x < \epsilon$. In this case we have $\frac{1}{\epsilon^2} < \frac{1}{(y-x)^2}$. If $0 < x < y \leq 1$ and $1 - \epsilon < x \leq 1$, then we have $0 < y - x < \epsilon$. In this case we have $\frac{1}{\epsilon^2} < \frac{1}{(y-x)^2}$. Therefore for any (x, y) in Z with $0 < x < y \leq 1$ we have

$$\varphi_\epsilon(x, y) = \frac{2}{\epsilon(2-\epsilon)} \chi_{S_\epsilon}(x, y) \leq \frac{1}{\epsilon^2} < \frac{1}{(y-x)^2} = h(x, y).$$

Hence we have $\theta_\epsilon \in \Lambda(Z)$.

For any $k \in B^b(Z)$ put

$$P(k) = \sup\{\int_Z k\ d\theta; \theta \in \Lambda(Z)\}.$$

Lemma 2.2. *For any* $f \in B^b(X)$ *and* $g \in B^b(Y)$ *one has*

$$\int_0^1 f(x)\ d\lambda(x) + \int_0^1 g(y)\ d\lambda(y) \leq P(f \circ \pi_X + g \circ \pi_Y).$$

Proof. Since the measures treated in this lemma are probability measures, it is sufficient for us to prove the inequality for non-negative functions f and g.

$$(1) \qquad 2\int_0^1 \int_0^1 \chi_{S_\epsilon}(x, y) f(x)\ d\lambda \otimes \lambda(x, y)$$

$$= 2\int_0^{1-\epsilon} f(x)\ d\lambda(x)\int_x^{x+\epsilon} d\lambda(y) + 2\int_{1-\epsilon}^1 f(x)\ d\lambda(x)\int_x^1 d\lambda(y)$$
$$= 2\epsilon\int_0^{1-\epsilon} f(x)\ d\lambda(x) + 2\int_{1-\epsilon}^1 (1-x)f(x)\ d\lambda(y)$$

and

$$(2) \qquad \epsilon(2-\epsilon)\int_0^1 f(x)\ d\lambda(x) = \epsilon(2-\epsilon)(\int_0^{1-\epsilon} f(x)\ d\lambda(x) + \int_{1-\epsilon}^1 f(x)\ d\lambda(x)).$$

Hence we have

$$2\int_0^1\int_0^1 \chi_{S_\epsilon}(x,y)f(x)\ d\lambda\otimes\lambda(x,y) - \epsilon(2-\epsilon)\int_0^1 f(x)\ d\lambda(x)$$
$$\geq \epsilon^2\int_0^{1-\epsilon} f(x)\ d\lambda(x) - \epsilon(2-\epsilon)\int_{1-\epsilon}^1 f(x)\ d\lambda(x).$$

Accordingly we have

$$(3) \qquad \int_Z f\circ\pi_X\ d\theta_\epsilon - \int_0^1 f(x)\ d\lambda(x)$$
$$= \frac{2}{\epsilon(2-\epsilon)}\int_0^1\int_0^1 \chi_{S_\epsilon}(x,y)f(x)\ d\lambda\otimes\lambda(x,y) - \int_0^1 f(x)\ d\lambda(x)$$
$$\geq \frac{\epsilon}{(2-\epsilon)}\int_0^{1-\epsilon} f(x)\ d\lambda(x) - \int_{1-\epsilon}^1 f(x)\ d\lambda(x).$$

By a similar argument we have

$$(4) \qquad \int_Z g\circ\pi_Y\ d\theta_\epsilon - \int_0^1 g(y)\ d\lambda(y) \geq \frac{\epsilon}{(2-\epsilon)}\int_\epsilon^1 g(y)\ d\lambda(y) - \int_0^\epsilon g(y)\ d\lambda(y).$$

Since $\theta_\epsilon \in \Lambda(Z)$, we have

$$P(f\circ\pi_X + g\circ\pi_Y) - \Big(\int_0^1 f(x)\ d\lambda(x) + \int_0^1 g(y)\ d\lambda(y)\Big)$$
$$\geq \frac{\epsilon}{(2-\epsilon)}\Big(\int_0^{1-\epsilon} f(x)\ d\lambda(x) + \int_\epsilon^1 g(y)\ d\lambda(y)\Big) - \Big(\int_{1-\epsilon}^1 f(x)\ d\lambda(x) + \int_0^\epsilon g(y)\ d\lambda(y)\Big).$$

Letting $\epsilon \to 0$, then we have

$$P(f\circ\pi_X + g\circ\pi_Y) - \Big(\int_0^1 f(x)\ d\lambda(x) + \int_0^1 g(y)\ d\lambda(y)\Big) \geq 0.$$

This shows that Lemma 2.2 holds.

In this example there does not exist a θ in $\Lambda(Z)$ having $\mu = \lambda$ and $\nu = \lambda$ as marginals (see Kellerer [1], page 196). This is a counterexample for Corollary 2.7 in [5]. It follows that Theorem 2.4 in [5] is not true.

We correct Theorem 2.4 in [5] as follows.

Theorem A *Let X and Y be Polish spaces and let $Z = X \times Y$. Let ρ be a measure on $\beta(Z)$ such that the restriction ρ_0 of ρ to $R(Z)$ is σ-finite on $R(Z)$ and $\rho(Z) \geq 1$. Given*

$\mu \in P(X)$ and $\nu \in P(Y)$, then the following two conditions are equivalent to each other:
(1) There exists a $\theta \in \Lambda(Z)$ having μ and ν as marginals.
(2) For any functions $f \in \beta(X)$ and $g \in \beta(Y)$ one has

$$\int_X f\ d\mu + \int_Y g\ d\nu \leq \sup\{\int_Z (f \circ \pi_X + g \circ \pi_Y)\ d\lambda; \lambda \in \Lambda(Z)\}.$$

Proof. The implication (1) $\rightarrow$ (2) is almost obvious. We shall show the converse. Put

$$B_0^b(Z) = \{f \circ \pi_X + g \circ \pi_Y; f \in B^b(X) \text{ and } g \in B^b(Y)\}.$$

We define a linear functional W_0 on $B_0^b(Z)$ by the equation

$$W_0(f \circ \pi_X + g \circ \pi_Y) = \int_X f\ d\mu + \int_Y g\ d\nu.$$

For any function $h \in B^b(Z)$ put

$$P_Z(h) = \sup\{\int_Z h\ d\lambda; \lambda \in \Lambda(Z)\}.$$

W_0 is a linear functional on $B_0^b(Z)$ satisfying $W_0 \leq P_Z$ on $B_0^b(Z)$. Since P_Z is subadditive and positively homogeneous on $B^b(Z)$, W_0 can be extended, by the Hahn Banach theorem, to a linear functional W on $B^b(Z)$ such that $W \leq P_Z$ on $B^b(Z)$. For any set $E \in R(Z)$ put

$$\theta_0(E) = W(\chi_E).$$

Clearly θ_0 is a finitely additive measure on $R(X)$ having μ and ν as marginals. Since X and Y are Polish spaces, μ and ν are Radon measures on X and Y, respectively. Accordingly θ_0 is a finite Radon measure on $R(Z)$. By Theorem 16 in [3] (page 51) the measure θ_0 can be extended to a probability measure θ on $\beta(Z)$. Since ρ_0 is a σ-finite measure on $R(Z)$, ρ_0 can be uniquely extended to the measure ρ^* on $\beta(Z)$. By the uniqueness of extension we have $\rho = \rho^*$ on $\beta(Z)$. Since $\theta_0 \leq \rho_0$ on $R(Z)$, we have $\theta \leq \rho^* = \rho$ on $\beta(Z)$. Clearly θ has μ and ν as marginals. Thus the theorem has been proved.

We correct Corollary 2.7 in [5] as follows.

Corollary A1 *Let X and Y be separable metric spaces and let $Z = X \times Y$. Let ρ be a rectangle-normal measure on $\beta(Z)$ with $\rho(Z) \geq 1$. Given $\mu \in P(X)$ and $\nu \in P(Y)$, then the following conditions are equivalent to each other:*
(1) There exists a measure $\theta \in \Lambda(Z)$ having μ and ν as marginals.
(2) For any functions $f \in B^b(X)$ and $g \in B^b(Y)$ one has

$$\int_X f\ d\mu + \int_Y g\ d\nu \leq \sup\{\int_Z (f \circ \pi_X + g \circ \pi_Y)\ d\lambda; \lambda \in \Lambda(Z)\}.$$

Proof. The implication (1) $\rightarrow$ (2) is almost obvious. We shall show the converse. Let $\widetilde{X}$ and $\widetilde{Y}$ be the completions of X and Y, respectively. Put $\widetilde{Z} = \widetilde{X} \times \widetilde{Y}$. Since ρ is rectangle-normal measure on Z, there exist disjoint sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_m\}_{m=1}^{\infty}$ of sets in $\beta(X)$ and $\beta(Y)$ respectively such that $X = \bigcup_{n=1}^{\infty} X_n$ and $Y = \bigcup_{m=1}^{\infty} Y_m$ and $\rho(X_n \times Y_m) < \infty, n, m =$

$1, 2, \ldots$. For any positive integer n there exists a set $A_n \in \beta(\widetilde{X})$ such that $A_n \cap X = X_n$. Put

$$\widetilde{X_n} = A_n - \bigcup_{i=1}^{n-1} A_i,\ n = 1, 2, \ldots .$$

$\{\widetilde{X_n}\}_{n=1}^{\infty}$ is a disjoint sequence of sets in $\beta(\widetilde{X})$ such that $X_n = \widetilde{X_n} \cap X$. Put $\widetilde{X_0} = \widetilde{X} - \bigcup_{n=1}^{\infty} \widetilde{X_n}$. $\{X_n\}_{n=0}^{\infty}$ is a disjoint sequence of sets in $\beta(\widetilde{X})$ such that $\widetilde{X} = \bigcup_{n=0}^{\infty} \widetilde{X_n}$. Similarly there exists a disjoint sequence $\{\widetilde{Y_m}\}_{m=0}^{\infty}$ of sets in $\beta(\widetilde{Y})$ such that $\widetilde{Y} = \bigcup_{m=0}^{\infty} \widetilde{Y_m}$ and for any positive integer m $Y_m = \widetilde{Y_m} \cap Y$ and $\widetilde{Y_0} \cap Y = \emptyset$. For any $\widetilde{E} \in \beta(\widetilde{Z})$ put $\widetilde{\rho}(\widetilde{E}) = \rho(\widetilde{E} \cap Z)$. $\widetilde{\rho}$ is a rectangle-normal measure on $\widetilde{Z}$. In fact, we have

$$\widetilde{\rho}(\widetilde{X_0} \times \widetilde{Y_m}) = \rho((\widetilde{X_0} \cap X) \times (\widetilde{Y_m} \cap Y)) = \rho(\emptyset \times Y_m) = 0,\ m = 0, 1, \ldots$$

and, for any positive integer n,

$$\widetilde{\rho}(\widetilde{X_n} \times \widetilde{Y_m}) = \rho(X_n \times Y_m) < \infty,\ m = 0, 1, \ldots .$$

Hence $\widetilde{\rho}$ is a rectangle-normal measure on $\widetilde{Z}$. i_X, etc., denote the canonical injection from X into $\widetilde{X}$. For every $\mu \in P(X)$ we denote by $i_X(\mu)$ the image measure of μ by i_X. Put $i_X(\mu) = \widetilde{\mu}$ and $i_Y(\nu) = \widetilde{\nu}$. We have $i_Z(\Lambda(Z)) = \{\widetilde{\theta} \in P(\widetilde{Z}); \widetilde{\theta} \leq \widetilde{\rho}\} = \Lambda(\widetilde{Z})$. In fact, it is obvious that $i_Z(\Lambda(Z))$ is contained in $\Lambda(\widetilde{Z})$. We shall show that the converse. Let $\widetilde{\theta} \in \Lambda(\widetilde{Z})$. For any set $E \in \beta(Z)$ there exists an $\widetilde{E} \in \beta(\widetilde{Z})$ such that $E = \widetilde{E} \cap Z$. Put $\theta(E) = \widetilde{\theta}(\widetilde{E})$. θ is well defined, that is, if $E = \widetilde{E} \cap Z = \widetilde{F} \cap Z$ ($\widetilde{E},\ \widetilde{F} \in \beta(\widetilde{Z})$), then we have $\widetilde{\theta}(\widetilde{E}) = \widetilde{\theta}(\widetilde{F})$. In fact, since

$$(\widetilde{E} \bigtriangleup \widetilde{F}) \cap Z = E \bigtriangleup E = \emptyset,$$

we have

$$\widetilde{\theta}(\widetilde{E} \bigtriangleup \widetilde{F}) \leq \widetilde{\rho}(\widetilde{E} \bigtriangleup \widetilde{F}) = \rho((\widetilde{E} \bigtriangleup \widetilde{F}) \cap Z) = \rho(\emptyset) = 0,$$

where $\widetilde{E} \bigtriangleup \widetilde{F}$ denotes the symmetric difference of $\widetilde{E}$ and $\widetilde{F}$. Hence we have $\widetilde{\theta}(\widetilde{E}) = \widetilde{\theta}(\widetilde{F})$. Since, for any disjoint sequence $\{E_n\}_{n=1}^{\infty}$ of sets in $\beta(Z)$, there exists a disjoint sequence $\{\widetilde{E_n}\}_{n=1}^{\infty}$ of sets in $\beta(\widetilde{Z})$ such that $\widetilde{E_n} \cap Z = E_n,\ n = 1, 2, \ldots$, θ is a measure on Z. Hence we have $i_Z(\theta) = \widetilde{\theta}$. For any functions $\widetilde{f} \in B^b(\widetilde{X})$ and $\widetilde{g} \in B^b(\widetilde{Y})$ put $f = \widetilde{f} \circ i_X$ and $g = \widetilde{g} \circ i_Y$. Then we have

$$\begin{aligned}
\int_{\widetilde{X}} \widetilde{f}\ d\widetilde{\mu} + \int_{\widetilde{Y}} \widetilde{g}\ d\widetilde{\nu} &= \int_X \widetilde{f} \circ i_X\ d\mu + \int_Y \widetilde{g} \circ i_Y\ d\nu \\
&= \int_X f\ d\mu + \int_Y g\ d\nu \\
&\leq \sup\{\int_Z (f \circ \pi_X + g \circ \pi_Y)\ d\lambda; \lambda \in \Lambda(Z)\} \\
&= \sup\{\int_Z (\widetilde{f} \circ i_X \circ \pi_X + \widetilde{g} \circ i_Y \circ \pi_Y)\ d\lambda; \lambda \in \Lambda(Z)\}.
\end{aligned}$$

Since $\pi_{\widetilde{X}} \circ i_Z = i_X \circ \pi_X$ and $\pi_{\widetilde{Y}} \circ i_Z = i_Y \circ \pi_Y$ and $i_Z(\Lambda(Z)) = \Lambda(\widetilde{Z})$, we have

$$\int_{\widetilde{X}} \widetilde{f}\, d\widetilde{\mu} + \int_{\widetilde{Y}} \widetilde{g}\, d\widetilde{\nu} \leq \sup\{\int_{\widetilde{Z}} (\widetilde{f} \circ \pi_{\widetilde{X}} + \widetilde{g} \circ \pi_{\widetilde{Y}})\, d\widetilde{\lambda}; \widetilde{\lambda} \in \Lambda(\widetilde{Z})\}.$$

Since $\widetilde{X}$ and $\widetilde{Y}$ are Polish spaces, by Theorem A there exists a $\widetilde{\theta} \in \Lambda(\widetilde{Z})$ having $\widetilde{\mu}$ and $\widetilde{\nu}$ as marginals. Since $i_Z(\Lambda(Z)) = \Lambda(\widetilde{Z})$, there exists a $\theta \in \Lambda(Z)$ such that $i_Z(\theta) = \widetilde{\theta}$. Then we have

$$i_X(\mu) = \widetilde{\mu} = \pi_{\widetilde{X}}(\widetilde{\theta}) = \pi_{\widetilde{X}} \circ i_Z(\theta) = i_X \circ \pi_X(\theta).$$

Since i_X is an injective map from $P(X)$ into $P(\widetilde{X})$, we have $\mu = \pi_X(\theta)$. Similary we have $\nu = \pi_Y(\theta)$. Since $\widetilde{\theta} \in \Lambda(\widetilde{Z})$, we have $\widetilde{\theta} \leq \widetilde{\rho}$. Accordingly $\theta \leq \rho$ holds. Thus Corollary A1 has been proved.

In the case where component spaces are separable measurable spaces Corollary 2.8 in [5] is corrected as follows.

Corollary A2 *Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be separable measurable spaces and $(Z, \underline{\underline{A}} \otimes \underline{\underline{B}})$ the cartesian product of these spaces. Let ρ be a rectangle-normal measure on Z with $\rho(Z) \geq 1$. Given probability measures μ and ν on X and Y, respectively, then the following two conditions are equivalent to each other:*
(1) *There exists a probability measure θ on Z such that $\theta \leq \rho$ and $\pi_X(\theta) = \mu$ and $\pi_Y(\theta) = \nu$.*
(2) *For any real valued bounded measurable functions f and g on X and Y respectively one has*

$$\int_X f\, d\mu + \int_Y g\, d\nu \leq \sup\{\int_Z (f \circ \pi_X + g \circ \pi_Y)\, d\lambda; \lambda \in \Lambda(Z)\}.$$

Proof. Any separable mesurable space is isomorphic to a measurable space consisting of a subset of the closed interval [0, 1] and its Borel σ-algebra (see [2], page 5, 4°). Therefore this corollary can be proved by transforming to the problem into one for the case of separable metric spaces

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